Supplemental Propagator Derivation

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Step	Traditional Propagator	Eq Ref [1]	Supplemental Propagator
Ref	$\phi(x)=\underbrace{\int\frac{d^3\boldsymbol{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}a(\mathbf{k})e^{-ikx}}_{\phi^+\text{ (destroys } a\text{ particles)}}+\underbrace{\int\frac{d^3\boldsymbol{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}b^\dagger(\mathbf{k})e^{ikx}}_{\phi^-\text{ (creates } b\text{ antiparticles)}}$ $\phi^\dagger(x)=\underbrace{\int\frac{d^3\boldsymbol{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}b(\mathbf{k})e^{-ikx}}_{\phi^{\dagger+}\text{ (destroys } b\text{ antiparticles)}}+\underbrace{\int\frac{d^3\boldsymbol{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}a^\dagger(\mathbf{k})e^{ikx}}_{\phi^{\dagger-}\text{ (creates } a\text{ particles)}}$	(3-37) (a) (3-37) (b)	$\underline{\phi}(x)=\underbrace{\int\frac{d^3\boldsymbol{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}\underline{a}(\mathbf{k})e^{i\underline{kx}}}_{\underline{\phi}^-\text{ (destroys } \underline{a}\text{ particles)}}+\underbrace{\int\frac{d^3\boldsymbol{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}\underline{b}^\dagger(\mathbf{k})e^{-i\underline{kx}}}_{\underline{\phi}^+\text{ (creates } \underline{b}\text{ particles)}}$ $\underline{\phi}^\dagger(x)=\underbrace{\int\frac{d^3\boldsymbol{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}\underline{b}(\mathbf{k})e^{i\underline{kx}}}_{\underline{\phi}^{\dagger-}\text{ (destroys } \underline{b}\text{ antiparticles)}}+\underbrace{\int\frac{d^3\boldsymbol{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}\underline{a}^\dagger(\mathbf{k})e^{-i\underline{kx}}}_{\underline{\phi}^{\dagger+}\text{ (creates } \underline{a}\text{ particles)}}$
1	for $t_y<t_x$ (trad particle) $T\{\phi(x)\phi^\dagger(y)\}=\phi(x)\phi^\dagger(y)$ for $t_x<t_y$ (trad anti-particle) $T\{\phi(x)\phi^\dagger(y)\}=\phi^\dagger(y)\phi(x)$ $i\Delta_F(x-y)=\langle 0\vert T\{\phi(x)\phi^\dagger(y)\}\vert 0\rangle$	(3-116) (3-117) (3-121)	for $t_y<t_x$ (suppl particle) $T\{\underline{\phi}(x)\underline{\phi}^\dagger(y)\}=\underline{\phi}(x)\underline{\phi}^\dagger(y)$ for $t_x<t_y$ (suppl anti-particle) $T\{\underline{\phi}(x)\underline{\phi}^\dagger(y)\}=\underline{\phi}^\dagger(y)\underline{\phi}(x)$ $i\underline{\Delta}_F(x-y)=\langle 0\vert T\{\underline{\phi}(x)\underline{\phi}^\dagger(y)\}\vert 0\rangle$
2	for $t_y<t_x$ (trad particle) $i\Delta_F(x-y)=\langle 0\vert\underbrace{[\phi^+(x),\phi^{\dagger-}(y)]}_{i\Delta^+(x-y)}\vert 0\rangle$ for $t_x<t_y$ (trad anti-particle) $i\Delta_F(x-y)=\langle 0\vert\underbrace{[\phi^{\dagger+}(y),\phi^-(x)]}_{i\Delta^-(x-y)}\vert 0\rangle$	(3-124) (3-125)	for $t_y<t_x$ (suppl particle) $i\underline{\Delta}_F(x-y)=\langle 0\vert\underbrace{[\underline{\phi}^-(x),\underline{\phi}^{\dagger+}(y)]}_{i\underline{\Delta}^+(x-y)}\vert 0\rangle$ for $t_x<t_y$ (suppl anti-particle) $i\underline{\Delta}_F(x-y)=\langle 0\vert\underbrace{[\underline{\phi}^{\dagger-}(y),\underline{\phi}^+(x)]}_{i\underline{\Delta}^-(x-y)}\vert 0\rangle$
