Chapter 4 *print vers 3/17/14 copyright of Robert D. Klauber*

Spinors: Spin ½ Fields

Niels Bohr: "What are you working on Mr. Dirac?"
Paul Dirac: "I'm trying to take the square root of something"

4.0 Preliminaries

While it may seem humorous to think of a physics Nobel laureate struggling over a square root problem, Dirac's meaning here was actually quite deep.

The quotes above purportedly came during a break at a 1927 conference Bohr and Dirac attended. Dirac later recalled that he continued on by saying he was trying to find a relativistic quantum theory of the electron, and Bohr commented, "But Klein has already solved that problem." Dirac then tried to explain he was not satisfied with the (Klein-Gordon) solution because it involved a 2nd order equation in time. That led to negative energy solutions, and he sought a 1st order equation like the non-relativistic Schrödinger equation. But the conference reconvened just then, and the discussion ended.

Dirac sought a first order relativistic Schrödinger equation

4.0.1 Background

Recall from Chap. 3, Sects. 3.0.1 (pg. 40) and 3.1.2 (pg. 42) that we had to use H^2 to develop our relativistic wave equation, because the relativistic Hamiltonian H entailed the operator $\partial_i \partial_i$ under a square root sign, and that had no meaning. Dirac wanted to find a meaningful H to use in a relativistic Schrödinger equation of form

$$H\psi = i\frac{\partial}{\partial t}\psi \,, \qquad (4\text{-}1)$$

In other words, he sought a wave equation in H, not H^2

rather than

$$H^2\phi = -\frac{\partial^2}{\partial t^2}\phi \qquad \text{(Klein-Gordon eq)} \,. \qquad (4\text{-}2)$$

It is no secret that he succeeded, and his famous result, published in early 1928, is now known as the Dirac equation. We will study it in depth in this chapter.

It wasn't too long after Dirac's discovery of the correct form for (4-1), that people realized (4-2) actually describes scalars; and (4-1), spin ½ fermions, such as the electron. The mathematical nature of the Dirac equation (4-1) provided a good indication for this. That is, (4-1) turns out (as we will see) to be a matrix equation with H being a square matrix quantity and ψ, a column matrix.

His equation turned out to be specifically for spin ½ particles, not all particles

In NRQM, we represented up and down spin of particles via wave functions that had a two component column matrix "tacked on". $(1,0)^T$ represented spin up; and $(0,1)^T$, spin down. So, if ψ in (4-1) in RQM (and QFT) turns out to be a column matrix (and it does), then we could make a good bet that it will represent spinors, rather than scalars. We would be smart to make such a bet, as we would end up winning it.

The Dirac equation is a matrix equation

Interestingly, the column matrix solutions ψ to (4-1) turn out to have four components, rather than two. Given that the relativistic (scalar) solutions to the relativistic wave equation we found in Chap. 3 provided us with antiparticles, which essentially doubled our total number of fields/particles, this should not be too surprising. Four spin components is just what we need to represent particles with up or down spin (2 components) plus antiparticles with up or down spin (2 more components.)

4.0.2 Chapter Overview

Spinor theory development parallels scalar theory

Our approach to spin ½ fermions in this chapter will parallel that for spin 0 bosons. You may find it helpful to compare and contrast the bulleted material below with that of the Chapter Overview for scalars at the beginning of Chap. 3, pg. 41.

RQM first,

RQM overview (spinors)

where we will look at

- the lack of a classical theory of fermions (no macroscopic fermionic behavior observed) and thus, being unable to use a classical H in 1st quantization,
- deducing the Dirac equation, a relativistic Schrödinger equation in H, not H^2,
- solutions (states in RQM) to the Dirac equation,
- probability density and its connection to the normalization constant in the solutions,
- negative energies and the Dirac equation solutions,
- how the Dirac solutions (unexpectedly at first) represent spin ½ particles, and
- spin and the spin operator acting on the solutions (which we didn't have with scalars).

Then QFT,

QFT overview (spinors)

- noting the lack of classical, macroscopic fermionic fields and thus, being unable to use a classical $\mathcal{H}$ in 2nd quantization,
- assuming the RQM Dirac equation as the QFT field equation, with the same solution form,
- using the (Dirac) field equation to deduce the QFT $\mathcal{L}$ for spinors (the reverse route from the scalar case), and employing the Legendre transformation to get $\mathcal{H}$,
- assuming solution coefficients obey anti-commutation (instead of commutation) relations,
- determining relevant operators in QFT: $H = \int \mathcal{H}\, d^3x$, number, creation/destruction, etc.,
- showing this approach avoids real particle negative energy states,
- seeing how the vacuum is filled with spinor quanta of energy $-\frac{1}{2}\hbar\omega$,
- deriving other operators (probability density, 3-momentum, charge, spin), and
- showing spinors are fermions, and they won't work with commutation relations.

