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Vectors: Spin 1 Fields

"Three passions, simple but overwhelmingly strong, have governed my life: the longing for love, the search for knowledge, and unbearable pity for the suffering of mankind. These passions, like great winds, have blown me hither and thither, in a wayward course, over a great ocean …
… I have wished to understand the hearts of men. I have wished to know why the stars shine. And I have tried to apprehend the Pythagorean power by which number holds sway above the flux. A little of this, but not much, I have achieved."

Excerpts from "What I Have Lived For"
by Bertrand Russell

5.0 Preliminaries

Few of you who have come this far in this book do not share, in some part, Russell's passion for knowledge. It is my hope that, as each of us lives his or her life, we can also share in his other two, most noteworthy, passions.

With regard to the "power by which number holds sway above the flux", few have more ably demonstrated that power in their work than James Clerk Maxwell. His casting of the phenomena of electricity and magnetism into one elegant and holistic mathematical structure will forever remain one of the monumental achievements in the history of mankind.

Although his famous Maxwell equations were formulated for a classical world, they play an equally fundamental role quantum mechanically, as we shall soon see.

5.0.1 Background

Maxwell's equations in 4D format are the basis for QFT of photons

Maxwell first published his equations in 1864, well before the advent of special relativity, so they were framed in a distinctly 3D spatial plus time format. And that is how virtually every physics student first learns them. However, as QFT is a distinctly relativistic theory, we will need to work with Maxwell's equations in the more appropriate 4D format. This treatment of electromagnetism is typically reserved for graduate courses, after students have gained some level of comfort with the 3 + 1 dimensional approach. We will review the 4D approach, but hopefully, it is something that readers of this book have already been exposed to, as it serves as the bedrock for QFT of photons (massless spin 1 bosons).

5.0.2 Chapter Overview

Spin 1 boson theory development parallels spin 0 boson theory

Our approach to spin 1 bosons (called <u>vectors</u>, for reasons that will become apparent) in this chapter is three fold, including i) a review of classical electromagnetic theory, ii) RQM for photons, and iii) QFT for photons. As we will see, the second and third of these bear striking parallel to comparable aspects of spin 0 boson theory, and this will help to make our work easier.

Vector bosons, like scalars, can be massive or massless, but since our focus in this book is on quantum electrodynamics (QED), where force is mediated by photons, we will be virtually exclusively concerned with photons, which are massless vector bosons.

The following bulleted points provide an overview of this chapter. You may find it helpful to compare and contrast the material below for RQM and QFT with that of the Chapter Overview for scalars at the beginning of Chap. 3.

A review of classical e/m first,

where we will look at

Classical e/m overview

- the pre-relativistic version of Maxwell's equations, their (3D) vector and scalar potentials, and how they describe classical electromagnetic fields/waves,
- those same equations and e/m fields/waves represented covariantly (i.e., in special relativity) as a single equation for a 4D potential A^μ, and
- the classical relativistic Lagrangian density $\mathcal{L}$ for classical e/m fields.

Then RQM for photons (of which e/m waves are made),

RQM preview (*photons = massless vectors*)

- deducing the quantum Maxwell equation in terms of the 4D potential A^μ by applying 1st quantization to classical theory,
- solutions $|A^\mu\rangle$ to that equation (the 4D potential will represent a photon mathematically), and
- noting that those solutions parallel the Klein-Gordon solutions, and so, much of what we learned for scalars can be carried over directly to vectors with $|\phi\rangle \rightarrow |A^\mu\rangle$.

Then QFT for photons,

QFT preview (*photons = massless vectors*)

- from 2nd quantization, finding the same Maxwell equation, with the same mathematical form for the solutions A^μ, but this time the solutions are quantum fields, not states,
- using the classical relativistic $\mathcal{L}$ for e/m fields and the Legendre transformation to get $\mathcal{H}$ (Hamiltonian density),
- from 2nd quantization, finding the photon field A^μ commutation relations for QFT,
- determining relevant QFT operators for photons by a short cut method: comparing to similar operators for scalars: H, number, creation/destruction, momentum, charge, etc., and
- finding the Feynman propagator for photons by analogy with the scalar propagator.

