Chapter 18

Path Integrals in Quantum Theories: A Pedagogic First Step

The universe in each dimension
is vast beyond all comprehension.
A myriad of mysteries,
a multitude of histories …
From *Divine Intentions*
by R. Klauber

18.0 Preliminaries

As I mentioned on the first page of the book, I strongly believe it is far easier, and more meaningful, for students to learn quantum field theory (QFT) first by the canonical quantization method, and once that has been digested, move on to the path integral (functional integral, many paths, or sum over histories) approach (functional quantization). The rest of the book is devoted to the first of these; the present chapter, to a brief introduction to the second.

Two approaches to (ways to quantize) QFT:
1) canonical
2) path integral

#1 simpler, rest of book;
#2 introduced in this chapter

18.0.1 Chapter Overview

This chapter was composed so it can be read independently of (without reading) the rest of the book. So, some things may be defined/discussed again herein that are covered elsewhere in the text.

In this chapter, we will define

- the functional and
- the functional integral,

then, with regard to non relativistic quantum mechanicis (NRQM),

- transition amplitudes for position eigenstates,
- the role of the Lagrangian and the wave function peak,
- the central idea in Feynman's path integral approach,
- expressing that idea mathematically, including Feynmans' three postulates,
- comparing the path integral approach in NRQM to Schrödinger and Heisenberg's,
- determining the transition amplitude from the functional integral, and
- applying the theory to an example.

We'll examine path integrals:
- *math behind*
- *NRQM*
- *QFT*

Then, with regard to QFT, we will investigate

- comparing particle theory (NRQM) to field theory (QFT)
- "derivation" of the many paths approach to QFT, and
- deducing the form of the transition amplitude for QFT

18.1 Background Math

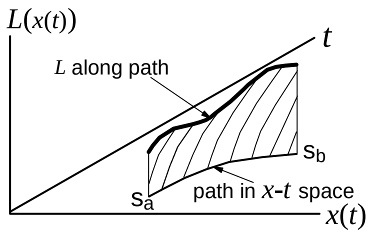
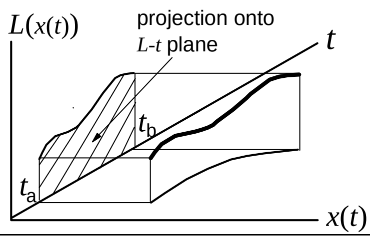
18.1.1 Integrating Functions of a Function

Functionals form the mathematical roots of Feynman's many paths approach to quantum theories. To help in understanding the concept, consider first a function of another function, such as the Lagrangian of a particle, which is typically a function of particle position x and its time derivative $\dot{x}$. Position x, in turn, is a function of time t, i.e., $x(t)$, and finding that functional dependence on time comprises typical problems to be solved.

Integrating a function of a function

There are several ways we can integrate such a function of another function, two being shown in Wholeness Chart 18-1 (Part A) below. The figures and comments in that chart should be self explanatory. Mathematically, L can be any function of a function, but for our purposes, it will generally be the Lagrangian.

Wholeness Chart 18-1. From a Function of a Function to the Functional Integral– Part A

Process		Graphically	Math	Comment
1.	Integration over the path in $x(t)$ vs t space = area shown		$\int_{s_a}^{s_b} L ds$ where s is spacetime distance along path	L is a function of the function x (and $\dot{x}$), and the functional dependence of x on t is typically the problem to be solved. Integration shown is not relevant for us.
2.	Integration over t = projection of the area in #1 onto the L-t plane		$F = \int_{t_a}^{t_b} L dt$ F is a <u>functional</u>	If L is the Lagrangian, then this integral $F = S$, the action. Classically, S = minimum (or stationary) for physical paths

18.1.2 Defining "Functional"

Our definition of a functional F tailored to quantum physics

In the path integral approach to quantum physics, we use a narrower definition of a functional than the general mathematical definition[1]. We define integration #2 above, the integral of the function (L) of a function ($x(t)$) with respect to the independent variable (t) between fixed limits t_a and t_b as a <u>functional</u>, and designate it as F. It is a number that depends on the form of the function $x(t)$, on t_a, and on t_b. It is different for different paths.

That definition

$$F = \int_{t_a}^{t_b} L dt \quad \text{(for a particular path)} \tag{18-1}$$

Functionals are symbolized by enclosing their arguments in square brackets.

$$\underline{\text{Symbolism}}: \qquad F\left[x(t)\right] \text{ or } F\left[x\right] , \tag{18-2}$$

though you may see functionals written with normal, rather than square, brackets.

If L = Lagrangian, F = S, the action

If L is the Lagrangian, then the functional $F = S$, the action.

