Simplified Guide to de Sitter and Anti-de Sitter Spaces

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NOTE: The main part of this article is an introduction to de Sitter and anti-de Sitter spaces with almost no math (two simple equations) for those who are not experts in the subject. The appendix provides background for some things that are simply stated without proof in the main text and is intended for readers with a physics or math background.

1 Background

You may have heard that our universe is flat (or so close to flat we can't tell the difference), but then been confused when you heard that we live in a de Sitter universe, which is curved.

This article is an attempt to explain, as simply as possible, this difference plus some aspects of anti-de Sitter space that may be relevant for the famed AdS/CFT (Anti-de Sitter space/conformal field theory) correspondence, also known as holographic duality or gauge/gravity duality.

2 The Simplest Answer

The simplest answer to whether our universe is curved or flat is this.

In 4D (3 space and 1 time, often called 3+1 space), our universe is curved. But for just the 3 dimensions of space, it is flat. (From now on, when we say our universe is "flat" we will mean as close to flat as we can currently measure. Any curvature, if it exists, is very small.) The 3 spatial dimensions of the universe are flat space-wise, but the whole thing is curved when we include the time dimension.

3 A More Complete Answer

3.1 Slicing Higher Dimensional Spaces into Lower Dimensional Ones

Note from the LHS of Figure 1, we can have a flat 3D space (a volume) with a curved 2D space (a surface) embedded in it. The classic example is of a sphere in our 3D spatially flat (Euclidean) universe. In the figure, we show a hemisphere, for ease of visualization. The surface of the (hemi)sphere is a curved 2D surface.

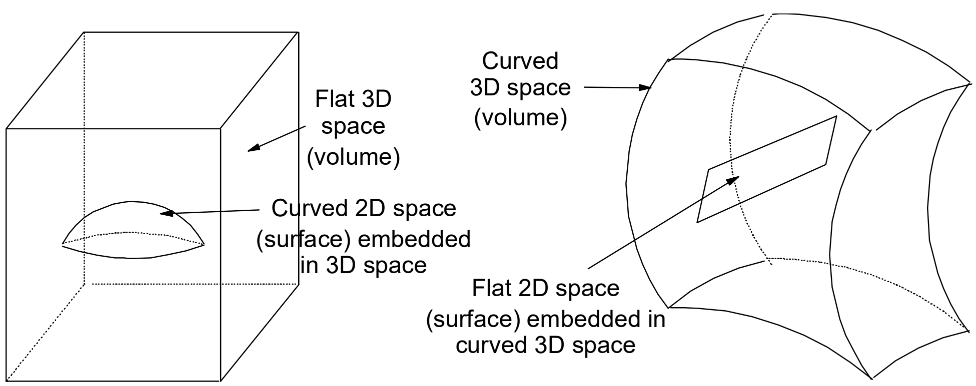

Figure 1. Lower Dimensional Spaces Embedded in a Space of One More Dimension

But the converse is also true. We can have a curved space with a lower dimensional flat space embedded in it, as shown heuristically in the RHS of Figure 1.[*] We can slice a 3D curved space (which is a little hard to visualize, though we tried in Figure 1) in such a way that the 2D space embedded in it is flat.

In essence, whether a space is curved or not, we can generally pick out a "slice" of it in one lower dimension that is either curved or flat. It all depends on how we want to define our slice. (The formal name for slicing in this

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[*] Note that the drawing is a bit misleading as we cannot readily depict a curved 3D space on the flat 2D space of this page. What we show is a curved 3D *object*, whereas the actual 3D space it is in would still be flat. But, hopefully, the basic idea that a flat 2D space can be embedded in a curved 3D space is conveyed.

context is "foliation". We foliate a higher D space to get a lower D space. For any given space, there are a slew [generally infinite] number of ways we can foliate to get lower dimension spaces of our choosing.)

We have shown 3D space foliation to get 2D spaces, but the same principle applies to any dimensions. For example, we can foliate our 4D universe and get our 3D spatial universe at one point in time.* Similarly, we can foliate a 5D space to get an embedded 4D space. Or a 6D space to get an embedded 5D space, etc.

When we talk, for example, of slicing (foliating) a 4D space to get a 3D space, we call the 4D space a hypervolume, and the 3D space, a hypersurface. Likewise, any space of dimension greater than three can be termed a hypervolume and the subspace obtained from foliating that, a hypersurface.

