# String Theory Gauges and Coordinate Systems

Augments Zwiebach Chaps 9-11. (See last paragraph of Sect 11.5, pg. 229)

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| | **GAUGES: Fixing $\tau$ and/or $\sigma$** | | | | | **Comment** |
|---|---|---|---|---|---|---|
| | **Static** | **Special Static** | **A General Type** | | **Light-Cone** | 4D here, also valid for higher D. |
| **Fix (define) $\tau$** | $\tau = t = X^0$<br>$-n \cdot X = \tau$ | as at left | $n \cdot X = \beta\alpha' (n \cdot p)\tau$<br>$p$ constant, any $n$.<br>This fixes $\tau$ for chosen $n$ | | At left, with $n$ aligned with light cone edge | $n$ and all vector quantities can be expressed in any coord system |
| **Fix (define) $\sigma$** | | $d\sigma = \frac{ds}{\sqrt{1-v_T^2/c^2}}$<br>From $d\sigma = \frac{dE}{T_0}$ | $n \cdot \mathcal{P}^\tau = \frac{\beta}{2\pi} n \cdot p$<br>Fixes $\sigma$ for chosen $n$, but not simple to see how | | At left with $n$ above | |
| **Other defined** | | $\dot{X} \cdot X' = 0$ | | All of below left column true for any $n$ | “ | Relations left & below good in any coord system |
| **Results** | | | $n \cdot \mathcal{P}^\sigma = 0$ | ← eq motion, $n \cdot \mathcal{P}^\tau$ above, BCs | “ | |
| Constraints | | | $\underbrace{\dot{X} \cdot X' = 0 \quad \dot{X}^2 + X'^2 = 0}_{\left(\dot{X} \pm X'\right)^2 = 0}$ | ← 1st from $n \cdot \mathcal{P}^\sigma = 0$, $\mathcal{P}^\sigma = \frac{\partial \mathcal{L}}{\partial X'}$ and $\mathcal{L}$; 2nd from 1st, $\mathcal{P}^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}}$ & 2 blocks, top left column | “ | |
| Conj mom densities | Complicated | $\mathcal{P}^\tau_\mu = \frac{T_0}{c^2} \dot{X}_\mu \quad \mathcal{P}^\sigma_\mu = -T_0 X'_\mu$ | $\mathcal{P}^\tau_\mu = \frac{1}{2\pi\alpha'} \dot{X}_\mu \quad \mathcal{P}^\sigma_\mu = \frac{1}{2\pi\alpha'} X'_\mu$ | ← $\mathcal{L}$ in $\mathcal{P}^\tau$, $\mathcal{P}^\sigma$ above | “ | |
| Eq motion | $\frac{\partial \mathcal{P}^\tau}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma}{\partial \sigma} = 0$<br>Details complicated | $\frac{1}{c^2} \ddot{X}_\mu - X''_\mu = 0$ | $\ddot{X}_\mu - X''_\mu = 0 \quad c = 1$ here | ← eq motion with $\mathcal{P}^\tau$, $\mathcal{P}^\sigma$ above left | “ | |
| Solution | Complicated | | $X^\mu$ sol to above (Neumann BC), (9.56) pg 186, | | “ | This soln good in any coord system |

NOTE: Each of the above possible gauges can be expressed in any coordinate system. The two coordinate systems we work with most commonly are the observer spacetime (Minkowski) system with $x^\mu = (ct, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3)$ and the light-cone coordinate system $x^\mu = (x^+, x^-, x^2, x^3)$.

Quantities like $\mathcal{P}^\tau$ and $\mathcal{P}^\sigma$ are different things in each gauge, because they are defined in terms of derivatives of the Lagrangian with respect to $\tau$ and $\sigma$ derivatives of $X^\mu$. $\tau$ and $\sigma$ are defined differently in different gauges, so $\mathcal{P}^\tau$ and $\mathcal{P}^\sigma$ are different things in different gauges.

For a given gauge, the four vectors $\mathcal{P}^\tau$ and $\mathcal{P}^\sigma$ ($\mu$ index suppressed) can be expressed in different coordinate systems. They are the same thing physically in a given gauge, regardless of coordinate system chosen. But their components in different coordinate systems will be different. This is like a 3D velocity vector that will have one set of components in one coordinate system and another set in a different coordinate system (say, one rotated with respect to the first system). It is the same thing expressed in two different coordinate systems.

Similarly, our choice of the four-vector $n$ determines our gauge. But $n$ has different components in different coordinate systems, even though (for the same gauge choice for $n$), it is the same physical entity regardless of the coordinate system chosen.

If we go with a particular gauge (like the light-cone gauge), we can express everything we deal with in any coordinate system. But, it turns out the light-cone coordinate system is the easiest to use with the light-cone gauge.

| | **COORDINATE SYSTEMS WITH DIFFERENT GAUGES** | | | | | |
|---|---|---|---|---|---|---|
| | **Static Gauge** | | **Our General Type Gauge** | | **Light-Cone Gauge** | |
| **Minkowski Coords** | $n = n^\mu = (n^0,n^1,n^2,n^3) = (1,0,0,0)$ | | arbitrary $n = n^\mu = (a,b,c,d)$ | | $n = n^\mu = (n^+,n^-,n^2,n^3) = = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$ | |
| | $-n \cdot X = \tau \rightarrow t = \tau$ | | $n \cdot X = \beta\alpha' (n \cdot p)\tau$ | | $n \cdot X = \beta\alpha' (n \cdot p)\tau \rightarrow \frac{-X^0 + X^1}{\sqrt{2}} = \beta\alpha'\left(-p^0 + p^1\right)\tau$ | |
| | | | | | | |
| **Light-cone coords** | $n = n^\mu = (n^+,n^-,n^2,n^3) = \left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0\right)$ | | $n = n^\mu = \left(\frac{1}{\sqrt{2}}(a+b),\frac{1}{\sqrt{2}}(a-b),c,d\right)$ | | $n = n^\mu = (n^+,n^-,n^2,n^3) = (1,0,0,0)$ | |
| | $-n \cdot X = \tau \rightarrow \frac{X^0 - X^1}{\sqrt{2}} = \tau$ | | $n \cdot X = \left(\frac{1}{\sqrt{2}}(a+b),\frac{1}{\sqrt{2}}(a-b),c,d\right)\begin{bmatrix} -X^1 \\ -X^0 \\ X^2 \\ X^3 \end{bmatrix}$ | | $n \cdot X = \beta\alpha' (n \cdot p)\tau \rightarrow X^+ = \beta\alpha' p^+ \tau$ | |
| | | | | | $n \bullet \mathcal{P}^\tau = \frac{\beta}{2\pi} n \bullet p \rightarrow \mathcal{P}^{\tau +} = \frac{\beta}{2\pi} p^+$ | |
| | | | | | Similarly easier for other quantities. | |