And then,

- finding the spinor Feynman propagator.

Free (no force) Fields

Still only free particles/fields in this chapter

As in Chap. 3, we look herein only at free spinors.

4.1 Relativistic Quantum Mechanics for Spinors

4.1.1 No Classical Spinor Fields: Can We Quantize?

No classical theory for spin ½ particles/fields, because fermions can't occupy same state

In Chap. 3, Wholeness Chart 3.1 (pg. 65), we recalled that, via the Pauli exclusion principle, fermions cannot occupy the same state within the same macro system. So, whereas photons (bosons) can occupy the same state and a lot of them can therefore reinforce one another to produce a macroscopic electromagnetic field, spinors (fermions) cannot do so. In other words, we have no classical macroscopic spinor fields to sense, interact with, and study experimentally. And thus, we have no classical theory of spinors.

First quantization started with the classical Hamiltonian (or equivalently, the Lagrangian) and used that as the quantum Hamiltonian. But we have no classical spinor theory and thus no classical spinor Hamiltonian. Precisely parallel statements can be made for 2nd quantization. There is simply no classical theory with spinor Hamiltonian and Lagrangian densities.

So we can't do 1st or 2nd quantization

So we can't do 1st or 2nd quantization for spinors in the way it was advertised earlier, i.e., as THE way to obtain a good quantum theory. (My apologies for the false advertising, but you would have been confused at the time, otherwise.)

Dirac found another way

So how do we deduce a relativistic spinor quantum theory? We answer this question in the next section by showing how Dirac did it (though he was actually trying to do something else.)

4.1.2 Dirac's Approach to RQM: Another History Lesson

Dirac's primary goal was a 1st order relativistic Schrödinger equation, and he postulated that if it existed, it must have the general form (where, as before, we use the ket form symbolism for the wave equation solution in particle quantum theory)

General form a 1st order RQM equation must have

$$i\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle = (\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)|\psi\rangle . \tag{4-3}$$

In (4-3), $\mathbf{p}$ is particle three momentum, and the vector $\boldsymbol{\alpha}$ and the scalar β would have to be determined. Thus, the equation would be first order in the time derivative (and hopefully yield only positive energy solutions). Also, the relativistic free particle H would be a linear function of both $\mathbf{p}$ and mass m. The key question then is 'what are $\boldsymbol{\alpha}$ and β?' in order for this equation to be true.

Square of this equation must equal K-G eq

To find the answer, Dirac reasoned that H^2 and $|\psi\rangle$ must also satisfy the usual relativistic energy momentum relation (and therefore the Klein-Gordon equation)

$$-\frac{\partial^2}{\partial t^2}|\psi\rangle = H^2|\psi\rangle = \left(\mathbf{p}^2 + m^2\right)|\psi\rangle . \tag{4-4}$$

Squaring the operators in (4-3) and inserting the results into (4-4), we get

$$\begin{aligned} -\frac{\partial^2}{\partial t^2}|\psi\rangle &= H^2|\psi\rangle = (\alpha_i p_i + \beta m)\left(\alpha_j p_j + \beta m\right)|\psi\rangle \\ &= \left(\alpha_i^2 p_i^2 + \underbrace{\left(\overbrace{\alpha_i\alpha_j + \alpha_j\alpha_i}^{i>j \text{ here}}\right)}_{\text{must}=0,} p_i p_j + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{\text{must}=0} p_i m + \beta^2 m^2\right)|\psi\rangle , \end{aligned} \tag{4-5}$$

This squaring restricts form of terms in general equation

where comparison with the RHS of (4-4) shows the bracketed quantities in the lower line above must equal zero. That comparison also shows that $\alpha_i^2 = 1$ and $\beta^2 = 1$. In summary, where anti-commutators are defined as $[\alpha_i,\alpha_j]_+ = \alpha_i\alpha_j + \alpha_j\alpha_i$,

$$\begin{aligned} &\left[\alpha_i,\alpha_j\right]_+ = [\alpha_i,\beta]_+ = 0 \quad i \neq j \quad \alpha_1,\alpha_2,\alpha_3,\beta \text{ all anti-commute with each other,} \\ &(\alpha_1)^2 = (\alpha_2)^2 = (\alpha_3)^2 = (\beta)^2 = 1 \text{ (the identity matrix).} \end{aligned} \tag{4-6}$$

