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Symmetry, Invariance, and Conservation for Free Fields

"The time has come", the walrus said, "to speak of many things,
of symmetries, Lagrangians, and changeless transformings."

Re-rendering of Lewis Carroll
by R. Klauber

6.0 Preliminaries

My apologies to Lewis Carroll for the liberties taken with his great work, but the Jabberwockian, oxymoron-like phrase "changeless transforming" will come to have deep significance for us. We will find it central to our understanding of symmetry in general, and more specifically, in our study of quantum field theory.

6.0.1 Background

Symmetry is one of the most aesthetically captivating and philosophically meaningful concepts known to mankind. Rooted originally in the arts, it has evolved and re-emerged in our modern age as a unified and holistic structural basis for all of science.

But if so, what then, particularly in mathematical terms, is it? If, in a work of art, it is a quality, perhaps somewhat abstract and related closely to feeling and emotion, how does it relate to physics? Can it be defined precisely?

We begin our answer to these questions after the chapter preview below.

6.0.2 Chapter Overview

A simple definition of symmetry with examples

First, an introduction to symmetry,

where we will look at

- a simple definition of symmetry without math,
- examples of symmetry, and
- a mathematical definition of symmetry.

Symmetry in classical mechanics

Then, symmetry in classical physics, including

- laws of nature symmetric under Lorentz transformation, i.e., laws are invariant in spacetime (same for all inertial observers)
- symmetry in the Lagrangian $L \rightarrow$ a related quantity is conserved

Symmetry in QFT

Then, symmetry in quantum field theory, including

- field equations symmetric under Lorentz transformation, i.e., they are invariant in spacetime (same for all inertial observers)
- symmetry in the Lagrangian density $\mathcal{L} \rightarrow$ a related quantity is conserved
- symmetry, gauges, and gauge theories

Free vs interacting fields

We will deal primarily with free particles and fields in this chapter, but the principles will apply in general, as we shall see when we investigate interactions.

Symmetry principles apply to free and interacting cases, but only free in this chapter

6.1 Introduction to Symmetry

6.1.1 Symmetry Simplified

Each of us has some intuitive feel for what symmetry is, though most might, at least at first, have some difficulty coming up with a very precise definition. Certainly snowflakes have symmetry, and so do cylinders and beach balls. A map of New York probably does not. Just what exactly is it that we sense about an object that causes us to deem it symmetric?

To see what that certain something is, imagine yourself looking at a real life version of the cylinder depicted in the figure below. Then imagine closing your eyes for a moment, and during the time you can't see, someone else rotates the cylinder about the vertical axis shown in the figure. When you open your eyes is there any way you could tell that the rotation had taken place? The answer, of course, is no, but what does that mean?

It means that even though something changed (the rotational position of the cylinder), something else remained unchanged. The form we perceive, the wholeness that is the cylinder, looks exactly the same. The act of moving or "transforming" the cylinder simultaneously exhibits the qualities of both change (transformation) and non-change (invariance).

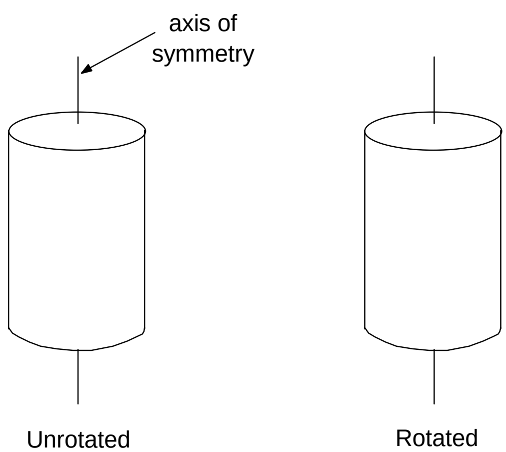

Figure 6-1. Symmetry of a Cylinder

Symmetry is the propensity for non-change with change

So what then is *symmetry*? It is simply the propensity *for non-change with change, for invariance under transformation.* In many cases, such as this one, it is a relationship between the whole and the parts in which the whole can exhibit changelessness while the component parts change. In virtually every case, it involves superficial change with more profound non-change.