As in Chaps. 3 and 4, in this chapter, we will deal only with free particles/fields.

Only free photons in this chapter

SEE THE TEXT FOR THE REMAINDER OF THIS CHAPTER

NOTE THAT WHOLENESS CHART SUMMARIZING CHAPS. 3 TO 5 IS INCLUDED BELOW

Summary Chart

Chaps. 3, 4, and 5 are summarized in Wholeness Chart 5-4 that follows.

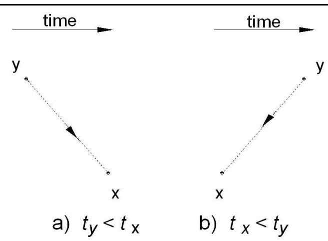
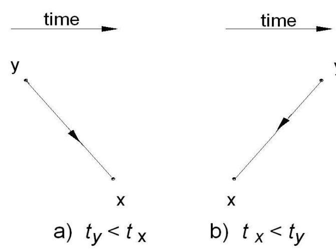
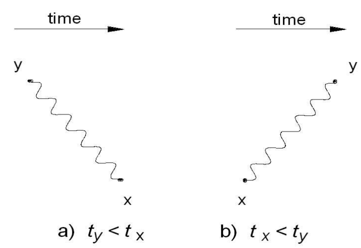
QED/FIELD THEORY OVERVIEW: PART 1

Wholeness Chart 5-4. From Field Equations to Propagators and Observables
Heisenberg Picture, Free Fields

	Spin 0	**Spin ½**	**Spin 1**
Classical Lagrangian density, free	$\mathcal{L}_0^0 = K\left(\partial_\alpha \phi \partial^\alpha \phi - \mu^2 \phi\phi\right)$	None. Macroscopic spinor fields not observed.	$\mathcal{L}_0^1 = \underbrace{\frac{\mu^2}{2} A^\mu A_\mu}_{\mu=0 \text{ for photons}} - \frac{1}{2}\left(\partial_\nu A_\mu\right)\left(\partial^\nu A^\mu\right)$
2nd quantization, Postulate #1	Bosons: Quantum field $\mathcal{L}$ (or equivalently, $\mathcal{H}$) same as classical, fields are complex, and K =1. Spinors: Dirac eq from RQM with states → fields. Deduce $\mathcal{L}$ from Dirac eq; $\mathcal{H}$ from Legendre transf.		
QFT Lagrangian density, free	$\mathcal{L}_0^0 = \left(\partial_\alpha \phi^\dagger \partial^\alpha \phi - \mu^2 \phi^\dagger \phi\right)$	$\mathcal{L}_0^{1/2} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \qquad \partial\!\!\!/ = \gamma^\alpha \partial_\alpha$	As above for classical.
	$\mathcal{L}$ ↑ into the Euler-Lagrange equation yields ↓		
Free field eqs	$\left(\partial_\alpha \partial^\alpha + \mu^2\right)\phi = 0$ $\left(\partial_\alpha \partial^\alpha + \mu^2\right)\phi^\dagger = 0$	$(i\gamma^\alpha \partial_\alpha - m)\psi = 0$ $(i\partial_\alpha \bar{\psi}\gamma^\alpha + m\bar{\psi}) = 0 \qquad \bar{\psi} = \psi^\dagger \gamma^0$	$\left(\partial_\alpha \partial^\alpha + \mu^2\right)A^\mu = 0$ photon $\mu = 0$ $A^{\mu\dagger} = A^\mu$ for chargeless (photon)
Conjugate momenta	$\pi_0^0 = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} = \dot{\phi}^\dagger;\ \pi_0^{0\dagger} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}^\dagger} = \dot{\phi}$	$\pi^{1/2} = i\psi^\dagger;\ \ \bar{\pi}^{1/2} = 0$	$\pi_\mu^1 = -\dot{A}_\mu$
Hamiltonian density	$\mathcal{H}_0^0 = \pi_0^0 \dot{\phi} + \pi_0^{0\dagger} \dot{\phi}^\dagger - \mathcal{L}_0^0$ $= \left(\dot{\phi}\dot{\phi}^\dagger + \nabla\phi^\dagger \cdot \nabla\phi + \mu^2 \phi^\dagger \phi\right)$	$\mathcal{H}_0^{1/2} = \pi^{1/2}\dot{\psi} - \mathcal{L}_0^{1/2}$	$\mathcal{H}_0^1 = \pi_\mu^1 \dot{A}^\mu - \mathcal{L}_0^1$
Free field solutions	$\phi = \phi^+ + \phi^-$ $\phi^\dagger = \phi^{\dagger +} + \phi^{\dagger -}$	$\psi = \psi^+ + \psi^-$ $\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^-$	$A^\mu = A^{\mu +} + A^{\mu -}$ (photon)
Discrete eigenstates (Plane waves, constrained to volume V)	$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx}\right)$ $\phi^\dagger(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}\right)$	$\psi = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}}\left(c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right)$ $\bar{\psi} = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}}\left(d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx}\right)$	$A^\mu = \sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx}\right)$
Continuous eigenstates (Plane waves, no volume constraint)	$\phi(x) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx}\right)$ $\phi^\dagger(x) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3 \omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx}\right)$	$\psi = \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3\mathbf{p}}{\sqrt{E_{\mathbf{p}}}}\left(c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx}\right)$ $\bar{\psi} = \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3\mathbf{p}}{\sqrt{E_{\mathbf{p}}}}\left(d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx}\right)$ spinor indices on u_r, v_r, and ψ suppressed. r = 1,2.	$A^\mu = \sum_r \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d\mathbf{k}}{\sqrt{\omega_{\mathbf{k}}}}\left(\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx}\right)$ $r = 0,1,2,3$ (4 polarization vectors)