The α_i, β thus must be matrices with certain properties

If α_i and β were numbers they would have to commute and could not possibly anti-commute. Hence, they can only be matrices. Since these matrices are operators operating on $|\psi\rangle$, then $|\psi\rangle$ itself must be a multicomponent object (i.e., a column matrix, at least.)

Using (4-6), one can show that the α_i and β matrices are traceless, hermitian, have ±1 eigenvalues, and must have an even dimension of at least four. It will save time if you can simply accept these results. If not, then please prove them to yourself. I do note that I, myself, have never done so.

Dirac & Pauli found a set of 4X4 matrices that worked

Choosing the minimum dimension case (four), Dirac and Pauli came up with a set of matrices which solve all of the above conditions (specifically (4-6)) which is now called the standard (or Dirac-Pauli) representation, and which we will study in some depth in this chapter. There are, however, other possible choices for α_i and β that satisfy the same conditions. Two of these, called the Weyl and Majorana representations, are also four dimensional and can be convenient for some advanced applications, but we will ignore them herein.

The 4D abstract space of the solutions is called spinor space

Square matrices in a 4D space must be 4X4, and thus from (4-3), if $|\psi\rangle$ is a column matrix (a vector), it must have four components (a 4D vector). Take care to note that the 4D space we are talking about here is *not* the four dimensional physical space of relativity theory, but an abstract space, often called spinor space.

The matrices Dirac and Pauli found for spinor space are

$$\beta=\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix}\quad \alpha_1=\begin{bmatrix}&&&1\\&&1&\\&1&&\\1&&&\end{bmatrix}\quad \alpha_2=\begin{bmatrix}&&&-i\\&&i&\\&-i&&\\i&&&\end{bmatrix}\quad \alpha_3=\begin{bmatrix}&&1&\\&&&-1\\1&&&\\&-1&&\end{bmatrix},\quad (4\text{-}7)$$

Form of the α_i, β matrices

where blank components equal zero. (4-7) is commonly written using the 2X2 Pauli matrices σ_i as

$$\beta=\begin{bmatrix}I&0\\0&-I\end{bmatrix}\quad \alpha_1=\begin{bmatrix}0&\sigma_1\\\sigma_1&0\end{bmatrix}\quad \alpha_2=\begin{bmatrix}0&\sigma_2\\\sigma_2&0\end{bmatrix}\quad \alpha_3=\begin{bmatrix}0&\sigma_3\\\sigma_3&0\end{bmatrix}, \qquad (4\text{-}8)$$

where 0 represents the 2X2 null matrix.

Note that the Klein-Gordon equation can be considered the "square" of the Dirac equation and hence any solution $|\psi\rangle$ which solves the Dirac equation also solves the Klein-Gordon equation.

Solutions to Dirac equation also solve K-G equation

4.1.3 More Convenient Way to Express the Matrices

The Dirac equation can be expressed in a more convenient way by premultiplying (4-3) by β. To help when we do that, we define four matrices, called Dirac matrices or gamma matrices, as

$$\gamma^0=\beta \qquad \gamma^1=\beta\alpha_1 \qquad \gamma^2=\beta\alpha_2 \qquad \gamma^3=\beta\alpha_3\ , \qquad (4\text{-}9)$$

Dirac matrices γ^μ, found from α_i and β, are better to work with

where you can do Prob. 2 to show these equal

$$\gamma^0=\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix}\quad \gamma^1=\begin{bmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{bmatrix}\quad \gamma^2=\begin{bmatrix}&&&-i\\&&i&\\&i&&\\-i&&&\end{bmatrix}\quad \gamma^3=\begin{bmatrix}&&1&\\&&&-1\\-1&&&\\&1&&\end{bmatrix},(4\text{-}10)$$

or commonly, as

Form of Dirac matrices

$$\gamma^0=\begin{bmatrix}I&0\\0&-I\end{bmatrix}\quad \gamma^1=\begin{bmatrix}0&\sigma_1\\-\sigma_1&0\end{bmatrix}\quad \gamma^2=\begin{bmatrix}0&\sigma_2\\-\sigma_2&0\end{bmatrix}\quad \gamma^3=\begin{bmatrix}0&\sigma_3\\-\sigma_3&0\end{bmatrix}. \qquad (4\text{-}11)$$

From henceforth, we will do virtually nothing with the α_i and β matrices, and focus on the γ^μ matrices, instead.

