Reducible and Irreducible Groups:

Direct Sums, Invariant Subspaces, and Reducibility

Consider a group **C** represented by the 5-dimensional matrices C of the form exhibited in (1) (where the matrix components could be real or complex and could be a function of one or more parameters). Components left blank signify zero values. Note the meaning of the $\oplus$ sign, which implies what is called a direct sum.

$$C=\begin{bmatrix} a_{11} & a_{12} & a_{13} & & \\ a_{21} & a_{22} & a_{23} & & \\ a_{31} & a_{32} & a_{33} & & \\ & & & b_{11} & b_{12} \\ & & & b_{21} & b_{22} \end{bmatrix}=A\oplus B \quad \text{where } A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{1}$$

Direct sum of matrices nXn and mXm yields (n+m)X(n+m) matrix

When the matrix C operates on a five-component vector v, the A submatrix only acts on the top three components and has no effect on the bottom two. Similarly, the B submatrix only acts on the bottom two components and does nothing to the top three.

$$\mathbf{Cv}=\mathbf{v}' \xrightarrow[\text{coordinate system}]{\text{in some specific}} Cv=\begin{bmatrix} a_{11} & a_{12} & a_{13} & & \\ a_{21} & a_{22} & a_{23} & & \\ a_{31} & a_{32} & a_{33} & & \\ & & & b_{11} & b_{12} \\ & & & b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}=\begin{bmatrix} v_1' \\ v_2' \\ v_3' \\ v_4' \\ v_5' \end{bmatrix}=v'. \tag{2}$$

nXn and mXm sub-matrices act independently on vector components in independent subspaces

The A and B matrices act independently on different components of a vector and can be considered independent matrices acting on independent vector spaces (of dimensions 3 and 2, respectively.) A and B are called submatrices of C. The form of the matrix C in (1) and (2) is said to be block diagonal, for what hopefully is a fairly obvious reason, i.e., non-zero submatrix blocks along the diagonal and zeroes everywhere else.

The dimension of the new matrix C is 5 which equals the dimension of A (=3) plus the dimension of B (=2). More generally, in direct summing, the resulting matrix dimension equals the sum of the submatrix dimensions. A operates in a subspace of the 5-dimensional space. That subspace has 3 dimensions. B operates in a subspace of 2 dimensions.

Direct sum matrix dimension = sum of submatrix dimensions

Now consider a similarity transformation T, which acts on the matrix operator C, that fills up at least some of the original zero value matrix components of C.

$$TCT^{-1}=\begin{bmatrix} \underset{\sim}{c}_{11} & \underset{\sim}{c}_{12} & \underset{\sim}{c}_{13} & \underset{\sim}{c}_{14} & \underset{\sim}{c}_{15} \\ \underset{\sim}{c}_{21} & \underset{\sim}{c}_{22} & \underset{\sim}{c}_{23} & \underset{\sim}{c}_{24} & \underset{\sim}{c}_{25} \\ \underset{\sim}{c}_{31} & \underset{\sim}{c}_{32} & \underset{\sim}{c}_{33} & \underset{\sim}{c}_{34} & \underset{\sim}{c}_{35} \\ \underset{\sim}{c}_{41} & \underset{\sim}{c}_{42} & \underset{\sim}{c}_{43} & \underset{\sim}{c}_{44} & \underset{\sim}{c}_{45} \\ \underset{\sim}{c}_{51} & \underset{\sim}{c}_{52} & \underset{\sim}{c}_{53} & \underset{\sim}{c}_{54} & \underset{\sim}{c}_{55} \end{bmatrix}=\underset{\sim}{C} \qquad \begin{array}{c}\text{Same abstract operation } \mathbf{C}\text{, expressed} \\ \text{in different coordinate system.}\end{array} \tag{3}$$

In other coordinate systems, the subspace independence may not be obvious

We can think of the $\underset{\sim}{C}$ matrix as representing the same group **C**, just expressed in different form. That is, T has essentially changed our coordinate system (a passive transformation). Matrices and vectors in the new coordinate system are denoted with "squiggles" underneath; those in the old system, as plain letters. So, for vectors,

$$\underset{\sim}{v}=Tv \qquad \text{Same abstract vector } \mathbf{v}\text{, expressed in different coordinate system} \tag{4}$$

$$\text{We still have} \qquad \mathbf{Cv}=\mathbf{v}' \xrightarrow[\text{coordinate system}]{\text{but in this new}} \underset{\sim}{C}\,\underset{\sim}{v}=\underset{\sim}{v}' \quad \left(\text{where } \underset{\sim}{v}'=Tv'\right). \tag{5}$$

The T transformation gives us different components for the matrices and the column vectors, even though the abstract operation **C** carries out the same operation on the same abstract vector **v**.