2^{nd} quantization Postulate #2	Bosons: $\left[\phi^r(\mathbf{x},t),\pi_s(\mathbf{y},t)\right]=\left[\phi^r\pi_s-\pi_s\phi^r\right]=i\delta^r{}_s\delta(\mathbf{x}-\mathbf{y})$, ϕ^r = any field, other commutators = 0. Spinors: Coefficient anti-commutation relations parallel coefficient commutation relations for bosons.		
	Bosons: using conjugate momenta expressions in ↑ yields ↓		
Equal time commutators (intermediate step only)	$\left[\phi(\mathbf{x},t),\dot{\phi}^\dagger(\mathbf{y},t)\right]=i\delta(\mathbf{x}-\mathbf{y})$	Not needed for spinor derivation.	$\left[A^\mu(\mathbf{x},t),\dot{A}^\nu(\mathbf{y},t)\right]$ $=-ig^{\mu\nu}\delta(\mathbf{x}-\mathbf{y})$
	Bosons: Using free field solutions in ↑ with 3D Dirac delta function (e.g., for discrete solutions, $\delta(\mathbf{x}-\mathbf{y})=\frac{1}{2V}\sum_{n=-\infty}^{+\infty}\left(e^{-i\mathbf{k}_n\cdot(\mathbf{x}-\mathbf{y})}+e^{i\mathbf{k}_n\cdot(\mathbf{x}-\mathbf{y})}\right)$), and matching terms, yields the coefficient commutators ↓.		
Coefficient commutators	$\left[a(\mathbf{k}),a^\dagger(\mathbf{k}')\right]=\left[b(\mathbf{k}),b^\dagger(\mathbf{k}')\right]$	$\left[c_r(\mathbf{p}),c_s^\dagger(\mathbf{p}')\right]_+=\left[d_r(\mathbf{p}),d_s^\dagger(\mathbf{p}')\right]_+$	$\left[a_r(\mathbf{k}),a_s^\dagger(\mathbf{k}')\right]$
discrete	$=\delta_{\mathbf{kk}'}$	$=\delta_{rs}\delta_{\mathbf{pp}'}$	$=\zeta_{\underline{r}}\delta_{\underline{r}s}\delta_{\mathbf{kk}'}$ $\zeta_0=-1,\ \zeta_{1,2,3}=1$
continuous	$=\delta(\mathbf{k}-\mathbf{k}')$	$=\delta_{rs}\delta(\mathbf{p}-\mathbf{p}')$	$=\zeta_{\underline{r}}\delta_{\underline{r}s}\delta(\mathbf{k}-\mathbf{k}')$
Other coeffs	All other commutators = 0	All other anti-commutators = 0	All other commutators = 0
The Hamiltonian Operator			
	Substituting the free field solutions into the free Hamiltonian density $\mathcal{H}_0$, integrating $H_0=\int\mathcal{H}_0\,d^3x$, and using the coefficient commutators ↑ in the result, yields ↓. Acting on states with H_0 yields number operators.		