3.2 Our Flat 3D Spatial Universe is Embedded in a Curved 4D Spacetime Universe

So, our universe is a curved hypervolume when we consider it in the full four dimensions of space and time. When we consider only the 3D spatial part, as a hypersurface embedded in 4D spacetime, that hypersurface is flat.

Note, we can also conceptualize our curved 4D universe as embedded in a flat 5D space. From that perspective, we have a 5D flat space with an embedded 4D curved space. But then, the 4D curved space has a flat 3D (spatial) space embedded in it.

4 Positive vs Negative Curvature

4.1 The General Idea

The 2D surface of a sphere is deemed by convention to have positive curvature. If you start at any point on it, and move in any direction, you will curve "downward" in the same way.

The 2D surface of a saddle, on the other hand, is different. See Figure 2. If you start at some point, such as the middle of the saddle, and move in one direction, you will curve downward. If you start at the same point, but move in a perpendicular direction, you will curve upward. This kind of curvature is called negative curvature.

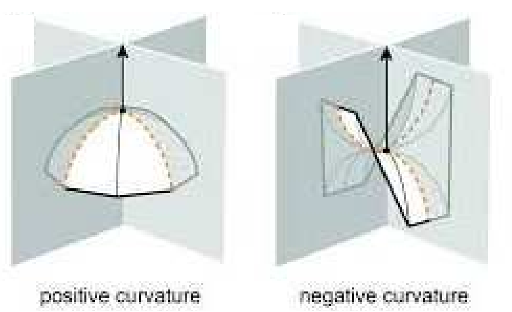

Figure 2. Positively and Negatively Curved 2D Surfaces

While we can visualize positive and negative curvature for 2D surfaces, it is not easy to do so in higher dimensions. However, there are mathematical descriptions of positive and negative curvature in 2D, and those can

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* I am trying to keep this as simple as possible. But, for the purists, we can define time throughout our universe in many ways. That is, we can consider standard clocks at every 3D point in our universe, where we can set each clock however we like. But in the common Friedman-Lemaitre-Robertson-Walker (FLRW) model of the universe, the time taken is that which we would have if, at the big bang, one clock at the initial singularity split into many clocks all having the same time on them. So, as the universe evolved, when each of these clocks at each point in the universe has the same time on it, we would consider that to be the time coordinate we keep constant in our foliation. That is, at every point in our 3D space, the clock would read the same, say t_0. Essentially, we slice our 4D space such that everywhere in our 3D space, the time value is t_0, and that is what cosmologists are using in the FLRW model when they say 3D space is flat. It doesn't have to be flat, it just turns out that way in our universe for this particular choice of foliation.

A more complicated way of saying the same thing is that we link the time on the clocks everywhere to the temperature of the cosmic microwave background radiation (CMBR). Wherever, over a spatial region, that temperature is the same, we consider the time to be the same over that spatial region. This works the same as the big bang clock splitting into many scenario, since the CMBR expands (and cools) in tandem with the time that would be on said clocks.

This particular choice of the time parameter is the most natural way to foliate the 4D universe.

be extrapolated to define curvature in a similar way in higher dimensions. That is, start at a given point. If you move in two directions perpendicular to each other, do you curve in the same way, or opposite ways? If the former, the space is positively curved. If the latter, negatively.

Of course, if you are on a flat 2D surface, you won't curve at all in any direction. That is considered zero curvature. And similarly, for higher dimension spaces. No curving away, no curvature, flat space.

And there are degrees of curvature. The surface of a beach ball, for example, is highly curved. If you start at any point and move away, you rapidly move downward. However, on the surface of the Earth, when you move away from one point, it seems pretty much like you are moving in a straight line, and not a curved path. Only after you travel a long distance, do you actually start curving downward by any significant amount. The beach ball surface has high curvature; the Earth surface, low.

4.2 Quantifying Curvature

It turns out, after cranking all the math, that the mathematical value for curvature of a 2D surface on a 3D sphere is inversely proportional to the sphere radius squared. (For the purists, we are talking about *scalar* curvature here.) The symbol R is used to designate the numerical value of the curvature.

So, for a sphere of radius r (where $\propto$ means "proportional to") *,

$$R \propto \frac{1}{r^2} \quad \text{(positive curvature).} \tag{1}$$

For higher dimensional spaces, curvature R is inversely proportional to the square of a radius-like number, which would correspond to a "sphere" in higher dimensions than 3. Again, this is hard to visualize, but we can extrapolate from what we can visualize in 3D and 2D. A "sphere" in a 4D space is called a *4-sphere*, and in general, for a D dimensional space, a D-sphere.