3	for $t_y<t_x$ (trad particle) $i\Delta^+(x-y)=[\phi^+(x),\phi^{\dagger-}(y)]$ $=\frac{1}{2(2\pi)^3}\iint\underbrace{[a(\mathbf{k}),a^\dagger(\mathbf{k}')]}_{\delta(\mathbf{k}-\mathbf{k}')}\frac{e^{-ikx}e^{ik'y}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}d^3\mathbf{k}\,d^3\mathbf{k}'$ $=\frac{1}{2(2\pi)^3}\int\frac{e^{-ik(x-y)}}{\omega_{\mathbf{k}}}d^3\mathbf{k}$ (just a number, no operators)	(3-127) to (3-129) KEY DIFF, MINUS SIGN	for $t_y<t_x$ (suppl particle) $i\underline{\Delta}^+(x-y)=[\underline{\phi}^-(x),\underline{\phi}^{\dagger+}(y)]$ $=\frac{1}{2(2\pi)^3}\iint\underbrace{[\underline{a}(\mathbf{k}),\underline{a}^\dagger(\mathbf{k}')]}_{-\delta(\mathbf{k}-\mathbf{k}')}\frac{e^{i\underline{kx}}e^{-i\underline{k'y}}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}d^3\mathbf{k}\,d^3\mathbf{k}'$ $=\frac{-1}{2(2\pi)^3}\int\frac{e^{i\underline{k}(\underline{x}-\underline{y})}}{\omega_{\mathbf{k}}}d^3\mathbf{k}$ (just a number, no operators)

	for $t_x < t_y$ (trad anti-particle) $i\Delta^{-}(x-y) = \left[\phi^{\dagger +}(y), \phi^{-}(x)\right]$ $= \frac{1}{2(2\pi)^3}\iint \underbrace{\left[b(\mathbf{k}), b^\dagger(\mathbf{k}')\right]}_{\delta(\mathbf{k}-\mathbf{k}')} \frac{e^{ikx}e^{-ik'y}}{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}} d^3\mathbf{k}\, d^3\mathbf{k}'$ $= \frac{1}{2(2\pi)^3}\int \frac{e^{ik(x-y)}}{\omega_\mathbf{k}} d^3\mathbf{k}$ (just a number, no operators)	(3-130) SAME DIFF AS ABOVE	for $t_x < t_y$ (suppl anti-particle) $i\underline{\Delta}^{-}(x-y) = \left[\underline{\phi}^{\dagger -}(y), \underline{\phi}^{+}(x)\right]$ $= \frac{1}{2(2\pi)^3}\iint \underbrace{\left[\underline{b}(\mathbf{k}), \underline{b}^\dagger(\mathbf{k}')\right]}_{-\delta(\mathbf{k}-\mathbf{k}')} \frac{e^{-i\underline{kx}}e^{i\underline{k'y}}}{\sqrt{\omega_\mathbf{k}\omega_{\mathbf{k}'}}} d^3\mathbf{k}\, d^3\mathbf{k}'$ $= \frac{-1}{2(2\pi)^3}\int \frac{e^{-i\underline{k}(\underline{x}-\underline{y})}}{\omega_\mathbf{k}} d^3\mathbf{k}$ (just a number, no operators)
	for $t_y < t_x$ (trad particle) $i\Delta_F(x-y) = \langle 0 \vert \underbrace{i\Delta^{+}(x-y)}_{\text{a number}} \vert 0\rangle$ $= i\Delta^{+}(x-y) = \frac{1}{2(2\pi)^3}\int \frac{e^{-ik(x-y)}}{\omega_\mathbf{k}} d^3\mathbf{k}$ for $t_x < t_y$ (trad antipart) $i\Delta_F(x-y) = \langle 0 \vert \underbrace{i\Delta^{-}(x-y)}_{\text{a number}} \vert 0\rangle$ $= i\Delta^{-}(x-y) = \frac{1}{2(2\pi)^3}\int \frac{e^{ik(x-y)}}{\omega_\mathbf{k}} d^3\mathbf{k}$		