H_0	$\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(N_a(\mathbf{k})+\frac{1}{2}+N_b(\mathbf{k})+\frac{1}{2}\right)$	$\sum_{\mathbf{p},r}E_{\mathbf{p}}\left(N_r(\mathbf{p})-\frac{1}{2}+\bar{N}_r(\mathbf{p})-\frac{1}{2}\right)$	$\sum_{\mathbf{k},r}\omega_{\mathbf{k}}\left(N_r(\mathbf{k})+\frac{1}{2}\right)$
Number operators	$N_a(\mathbf{k})=a^\dagger(\mathbf{k})a(\mathbf{k})$ $N_b(\mathbf{k})=b^\dagger(\mathbf{k})b(\mathbf{k})$	$N_r(\mathbf{p})=c_{\underline{r}}^\dagger(\mathbf{p})c_{\underline{r}}(\mathbf{p})$ $\bar{N}_r(\mathbf{p})=d_{\underline{r}}^\dagger(\mathbf{p})d_{\underline{r}}(\mathbf{p})$	$N_r(\mathbf{k})=\zeta_{\underline{r}}\,a_{\underline{r}}^\dagger(\mathbf{k})a_{\underline{r}}(\mathbf{k})$
Creation and Destruction Operators			
	Evaluating $N_a(\mathbf{k})\,a(\mathbf{k})\,\|n_{\mathbf{k}}\rangle$ (similar for other particle types) with ↑ and the coefficient commutators yields ↓		
creation	$a^\dagger(\mathbf{k}),\ b^\dagger(\mathbf{k})$	$c_r^\dagger(\mathbf{p}),\ d_r^\dagger(\mathbf{p})$	$a_r^\dagger(\mathbf{k})$
destruction	$a(\mathbf{k}),\ b(\mathbf{k})$	$c_r(\mathbf{p}),\ d_r(\mathbf{p})$	$a_r(\mathbf{k})$
Normaliz factors lowering	$a(\mathbf{k})\|n_k\rangle=\sqrt{n_k}\,\|n_k-1\rangle$	$c_r(\mathbf{p})\left\|\psi_{r,\mathbf{p}}\right\rangle=\|0\rangle$	as with scalars
raising	$a^\dagger(\mathbf{k})\|n_k\rangle=\sqrt{n_k+1}\,\|n_k+1\rangle$	$c_r^\dagger(\mathbf{p})\|0\rangle=\left\|\psi_{r,\mathbf{p}}\right\rangle$	as with scalars
tot particle num	$N(\phi)=\sum_{\mathbf{k}}\left(N_a(\mathbf{k})-N_b(\mathbf{k})\right)$	$N(\psi)=\sum_{\mathbf{p},r}\left(N_r(\mathbf{p})-\bar{N}_r(\mathbf{p})\right)$	$N\left(A^\mu\right)=\sum_{\mathbf{k},r}N_r(\mathbf{k})$
tot particle num: lowering	$\phi=\phi^+ + \phi^-$	$\psi=\psi^+ + \psi^-$	$A^{\mu+}$
raising	$\phi^\dagger=\phi^{\dagger+}+\phi^{\dagger-}$	$\bar{\psi}=\bar{\psi}^+ + \bar{\psi}^-$	$A^{\mu-}$