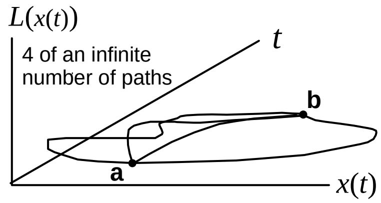
18.2 Defining Functional Integral

The functional integral is an integral (over all paths) of the functional F (itself an integral)

A functional (our definition) is a definite integral, i.e., a number obtained by integrating between the end points of a certain path. Yet, because we get a different such number for each different path in x-t space, we can integrate those numbers over all possible paths. In other words, the functional, an integral for us, can itself be integrated. Such integrations are not simple, nor is their purpose at all obvious at this point. They are visualized in cases #4 and #7 below and are called <u>functional integrals</u>. We devote much of this chapter to explaining their origin, value, and means to evaluate. For now, just let the general concept sink in, without straining to analyze it too much.

[1] Mathematically, a functional is a function of a vector space to a scalar field, i.e., a functional maps a vector to a scalar. Spatial functions of time, i.e., paths, form a vector space by themselves, so our narrower definition is in line with the general definition. In our case, the mapping involves an integration.

Wholeness Chart 18-1 (continued). From a Function of a Function to the Functional Integral – Part B

Procedure		Graphically	Math	Comment
3.	Sum F values as in #2 above for a number of discrete paths between **a** and **b**.		$\sum_{n=1}^{4} F_n = \sum_{n=1}^{4} \int_{t_a}^{t_b} L_n dt$	Not relevant for us.
4.	Integrate F over all possible (continuous range of) paths between **a** and **b**.	Hard to show visually.	$\int_{x_a}^{x_b} F \mathcal{D}x(t)$	Not relevant for us. $\mathcal{D}x(t)$ implies all paths.
5.	Another function of F (i.e., where F is the argument), e.g. exponentiation of F.	Not graphic. Raise e to i times value F for a given path.	$e^{iF[x(t)]} = e^{i\int_{t_a}^{t_b} Ldt}$	Relevant for us.
6.	Sum e^{iF} values for a number of discrete paths, like in #3 above.	Same paths as in #3.	$\sum_{n=1}^{4} e^{iF_n} = \sum_{n=1}^{4} e^{i\int_{t_a}^{t_b} L_n dt}$	Relevant for us.
7.	Integration like #4 above over all possible paths in $x(t)$ vs t space.	Hard to show visually. Same paths as in #4.	$\int_{x_a}^{x_b} e^{iF} \mathcal{D}x(t)$	Feynman QM path integral approach. All paths, not just classical.

Adding F for all paths → integration over all paths = functional integration

The chart above should be relatively self explanatory. In summary, we can add the values F_n for a discrete number of paths N (= 4 in #3). In the limit of adding all paths, we pass to an integral (don't worry how for now), where we use the <u>symbol $\mathcal{D}x(t)$</u> to represent that functional integration.

$$\sum_{n=1}^{N} F_n = \sum_{n=1}^{N} \int_{t_a}^{t_b} L_n dt \xrightarrow[\text{total paths } N \to \infty]{\text{limit as}} \int_{x_a}^{x_b} F \mathcal{D}x(t) \qquad \text{(not relevant for us)}. \qquad (18\text{-}3)$$

Instead of F, do functional integration of e^{iF} over all paths. This is important in quantum physics

Alternatively, we can do the same thing for a function of F, such as e^{iF} (as in #6 and #7 above). Note that e^{iF} can itself be considered a functional, as it comprises a mapping from $x(t)$ to a (complex) scalar.

$$\sum_{n=1}^{N} e^{iF_n} = \sum_{n=1}^{N} e^{i\int_{t_a}^{t_b} L_n dt} \xrightarrow[\text{total paths } N \to \infty]{\text{limit as}} \int_{x_a}^{x_b} e^{iF} \mathcal{D}x(t) \qquad \text{(will be relevant for us)}. \qquad (18\text{-}4)$$

We will evaluate (18-4) for a free quantum particle later in this chapter.

<u>Alternative nomenclature:</u> Because <u>functional integration</u> involves integration over paths (in x-t space), Feynman's approach is often also referred to as the <u>path integral</u> approach.

18.3 The Transition Amplitude

18.3.1 General Wave Functions (States)

Review of states, amplitudes, & probability

Recall from QM wave mechanics, that for a general normalized wave function ψ equal to a superposition of energy eigenfunction waves (which are each also normalized),

$$\psi = A_1\psi_1 + A_2\psi_2 + A_3\psi_3 \,, \qquad (18\text{-}5)$$

$\psi \to \psi_1$ transition amplitude = A_1

A_1 is the amplitude of ψ_1, so the probability of finding ψ_1 upon measuring is

$$A_1^* A_1 = |A_1|^2 \,. \qquad (18\text{-}6)$$

Probability of transition = $|A_1|^2$

If we were to start with ψ initially, and measure ψ_1 later, the wave function would have collapsed, i.e., underwent a transition to a new state. (18-6) would be the transition probability.