For example, the operation carried out by the C group could be rotation of a 5D vector in the 5D space. Physically (imagining a 5D space) the vector would be an independent physical quantity like position rotated through a particular angle. The angle and the vector length remain the same physically regardless of which coordinate system we prefer to view them in. But the components of the matrix representing the rotation, and the components of the 5D vector, will be different in different coordinate systems. The right-hand sides of (2) and (5) represent that same rotation as observed in different coordinate systems. The T transformation changes the coordinate system.

The subspace independence is still there in the coordinate independence sense

Of course, the operation represented by the matrix C could be any number of things, not just rotation. But similar logic applies, regardless of the particular operation **C** carries out.

Note that if we had started with (3), instead of (1), we would not be immediately aware that there were two independent subspaces in which the **C** operation operates. They would still be there, but it would not be obvious. By operating on $\underset{\sim}{C}$ with T^{-1}, i.e., $T^{-1}\underset{\sim}{C}T$ we would get C, and then it would be obvious.

In going from $\underset{\sim}{C}$ to C, we have *reduced* the matrix to two sub-matrices, i.e., reduced the group representation to two (independent) submatrices. One could then imagine a scenario where either A or B matrices could be further reduced to block diagonal form with submatrices within them, such as

$$\underset{\sim}{A} = \hat{T} A \hat{T}^{-1} = \begin{bmatrix} \underset{\sim}{a}_{11} & \underset{\sim}{a}_{12} & \\ \underset{\sim}{a}_{21} & \underset{\sim}{a}_{22} & \\ 0 & 0 & \underset{\sim}{a}_{33} \end{bmatrix} . \qquad (6)$$

However, if we cannot further reduce A or B, in a manner such as that of (6), then we say they are irreducible. An irreducible matrix has no submatrices into which it could be reduced. A reducible matrix does have such submatrices. $\underset{\sim}{C}$ of (3) is reducible because it can be similarity transformed to block diagonal form C of (1). Note that a direct sum of matrices, as in (1), always gives us a reduced matrix.

A reducible matrix is transformable to block diagonal form

Two representations of a given group, such as C in (2) and $\underset{\sim}{C}$ in (3), are said to be equivalent representations if they are related by a similarity transformation, such as T in (3). Note the matrix group may or may not be reducible, yet still have different, equivalent representations of the group. To be equivalent only means one can be transformed into the other. Equivalence of D and D' only means there is some transformation T such that $D' = TDT^{-1}$, nothing more (i.e., no block diagonal necessarily implied.)

D and TDT^{-1} are equivalent representations

The set of all possible vector components v_1, v_2, v_3 in (2) comprise what is termed an invariant subspace (of vectors in a vector space) under C because any action of C on the five component vector v_i will not involve any of the three components v_1, v_2, v_3 in determining v'_4 or v'_5. Likewise, the set of all v_4, v_5 components will not, under C, play any role in determining v'_1, v'_2, v'_3, so it is also an invariant subspace. Two invariant subspaces behave independently under C.

Invariant subspace = vector space outside of which group action does not take a vector

By way of our prior example of 5D rotation, any vector initially in a 2D plane spanned by the basis vectors along the 4th and 5th axes in (2) would be rotated by **C** inside that plane. **C** could never rotate such a vector outside the plane. Even if we change coordinate axes via the transformation T of (3), that same physical vector would not be rotated by **C** outside of that original physical 2D plane. It may be rotated out of a coordinate plane defined by the x'_4, x'_5 axes in the new primed coordinate system, and have non-zero values in any or all of the five vector components in the new primed coordinates, but it would not be rotated out of the original 2D plane formed by the x_4, x_5 axes. That lack of ability of **C** to move the original vector out of the original plane indicates **C** has a sub-group that has its own independent action on vectors in the vector space.

Parallel logic applies, of course, to the other invariant subspace, the 3D volume formed by the x_1, x_2, x_3 axes outside of which **C** would not rotate any vector originally inside that volume.

For groups represented as matrices, we can define a reducible group as one for which a similarity transformation (a transformation such as T in (3) or $\hat{T}$ in (6)) can result in block diagonal form (such as in (2) or (6)). An irreducible group could not be transformed to such form.

A reducible group contains subgroups
An irreducible group does not

Things to note

The 3D rotation group $SO(3)$ representation of eq (2-11) in *SFQFT* Vol. 2, Chapter 2 on Group Theory, Sect. 2.2.5 (as of December 19, 2019) is irreducible because any vector in the 3D space can be rotated into any other vector. Nothing in eq (2-11) of that chapter constricts any 3D vector to any particular 2D plane, i.e., any particular subspace. That group representation cannot be block diagonalized into submatrices, and the 3D vector space it acts on is an invariant subspace.