Four Currents and Probability			
Four currents (operators) $j^{\mu}{}_{,\mu}=0$	$j^{\mu}=(\rho,\mathbf{j})=-i\left(\phi^{\dagger,\mu}\phi-\phi^{,\mu}\phi^{\dagger}\right)$	$j^{\mu}=(\rho,\mathbf{j})=\bar{\psi}\gamma^{\mu}\psi$	$j^{\mu}=-i\left(A_{\alpha}{}^{,\mu\dagger}A^{\alpha}-A_{\alpha}{}^{,\mu}A^{\alpha\dagger}\right)$ $=0$ for photons $\left(A_{\alpha}{}^{\dagger}=A_{\alpha}\right)$
	Emphasis in field theory is usually on the number of particles ($N(\mathbf{k})$ operator), and particle probability densities are rarely used. For completeness, however, and to make the connection with quantum mechanics, they are included below. (Antiparticles would have negative values of those below!)		
Single particle probability density (not operator)	$\rho(\mathbf{x},t)=$ $\langle\phi(\mathbf{x}',t)\vert\, j^{0}(\mathbf{x},t)\vert\phi(\mathbf{x}',t)\rangle$ Note integration over $\mathbf{x}'$, not $\mathbf{x}$. For type a plane wave, $\rho=\frac{1}{V}$	As at left, but with Dirac j^{0} above.	= 0 for chargeless particles.
Charge, not probability	Scalar type b particle → negative ρ. Photons → $\rho=0$. Led to conclusion that j^{0} is really proportional to *charge* probability density.		
Observables			
	Observable operators like total energy, three momentum, and charge are found by integrating corresponding density operators over all 3-space. (For spin ½, electrons assumed below with $q = -e$)		
H	$P_0=\sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(N_a(\mathbf{k})+N_b(\mathbf{k})\right)$	$P_0=\sum_{\mathbf{p},r}E_{\mathbf{p}}\left(N_r(\mathbf{p})+\bar{N}_r(\mathbf{p})\right)$	$P_0=\sum_{\mathbf{k},r}\omega_{\mathbf{k}}N_r(\mathbf{k})$
P_i = 3-momentum	$\mathbf{P}=\sum_{\mathbf{k}}\mathbf{k}\left(N_a(\mathbf{k})+N_b(\mathbf{k})\right)$	$\mathbf{P}=\sum_{\mathbf{p},r}\mathbf{p}\left(N_r(\mathbf{p})+\bar{N}_r(\mathbf{p})\right)$	$\mathbf{P}=\sum_{\mathbf{k},r}\mathbf{k}\,N_r(\mathbf{k})$
s^{μ}	$q\,j^{\mu}=q(\rho,\mathbf{j})$	$q\left(j^{\mu}-(\text{constant})\right)\to\partial_{\mu}s^{\mu}=0$	0 for photons
Q	$\int s^0d^3x=$ $q\sum_{\mathbf{k}}\left(N_a(\mathbf{k})-N_b(\mathbf{k})\right)$	$\int s^0d^3x=$ $-e\sum_{\mathbf{p},r}\left(N_r(\mathbf{p})-\bar{N}_r(\mathbf{p})\right)$	0 for photons
Spin operator for RQM states and QFT fields	N/A	$\boldsymbol{\Sigma}=\Sigma_i=\frac{1}{2}\begin{bmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{bmatrix}\quad i=1,2,3$ $\sigma_i=\text{2D Pauli matrices}$	magnitude = 1 for photons,
Helicity operator for RQM states and QFT fields	N/A	$\frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{\vert\mathbf{p}\vert}$	helicity eigenstates
Spin operator for QFT states	N/A	$\int\psi^{\dagger}\boldsymbol{\Sigma}\psi d^3x$	magnitude = 1 for photons,
Helicity operator for QFT states	N/A	$\int\psi^{\dagger}\left(\frac{\boldsymbol{\Sigma}\cdot\mathbf{p}}{\vert\mathbf{p}\vert}\right)\psi d^3x$	helicity eigenstates