Symmetry manifests to greater or lesser degrees. A sphere, for instance, has more symmetry than a cylinder because it possesses innumerable (rather than only one) possible axes about which it could be rotated and still appear the same. A snowflake has even less symmetry than a cylinder, since there are only six discrete positions into which it could be rotated where no change could be discerned. A baseball glove has no symmetry whatever. There are absolutely no ways it could be rotated (not counting multiples of 360^{o}) without looking distinctly different.

Different degrees of non-change with change mean different degrees of symmetry

Symmetry extends beyond rotation. Consider an infinite length horizontal line. Translate it 10 meters to the right. It still looks the same. It has translational symmetry. Consider the human body where the right half is reflected to the left, and the left half reflected to the right. It still looks the same (to good approximation.) To high degree, our bodies have mirror, or reflection, symmetry.

Different kinds of symmetry: rotational, translational, reflection

There are continuous symmetries, like the cylinder of Fig. 6-1, a sphere, or the infinite straight line discussed above. For these, transformation is continuous. And there are discrete symmetries, like the snowflake, an infinite picket fence, or any reflection symmetry. For these, the transformation only maintains an invariant quality in certain discrete positions.

Symmetries can be continuous or discrete

Extrapolating these ideas beyond mere geometry and rotation, we can begin to understand why symmetry is considered so meaningful and fascinating. Non-change with change permeates many diverse phenomena. In many works of visual art, such as those of Escher or Indian mandalas, this principle is evident. In architecture, it has been pervasive throughout the ages. In music, the refrain, typically the essence of a song, remains the same, while other lyrics change. And that certain something we sense in the work of a great master is typically there throughout all of his or her individual pieces. We know that a Bach sonata, even if we have never heard it before, is by Bach. We know a Picasso painting, even if we have never seen it before, is by Picasso. We sense symmetry.

Symmetry plays a major role in the arts and elsewhere

6.1.2 Symmetry Mathematically

Mathematically, symmetry comprises invariance under transformation

In mathematical terms, the rotations, translations, and reflections we discussed in the previous section are known as transformations. Any transformation, by definition, is a change of something. If the transformation is symmetric, something else remains unchanged, or in math terms, invariant. Not all transformations are symmetric, of course. We will look at some mathematical examples below, but first we need to note one more thing.

The transformation depicted in Fig. 6-1 can be understood either as a rotation of the cylinder in one direction while we remain fixed (an active transformation, by name), or alternatively, as a rotation of our viewing frame of reference in the other direction while the cylinder remains fixed (a passive transformation). The same thing is true for snowflakes, the translation of a straight line, and more. Transformations typically involve a *change of perspective*, a change in the relationship between the observer and the thing being observed.

Transformation is change of object with observer fixed or vice versa.

Mathematically, when we change our position of observation, it is equivalent to using a new, different reference frame and coordinate system, oriented differently from, and/or displaced relative to, the original. So a transformation can be viewed simply as a change of coordinate system, and this is often represented as a shifting from unprimed to primed coordinates. We will focus on this (passive transformation) interpretation, the most common one in physics, and most relevant to QFT.

Changing observer = changing coordinate system, most useful interpretation

Example #1

So how about some simple examples? For starters, see Fig. 6-2, where on the left hand side we show the function