Note the Hermiticity conditions (which you can prove by doing Prob. 3),

$$\gamma^{\mu\dagger}=\gamma^0\gamma^\mu\gamma^0\ . \qquad (4\text{-}12)$$

Complex conjugate transpose relations for Dirac matrices

4.1.4 The Dirac Equation Expressed with Dirac Matrices

Dirac's original 1st order equation (4-3) in terms of $\boldsymbol{\alpha}$ and β, pre-multiplied by β, takes on the form

$$i\underbrace{\beta}_{\gamma^0}\frac{\partial}{\partial t}|\psi\rangle=\Big(\underbrace{\beta\alpha_i}_{\gamma^i}p_i+\underbrace{\beta^2}_{\mathrm{I}}m\Big)|\psi\rangle=\Big(-i\gamma^i\frac{\partial}{\partial x^i}+m\Big)|\psi\rangle, \qquad (4\text{-}13)$$

or rearranged as what is formally called the Dirac equation

$$\boxed{\sum_{\eta=1}^{4}\left(\sum_{\mu=0}^{3}i\left(\gamma^\mu\right)_{\kappa\eta}\partial_\mu-m\delta_{\kappa\eta}\right)|\psi\rangle_\eta=0 \qquad \kappa=1,2,3,4}, \qquad (4\text{-}14)$$

Dirac equation in terms of Dirac matrices & all indices written out

where we have written out the 4X4 spinor space indices in κ and η, and the summation signs, in order to make it explicitly clear what is going on in spinor space. Note that the Dirac equation is actually *four separate non-matrix equations*, one for each value of the index κ. And each of these equations entails a sum of matrix components (sum over μ), each post multiplied by one the four components (in η index) of the column vector $|\psi\rangle$. Yes, it seems complicated. But also yes, it works. And also, yes, it is considered beautiful by many.

Dirac equation is actually four non-matrix equations

You will get used to the complication in time. When you do, you should gain an appreciation for the beauty, as well. In the words of the equation's discoverer,

> *"The research worker, in his efforts to express the fundamental laws of Nature in mathematical form, should strive mainly for mathematical beauty. He should take simplicity into consideration in a subordinate way to beauty ... It often happens that the requirements of simplicity and beauty are the same, but where they clash, the latter must take precedence. "*
>
> — Paul A. M. Dirac

Common, short hand form of Dirac equation

You should do Prob. 4 to provide some practice with (4-14), and then note that the common way to write the Dirac equation is to hide the spinor space indices in κ and η, i.e.,

$$\boxed{\left(i\gamma^{\mu}\partial_{\mu}-m\right)|\psi\rangle=0}\,, \tag{4-15}$$

where you have to be vigilant to remember the implicit 4X4 spinor space matrix/column nature of (4-15) as expressed explicitly in (4-14).

Slash notation also very common in Dirac equation

Another notation commonly used, which is the most streamlined of all, is

$$\not{\partial}=\gamma^{\mu}\partial_{\mu} \quad \text{so, the Dirac equation}\rightarrow \left(i\not{\partial}-m\right)|\psi\rangle=0\,. \tag{4-16}$$

We note in passing that

$$m\rightarrow\frac{mc}{\hbar} \quad \text{in non-natural units in the Dirac equation}\,. \tag{4-17}$$

4.1.5 Solutions to the Dirac Equation

We can write out (4-15) fully as

$$i\gamma^{\mu}\partial_{\mu}|\psi\rangle=i\left(\gamma^{0}\partial_{0}+\gamma^{1}\partial_{1}+\gamma^{2}\partial_{2}+\gamma^{3}\partial_{3}\right)|\psi\rangle=m|\psi\rangle= \tag{4-18}$$