Bosons, Fermions, and Commutators			
	Operations on states with creation, destruction, and number operators above yield the properties below.		
Properties of states:	$n_a(\mathbf{k}) = 0,1,2,\ldots,\infty$ So spin 0 states bosonic.	$n_r(\mathbf{p}) = 0,1$ only So spin ½ states fermionic.	$n_r(\mathbf{k}) = 0,1,2,\ldots,\infty$ So spin 1 states bosonic.
Bosons can only employ commutators Fermions can only employ anti-commutators	If anti-commutators used instead of commutators with Klein-Gordon equation solutions, then observable (not counting vacuum energy) Hamiltonian operator would have form $H_0{}^0 = 0$ and $H_0{}^0 \, \lvert\phi_{\mathbf{k}}\rangle = 0$,i.e., all scalar particles would have zero energy. Hence, we cannot use anticommutators with spin 0 bosons.	Commutators lead to 2 or more identical particle states co-existing in same multiparticle state. Anti-commutators lead to only one given single particle state per multi-particle state. Therefore, commutators cannot be used with spin ½ fermions. This is further proof that we need commutators with bosons.	Same as spin 0.
The Feynman Propagator			
	Creation and destruction of free particles (& antiparticles) and their propagation visualized below.		
Feynman diagrams			
Step 1 Time ordered operator T	If $t_y < t_x$, $T\left\{\phi(x)\phi^{\dagger}(y)\right\} = \phi(x)\phi^{\dagger}(y)$, i.e., the $\phi^{\dagger}(y)$ operates first, and should be placed on the right. If $t_x < t_y$, $T\left\{\phi(x)\phi^{\dagger}(y)\right\} = \phi^{\dagger}(y)\phi(x)$, i.e, the $\phi(x)$ operates first, and should be placed on the right. Note that $\phi(x)$ commutes with $\phi^{\dagger}(y)$ for $x \neq y$. [Fermion fields anti-commute.]		
Transition amplitude (double density in x and y)	$\langle 0 \lvert T\left\{\phi(x)\phi^{\dagger}(y)\right\}\rvert 0\rangle = i\Delta_F$	$\langle 0 \lvert T\left\{\psi_{\alpha}(x)\bar{\psi}_{\beta}(y)\right\}\rvert 0\rangle = iS_{F\alpha\beta}$	$\langle 0 \lvert T\left\{A^{\mu}(x)A^{\nu}(y)\right\}\rvert 0\rangle = iD_F{}^{\mu\nu}$
	The above vacuum expectation values (transition amplitudes) represent both 1) creation of a particle at y, destruction at x, and 2) creation of an antiparticle at x, destruction at y } transition amplitude = Feynman propagator		
Step 2 Propagator in terms of two commutators	By adding a term equal to zero to the Feynman propagator above, it can be expressed as vacuum expectation values (VEVs) of two commutators		
	$i\Delta_F(x-y) =$ $\langle 0\lvert\left[\phi^{+}(x),\phi^{\dagger -}(y)\right]\rvert 0\rangle \;\; t_y < t_x$ $\langle 0\lvert\left[\phi^{\dagger +}(y),\phi^{-}(x)\right]\rvert 0\rangle \;\; t_x < t_y$	$iS_{F\alpha\beta}(x-y) =$ $\langle 0\lvert\left[\psi^{+}_{\alpha}(x),\bar{\psi}^{-}_{\beta}(y)\right]_{+}\rvert 0\rangle \;\; t_y < t_x$ $-\langle 0\lvert\left[\bar{\psi}^{+}_{\beta}(y),\psi^{-}_{\alpha}(x)\right]_{+}\rvert 0\rangle \;\; t_x < t_y$	$iD_F{}^{\mu\nu}(x-y) =$ $\langle 0\lvert\left[A^{\mu +}(x),A^{\nu -}(y)\right]\rvert 0\rangle \;\; t_y < t_x$ $\langle 0\lvert\left[A^{\nu +}(y),A^{\mu -}(x)\right]\rvert 0\rangle \;\; t_x < t_y$