$$f\left(x^1, x^2\right) = \left(x^1\right)^2 + \left(x^2\right)^2 . \tag{6-1}$$

Example of a function symmetric under rotation transformation

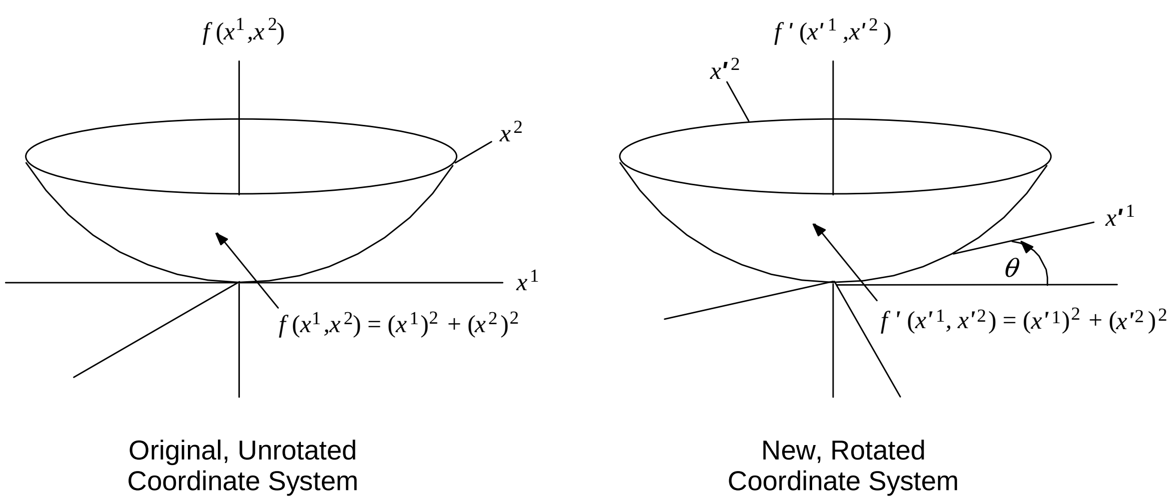

Figure 6-2. Example of a Function Symmetric Under Coordinate Transformation

We then change to a coordinate system rotated relative to the first, where our transformation from the first set of coordinates to the second is

$$x'^1 = x^1 \cos\theta + x^2 \sin\theta \qquad x'^2 = -x^1 \sin\theta + x^2 \cos\theta \,, \tag{6-2}$$

with the inverse transformation being

$$x^1 = x'^1 \cos\theta - x'^2 \sin\theta \qquad x^2 = x'^1 \sin\theta + x'^2 \cos\theta \,. \tag{6-3}$$

In matrix form, these are

$$\begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{T} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \qquad \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{T^{-1}=T^{\mathrm{T}}} \begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix}, \tag{6-4}$$

2D rotation transformation

where we designate the transformation by T, whose inverse is its own transpose.

Substituting (6-3) into (6-1) to express our function in the new system primed coordinates yields

$$\begin{aligned} f\left(x^1,x^2\right) &= \left(x^1\right)^2 + \left(x^2\right)^2 = \\ f'\left(x'^1,x'^2\right) &= \left(x'^1 \cos\theta - x'^2 \sin\theta\right)^2 + \left(x'^1 \sin\theta + x'^2 \cos\theta\right)^2 \\ &= \left(x'^1\right)^2\left(\cos^2\theta + \sin^2\theta\right) + \left(x'^2\right)^2\left(\cos^2\theta + \sin^2\theta\right) = \left(x'^1\right)^2 + \left(x'^2\right)^2 = f\left(x'^1,x'^2\right). \end{aligned} \tag{6-5}$$

Function has same form in original or primed coordinates → it is symmetric under the transformation

The function has exactly the same form in both coordinate systems, exactly the same form whether we express it in terms of the unprimed or primed coordinates. Given Fig. 6-2, this should not be much of a surprise.

The prime on f' is used to indicate it has, in general, different functional form from f, which is the case for non-symmetric functions. But since the function f here is symmetric under the transformation, the functional form of f and f' is the same, so we drop the prime. This can be more easily understood with the following example.

Example #2

Consider the function

$$g\left(x^1,x^2\right) = \left(x^2\right)^2 . \tag{6-6}$$

Example of function not symmetric under rotation transformation

Express (6-6) in the primed coordinate system by substituting (6-3) into it, and we get

$$g = \left(x^2\right)^2 = \left(x'^1 \sin\theta + x'^2 \cos\theta\right)^2 = \left(x'^1\right)^2 \sin^2\theta + \left(x'^2\right)^2 \cos^2\theta + 2x'^1x'^2 \sin\theta\cos\theta \neq \left(x'^2\right)^2 . \tag{6-7}$$

Thus, g has different form in the two systems and is *not* symmetric under the transformation T.

$$g\left(x^1,x^2\right) = g'\left(x'^1,x'^2\right) \neq g\left(x'^1,x'^2\right) \quad \text{but} \quad f\left(x^1,x^2\right) = f'\left(x'^1,x'^2\right) = f\left(x'^1,x'^2\right). \tag{6-8}$$

The transformed form of g, represented by g', has the same value at the same physical point, but it is not the same form in terms of the primed coordinates as g was in terms of the unprimed coordinates. But f', the transformed form of f, did have the same form in terms of both sets of coordinates, and thus, we dropped the prime on f on the RHS of (6-8).