However, we could imagine another $SO(3)$ rotation group matrix that would constrict rotation to a 2D plane, and such a matrix would be reducible to a 2D submatrix and a 1D submatrix. The 3D vector space would then not be an invariant subspace under the group.

Don't confuse eigenvalue matrix diagonalization with subgrouping

Also, one must be careful not to confuse eigenvector analysis with subgroups. Eigenvalues can be found by diagonalizing a matrix, via a suitable transformation (which is also applied to the vector). In that transformed state, the new basis vectors are eigenvectors. One might think we then have subgroups of the matrix, each subgroup of one dimension. But this is not the case. The diagonalized matrix is relevant for only the eigenvectors, *not all* vectors in the space. That is, only eigenvectors acted on by the matrix remain in the 1D space they started in. Other vectors are rotated to different alignments. None of these other vectors could be considered to be in one of the supposed invariant subspaces on which the supposed 1D subgroups act.

With diagonalized matrix, non-eigenvectors change subspace

All of the subgroups must, collectively, act on all the vectors in the space and keep those in their respective invariant subspaces within those subspaces. But in the eigenvalue diagonalized matrix situation, vectors that are not eigenvectors do not stay in a given subspace under the action of the group.

Symbol ⊕ means two subgroups in resulting group

This is not a group binary operation, but a combination of groups

When we see the symbol ⊕ for direct summing two groups together to form a larger group (as in (1)), we should recognize that this new larger group has two subgroups, the ones direct summed together via the ⊕ symbol. This is *not* a binary operation, as defined in *SFQFT* Vol. 2, Chapter 2 on Group Theory Wholeness Chart 2-1, pg. 9 (as of December 19, 2019). A binary operation occurs between members of a set. The symbol ⊕ means we are adding (direct summing) two different sets (sub-groups, really). It is an operation that combines groups, not an operation between elements of a given group.

After all this simplification with vectors, the real meaning of sub-group

We have visualized the action of sub-groups on vectors, as represented in (2) and (6), as a non-mixing of certain vector components. However, a group does not have to operate on a vector to be a group. Yet, it could still have sub-groups into which it is reducible.

Consider a particular group with two sub-groups. In (7), we show the (binary) group operation of two group members as one group member represented by a 4X4 matrix D matrix multiplied with another group member represented by another 4X4 matrix H. The sub-groups are represented as 2X2 matrices: F and G for group element D, and R and S for group element H.

$$\underbrace{\boldsymbol{DH}}_{\substack{\text{elements}\\\text{of group}}} \xrightarrow[\text{representation}]{\text{matrix}} \underbrace{\begin{bmatrix} F & \\ & G \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} R & \\ & S \end{bmatrix}}_{H} = \begin{bmatrix} f_{11} & f_{12} & & \\ f_{21} & f_{22} & & \\ & & g_{11} & g_{12} \\ & & g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & & \\ r_{21} & r_{22} & & \\ & & s_{11} & s_{12} \\ & & s_{21} & s_{22} \end{bmatrix}$$
$$= \begin{bmatrix} f_{11}r_{11}+f_{12}r_{21} & f_{11}r_{12}+f_{12}r_{22} & & \\ f_{21}r_{11}+f_{22}r_{21} & f_{21}r_{12}+f_{22}r_{22} & & \\ & & g_{11}s_{11}+g_{12}s_{21} & g_{11}s_{12}+g_{12}s_{22} \\ & & g_{21}s_{11}+g_{22}s_{21} & g_{21}s_{12}+g_{22}s_{22} \end{bmatrix} = \underbrace{\begin{bmatrix} FR & \\ & GS \end{bmatrix}}_{DH}. \tag{7}$$

Note that all the resulting elements in the upper left 2X2 block matrix in the last line of (7) are from the matrix multiplication of the F and R sub-matrices, which are members of one of the subgroups. Similarly, all the resulting elements in the lower right 2X2 block matrix are from the matrix multiplication of the G and S matrices, which are members of the other subgroup.

So, we see that just as the action of a subgroup on a vector does not mix certain components of the vector, so the group operation between elements of a group does not mix the subgroups together. They stay separate. In a binary group operation, no subgroup changes any part of any other subgroup. It only affects other members of its own subgroup.

Real meaning of subgroups = under group operation, no mixing of subgroups (*as opposed to vector subspaces*)

We introduced this concept originally with vectors in hopes of making it easier to understand. In these last few paragraphs, we have made it more precise, and more in the true spirit of group theory, which is defined in terms of group (binary) operations, not operations on vectors.