for $t_y < t_x$ (suppl particle) $i\underline{\Delta}_F(x-y) = \langle 0 \vert \underbrace{i\underline{\Delta}^{+}(x-y)}_{\text{a number}} \vert 0\rangle$ $= i\underline{\Delta}^{+}(x-y) = \frac{-1}{2(2\pi)^3}\int \frac{e^{i\underline{k}(\underline{x}-\underline{y})}}{\omega_\mathbf{k}} d^3\mathbf{k}$ for $t_x < t_y$ (suppl antipart) $i\underline{\Delta}_F(x-y) = \langle 0 \vert \underbrace{i\underline{\Delta}^{-}(x-y)}_{\text{a number}} \vert 0\rangle$ $= i\underline{\Delta}^{-}(x-y) = \frac{-1}{2(2\pi)^3}\int \frac{e^{-i\underline{k}(\underline{x}-\underline{y})}}{\omega_\mathbf{k}} d^3\mathbf{k}$
4	(trad particle) $i\Delta^{+}(x-y) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \underbrace{\left\{\frac{e^{-i\omega_\mathbf{k}(t_x-t_y)}}{2\omega_\mathbf{k}}\right\}}_{f(\omega_\mathbf{k})} d^3\mathbf{k}$ $f(k_0) = \frac{e^{-ik_0(t_x-t_y)}}{k_0+\omega_\mathbf{k}}$ $\left(\text{so } f(\omega_\mathbf{k}) = \frac{e^{-i\omega_\mathbf{k}(t_x-t_y)}}{\omega_\mathbf{k}+\omega_\mathbf{k}} \text{ as above}\right)$ Also $f(\omega_\mathbf{k}) = \frac{1}{i2\pi}\int_{C^+}\frac{f(k_0)}{k_0-\omega_\mathbf{k}}dk_0$	(3-133) (3-134) See Fig. 3-4 for complex integral diagram	(suppl particle) $i\underline{\Delta}^{+}(x-y) = \frac{-1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \underbrace{\left\{\frac{e^{i\omega_\mathbf{k}(t_x-t_y)}}{2\omega_\mathbf{k}}\right\}}_{f(\omega_\mathbf{k})} d^3\mathbf{k}$ $f(k_0) = \frac{e^{ik_0(t_x-t_y)}}{k_0+\omega_\mathbf{k}}$ $\left(\text{so } f(\omega_\mathbf{k}) = \frac{e^{i\omega_\mathbf{k}(t_x-t_y)}}{\omega_\mathbf{k}+\omega_\mathbf{k}} \text{ as above}\right)$ Also, same as at left $f(\omega_\mathbf{k}) = \frac{1}{i2\pi}\int_{C^+}\frac{f(k_0)}{k_0-\omega_\mathbf{k}}dk_0$

	$i\Delta^+(x-y) = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left\{\frac{1}{i2\pi}\int_{C^+}\frac{f(k_0)}{k_0-\omega_\mathbf{k}}dk_0\right\}d^3\mathbf{k}$ $= \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left\{\frac{1}{i2\pi}\int_{C^+}\frac{e^{-ik_0(t_x-t_y)}}{(k_0-\omega_\mathbf{k})(k_0+\omega_\mathbf{k})}dk_0\right\}d^3\mathbf{k}$ $= \frac{-i}{(2\pi)^4}\int_{C^+}\frac{e^{-ik(x-y)}}{(k_0)^2-(\omega_\mathbf{k})^2}d^4k.$ With $(k_0)^2=k^2+(\mathbf{k})^2$ and $\omega_\mathbf{k}^2=\mu^2+(\mathbf{k})^2$, so (trad particle) $i\Delta^+(x-y)=\frac{-i}{(2\pi)^4}\int_{C^+}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k$ Similarly, (trad antipart) $i\Delta^-(x-y)=\frac{i}{(2\pi)^4}\int_{C^-}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k$	(3-135) (3-136) and (3-137) (3-138) (3-139)	$i\underline{\Delta}^+(x-y) = \frac{-1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left\{\frac{1}{i2\pi}\int_{C^+}\frac{f(k_0)}{k_0-\omega_\mathbf{k}}dk_0\right\}d^3\mathbf{k}$ $= \frac{-1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left\{\frac{1}{i2\pi}\int_{C^+}\frac{e^{ik_0(t_x-t_y)}}{(k_0-\omega_\mathbf{k})(k_0+\omega_\mathbf{k})}dk_0\right\}d^3\mathbf{k}$ $= \frac{i}{(2\pi)^4}\int_{C^+}\frac{e^{i\underline{k}(\underline{x}-\underline{y})}}{(k_0)^2-(\omega_\mathbf{k})^2}d^4k$ With same relations as at left (suppl particle) $i\underline{\Delta}^+(x-y)=\frac{i}{(2\pi)^4}\int_{C^+}\frac{e^{i\underline{k}(\underline{x}-\underline{y})}}{k^2-\mu^2}d^4k$ Similarly, (suppl antipart) $i\underline{\Delta}^-(x-y)=\frac{-i}{(2\pi)^4}\int_{C^-}\frac{e^{i\underline{k}(\underline{x}-\underline{y})}}{k^2-\mu^2}d^4k$
5	$i\Delta_F(x-y)=\frac{i}{(2\pi)^4}\int_{C_F}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k \begin{cases}\text{for }\Delta_F=\Delta^+, C_F=C^+\\ \text{for }\Delta_F=\Delta^-, C_F=C^-\end{cases}$ $k^2-\mu^2=(k_0)^2-(\omega_\mathbf{k})^2$ With offset η from Re k_0 axis, $i\Delta_F(x-y)=\frac{i}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{-ik(x-y)}}{(k_0)^2-(\omega_\mathbf{k}-i\eta)^2}d^4k$ Or finally, (traditional propagator) $i\Delta_F(x-y)=\frac{i}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{-ik(x-y)}}{k^2-\mu^2+i\varepsilon}d^4k$ which has Fourier transform (traditional propagator) $i\Delta_F(k)=\frac{1}{k^2-\mu^2+i\varepsilon}$	(3-140) & Fig. 3-6 Figs. 3-7 and 3-8 (3-142) (3-143) (3-146)	$i\underline{\Delta}_F(x-y)=\frac{-i}{(2\pi)^4}\int_{C_F}\frac{e^{i\underline{k}(\underline{x}-\underline{y})}}{k^2-\mu^2}d^4k \begin{cases}\text{for }\underline{\Delta}_F=\underline{\Delta}^+, C_F=C^+\\ \text{for }\underline{\Delta}_F=\underline{\Delta}^-, C_F=C^-\end{cases}$ Same relation as at left for traditional particle With offset η from Re k_0 axis, $i\underline{\Delta}_F(x-y)=\frac{-i}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{i\underline{k}(\underline{x}-\underline{y})}}{(k_0)^2-(\omega_\mathbf{k}-i\eta)^2}d^4k=\frac{-i}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{i\underline{k}(\underline{x}-\underline{y})}}{k^2-\mu^2+i\varepsilon}d^4k$ Since $-k(x-y)=-k_0t+\mathbf{k}\cdot\mathbf{x}$ and $\underline{k}(\underline{x}-\underline{y})=k_0t+\mathbf{k}\cdot\mathbf{x}$, and the above is integrated over all positive and negative k_0, we have (supplemental progagator) $i\underline{\Delta}_F(x-y)=\frac{-i}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{-ik(x-y)}}{k^2-\mu^2+i\varepsilon}d^4k$ (supplemenal propagator) $i\underline{\Delta}_F(k)=\frac{-1}{k^2-\mu^2+i\varepsilon}$

	Bottom line: $\underline{\Delta}_F\left(x-y\right)=-\Delta_F\left(x-y\right)$ $\underline{\Delta}_F\left(k\right)=-\Delta_F\left(k\right)$ The supplemental propagator has same form as, but opposite sign from, the traditional propagator. Key difference in derivation is Step #3.

References

[1] Klauber, R. D., *Student Friendly Quantum Field Theory*, Sandtrove (2013)