In spite of its non-symmetry under rotation, g is symmetric under a different kind of transformation, the translation to a coordinate system which is displaced relative to the first along the x^1 axis, i.e., $x^1 \rightarrow x'^1 = x^1 + \text{constant}$, or

But same function is symmetric under a translation transformation

$$\begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix} = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} K \\ 0 \end{bmatrix} \qquad K = \text{constant} \,. \tag{6-9}$$

Substitution of (6-9) into (6-6) yields $g'(x'^1, x'^2) = (x'^2)^2$, having the same form in both systems.

If a coordinate is missing from f, then f is symmetric with respect to change of that coordinate

Lessons from the Examples

From Example #2, we can deduce the general rule that if a coordinate is missing in a given function, that function is invariant under a transformation solely in the direction of that coordinate (and also under multiplication of the coordinate by a constant, which will be less important for us.) The function is symmetric with respect to that transformation.

In both examples and in general, the value of a particular function at a given physical point in space is the same under any transformation, symmetric or not. The new coordinates are simply a new way to designate that particular point with different numbers, but it is the same point in space, and hence must have the same numeric value for functional there. If f or g were a physical entity, like pressure, simply changing our coordinates would not change the value of the pressure at any given point in space, even though the numbers describing that point's location are different.

Value of a scalar function at a physical point stays same under any transformation

So under *any* transformation of coordinate axes, the value at a physical point of every possible scalar function is invariant. Under a *symmetry* transformation the form of the function also is invariant. Under a non-symmetry transformation, the form of the function looks different in terms of the new coordinates, and we represent that functional difference with a prime on the function label.

Form of a scalar function stays same under a symmetry transformation

Scalars are Invariant, Vectors are Covariant

Consider a 2D position vector in physical space represented in the unprimed coordinates of Example #1 by $x^i = (x^1, x^2)$. Under the rotation transformation T, this becomes $x'^i = (x'^1, x'^2) \neq (x^1, x^2)$. A different (i.e., non-invariant) set of coordinates represents the exact same vector. But it is the same vector at the same physical location, and in fact, has the same length in each coordinate system equal to

Vector components change under transformation

$$|\mathbf{x}| = \left|x^i\right| = \sqrt{\left(x^1\right)^2 + \left(x^2\right)^2} = \sqrt{\left(x'^1\right)^2 + \left(x'^2\right)^2} = \left|x'^i\right|. \tag{6-10}$$

But vector length & direction in physical space unchanged for any coord system

So the scalar value at the point (equal to the length of the position vector at that point) is the same in both systems, but the coordinate values are not.

It is generally true of every vector **v**, not just the position vector shown here, that its physical, measurable length (a scalar value) remains unchanged under any coordinate transformation, but its component values change. This is called covariance. Scalar values are invariant under coordinate transformation; vector components are covariant. (Don't confuse this use of the word "covariant" with our use of the terms covariant and contravariant coordinates.)

Vectors are covariant under coordinate transformation

Parallel to scalars, if the vector components remain unchanged under a given transformation, then that transformation is a symmetry transformation, i.e., $v'^j(x'^i) = v^j(x'^i)$. One example is the **E** field around a point charge, which points radially outward from the point, described in a coordinate system with origin at the point. Rotating to a new coordinate system, we find the same functional dependence of the **E** field on the new coordinates. See Prob. 7.

Vector transformation symmetric if components unchanged

All of these conclusions are valid for any dimension space, and in particular for our purposes, the 4D spacetime of relativity theory. They are also valid for systems of generalized coordinates, not just Cartesian like those shown here, and for both particles and fields. Probs. 1 through 6 and Wholeness Chart 6-1 can help you gain more comfort with these concepts.

All of above true for 4D *and other spaces, as well*

Wholeness Chart 6-1. Symmetry Summary

	Non-Symmetric Transformation	**Symmetric Transformation**
Coordinate values change?	Yes	Yes
Scalar value at a physical point the same?	Yes	Yes
Form of function invariant?	No	Yes
Vector magnitude and direction at a physical point the same?	Yes	Yes
Vector components invariant?	No	Yes
Vector components vary covariantly?	Yes	No, invariant
General rule: If a function h is not a function of the jth coordinate x^j, then h is symmetric under the transformation $x^j \rightarrow x^j$ + constant		