Writing out Dirac equation

$$i\left(\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix}\partial_0+\begin{bmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{bmatrix}\partial_1+\begin{bmatrix}&&&-i\\&&i&\\&i&&\\-i&&&\end{bmatrix}\partial_2+\begin{bmatrix}&&1&\\&&&-1\\-1&&&\\&1&&\end{bmatrix}\partial_3\right)\begin{vmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{matrix}\right\rangle$$

$$=i\left(\begin{bmatrix}\partial_0&0&\partial_3&\partial_1-i\partial_2\\0&\partial_0&\partial_1+i\partial_2&-\partial_3\\-\partial_3&-\partial_1+i\partial_2&-\partial_0&0\\-\partial_1-i\partial_2&\partial_3&0&-\partial_0\end{bmatrix}\right)\begin{vmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{matrix}\right\rangle=m\begin{vmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{matrix}\right\rangle. \tag{4-19}$$

Note that the numeric subscripts on the ∂ symbols refer to derivatives with respect to time and space, whereas the numeric subscripts on the components of $|\psi\rangle$ refer to the respective components of the ket in spinor space.

(4-19) is a 4X4 matrix problem, for which we can try solutions of form $|\psi\rangle=\left|u_{\alpha}e^{\pm kx}\right\rangle$, where u_{α} is a four component spinor space column matrix. Doing this and carrying out the derivatives in (4-19), we end up with an 4X4 eigenvalue problem. This has four solutions $|\psi^{(n)}\rangle$, where n = 1,2,3,4, with each such solution having four spinor space components. We will not go through the tedium of doing this. Rather, I will simply provide the solutions, and you will do Prob. 5 to prove to yourself, by substitution, that they are indeed valid solutions to (4-19).

The Dirac equation solutions in the Dirac-Pauli (standard) representation are

$$\left|\psi^{(1)}\right\rangle = \underbrace{\sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \end{pmatrix}}_{\text{spinor } u_1 = \text{ part of solution in 4D spinor space}} \underbrace{e^{-ipx}}_{\text{4D physical space part}} = u_1 e^{-ipx} \qquad \left|\psi^{(2)}\right\rangle = \underbrace{\sqrt{\frac{E+m}{2m}}\begin{pmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{E+m} \\ \frac{-p^3}{E+m} \end{pmatrix}}_{\text{spinor } u_2} e^{-ipx} = u_2 e^{-ipx}$$

$$\left|\psi^{(3)}\right\rangle = \underbrace{\sqrt{\frac{E+m}{2m}}\begin{pmatrix} \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m} \\ 1 \\ 0 \end{pmatrix}}_{\text{spinor } v_2} e^{ipx} = v_2 e^{ipx} \qquad \left|\psi^{(4)}\right\rangle = \underbrace{\sqrt{\frac{E+m}{2m}}\begin{pmatrix} \frac{p^1-ip^2}{E+m} \\ \frac{-p^3}{E+m} \\ 0 \\ 1 \end{pmatrix}}_{\text{spinor } v_1} e^{ipx} = v_1 e^{ipx} . \qquad (4\text{-}20)$$

Solutions to Dirac equation (discrete, plane waves, **p** *eigenstates)*

Yes, again, these are more complicated than solutions we have dealt with in the past, but you will get used to them with time. Note several things in (4-20). Any constant instead of $\sqrt{(E+m)/2m}$ would suffice, but that choice was made because things will work better later on with it, as we will see. The symbol E is always a positive number of magnitude equal to the energy. p^i is positive if it points in the positive direction of its respective axis. These are plane wave solutions. We have defined new symbols $u_r(\mathbf{p})$ and $v_r(\mathbf{p})$ (r =1,2), which are the column vectors multiplied by the constant shown, are functions only of **p** for a given m (since $E=\sqrt{\mathbf{p}^2+m^2}$), and go by the name spinors, or four-spinors. Note that the particles represented by the $|\psi^{(n)}\rangle$ are also often called spinors. We will show shortly that r values represent different spin states (for example, u_1 represents spin up, and u_2 represents spin down in the particle at-rest system.) As you might expect, we will find the solutions containing $v_r(\mathbf{p})$ are associated with antiparticles; and those with $u_r(\mathbf{p})$, with particles. More on this later, but for now, take care to note the reverse order numbering on $v_{2,1}$ from $u_{1,2}$, which is customary.

Column vector parts of solutions called spinors

The solutions (4-20) are eigenstates of **p**, since every measurement of 3-momentum of the particles they represent would result in the value **p**. They are also eigenstates of energy, since, for given m, a free particle of 3-momentum **p** has a fixed E.

SEE TEXT FOR REMAINDER OF THIS CHAPTER