Step 3 As 3-momentum integrals	With the coefficient commutation relations, the above two commutators (for each spin type) can be expressed as two integrals over 3-momentum space		
Definition of symbols for commutators	$\left[\phi^{+}(x),\phi^{\dagger -}(y)\right]=i\Delta^{+}(x-y)$ $\left[\phi^{\dagger +}(y),\phi^{-}(x)\right]=i\Delta^{-}(x-y)$	$\left[\psi^{+}_{\alpha}(x),\bar{\psi}^{-}_{\beta}(y)\right]_{+}=iS^{+}_{\alpha\beta}(x-y)$ $-\left[\bar{\psi}^{+}_{\beta}(y),\psi^{-}_{\alpha}(x)\right]_{+}=iS^{-}_{\alpha\beta}(x-y)$	$\left[A^{\mu +}(x),A^{\nu -}(y)\right]=iD^{\mu\nu +}(x-y)$ $\left[A^{\nu +}(y),A^{\mu -}(x)\right]=iD^{\mu\nu -}(x-y)$
	$i\Delta^{\pm}=\frac{1}{2(2\pi)^3}\int\frac{e^{\mp ik(x-y)}}{\omega_{\mathbf{k}}}d^3\mathbf{k}$	$iS^{\pm}=\frac{\pm 1}{2(2\pi)^3}\int\frac{(\not{p}\pm m)e^{\mp ip(x-y)}}{E_{\mathbf{p}}}d^3\mathbf{p}$	$iD^{\mu\nu\pm}=-g^{\mu\nu}i\Delta^{\pm}$
	Δ^{+}, S^{+}, $D^{\mu\nu +}$ represent particles; Δ^{-}, S^{-}, $D^{\mu\nu -}$ represent anti-particles. Symbols $S^{\pm}=S^{\pm}_{\alpha\beta}$ Although fields such as ϕ are operators, because of their coefficient commutation relations, each integral above is a number, not an operator. The expectation value of a number X is simply the same number X. ($\langle 0\|X\|0\rangle=X\langle 0\|0\rangle=X$. So, the Feynman propagator will also be simply a number (no brackets needed.)		
Step 4 As contour integrals	Contour integral theory (integration in the complex plane) permits the above two integrals (for each spin type) over real 3-momentum space to be expressed as contour integrals.		
	$i\Delta^{\pm}=$ $\frac{\mp i}{(2\pi)^4}\int_{C^{\pm}}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k$	$iS^{\pm}=$ $\frac{\mp i}{(2\pi)^4}\int_{C^{\pm}}\frac{(\not{p}+m)e^{-ip(x-y)}}{p^2-m^2}d^4p$	$iD^{\mu\nu\pm}=$ $\frac{\mp ig^{\mu\nu}}{(2\pi)^4}\int_{C^{\pm}}\frac{e^{-ik(x-y)}}{k^2\underbrace{-\mu^2}_{\text{photon}=0}}d^4k$
Step 5 As one integral	Taking certain limits with contour integrals in the complex plane yields a single form for the Feynman propagator that works for any time ordering and will prove more convenient.		
in physical space	$\Delta_F(x-y)=$ $\frac{1}{(2\pi)^4}\int\frac{e^{-ik(x-y)}}{k^2-\mu^2+i\varepsilon}d^4k$	$S_{F\alpha\beta}(x-y)=$ $\frac{1}{(2\pi)^4}\int\frac{(\not{p}+m)e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}d^4p$	$D_F^{\mu\nu}(x-y)=$ $\frac{-g^{\mu\nu}}{(2\pi)^4}\int\frac{e^{-ik(x-y)}}{k^2+i\varepsilon}d^4k$
in momentum space	$\Delta_F(k)=\frac{1}{k^2-\mu^2+i\varepsilon}$	$S_{F\alpha\beta}(p)=\frac{\not{p}+m}{p^2-m^2+i\varepsilon}$	$D_F^{\mu\nu}(k)=\frac{-g^{\mu\nu}}{k^2+i\varepsilon}$