For a 2D saddle shape, where the upward degree of curvature aligned with the forward/backward direction of the saddle equals the downward degree of curvature in the sideways direction, we get a similar relationship, but with a negative sign. Recall saddle shapes have negative curvature. The r value here corresponds to what the radius would be of a circle whose circumference is aligned with either of the two perpendicular directions, i.e., the circle has the same curvature as that section of the saddle.

So, for a negatively shaped surface in any dimension, we will have curvature R

$$R \propto -\frac{1}{r^2} \quad \text{(negative curvature).} \tag{2}$$

Note that for a beach ball of radius .5 meters, from (1), we would get a relatively high value of R. For the Earth of radius 6 million meters, we would get a very small curvature, a very small R. Both values would be positive.

4.3 Constant vs Varying Curvature

4.3.1 Constant Curvature over Space

Note that the curvature of the surface of a sphere is constant everywhere. Wherever we measure we have the same value for radius r, and thus, via (1), the same curvature R.

For a surface that is not so uniform, however, such as an egg, the surface curvature R would be different at the ends than at the middle. Non-uniform (non-constant) curvature is, in fact, far more common than uniform (constant) curvature. Consider, for examples, a mountain's surface terrain, the surface of your body, a donut surface, or the surface of the bark on a tree. All have different curvatures at different places on the surface.

An object in any dimension is said to have maximal symmetry if the curvature everywhere is the same. Hence, a sphere surface has maximum symmetry, whereas that of an egg shell and a mountain's terrain do not.

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* To be more precise, $R = \frac{2}{r^2}$ for a 2D surface of a 3D sphere. In general, for hypersurface dimension D, $R = \frac{D(D-1)}{r^2}$, which is easily checked for the $D = 2$ surface of a sphere to be what we stated.

It is relatively easy to visualize positive curvature being constant everywhere over a hypersurface. It is not so easy visualizing negative curvature to be so. But such spaces with constant negative curvature do exist, and we will be looking at one shortly.

4.3.2 When Our Simple *R* Relations Don't Work

We now point out that the *R* values of (1) and (2) are for the simple cases where the curves traveled going outward from a point curve to the same degree in perpendicular directions. For other cases, such as around the center of an egg or an American football, the amount one curves along such different paths is different, then we cannot relate the *r* of (1) and (2) to a simple radius value. To have the maximal symmetry considered above, the degree of curvature one experiences traveling outward from a point would also have to be the same in all directions, not just at all points.

We will only be considering cases of maximal symmetry (the same curvature everywhere, in all directions) in what follows.

4.3.3 *R* Changing Over Time

Note that in time evolving spatial systems (consider a balloon being blown up), the curvature *R* can change with time. A smaller radius balloon will have larger curvature than a larger radius balloon.

5 Source of Curvature of Our 4D Universe

5.1 What Curves Our, or Any Other, Universe?

Anyone reading this already knows that mass-energy curves spacetime. This is the fundamental lesson from Einstein's theory of general relativity (GR).

For the universe, (if one cranks all the considerable physics and math involved) one finds the mass in it causes curvature of 4D spacetime, and this results in an attractive force tending to pull the universe closer together (tending to decelerate the universal expansion rate).

However, there can be another cause for spacetime curvature, something called the cosmological constant. This is part of another term in the Einstein field equation for gravity that, up until 1998, was generally considered to be zero. And so, up until 20 years or so ago, it was generally omitted from any general relativistic calculations. But astronomical observations since that time have revealed that this term is not zero, and the result has been Earth shaking for physics and cosmology.

It turns out that the cosmological constant can be positive or negative. A positive value leads to positive 4D curvature of the universe, and a repulsion gravity force (i.e., anti-gravity, in lay terms). A negative value yields negative 4D curvature and attractive gravity (like we are used to).

Remarkably, our universe seems to have a positive cosmological constant. This means matter in the universe is being repelled away from other matter, and the universal expansion rate is accelerating. This has flabbergasted the physics community 1) as it was totally unexpected, and 2) because no one, to this date, knows why this is or what causes it.

We summarize all this in Wholeness Chart 1.

Source of Curvature	**Cosmological Constant Sign**	**4D Curvature**	**Gravity Effect**	**Universe Effect**
Mass-energy density	N/A	Varies depending on pressure from the mass-energy	attractive	negative acceleration (decelerates)
Cosmological Constant	positive	positive	repulsive	positive acceleration (accelerates)
	negative	negative	attractive	negative acceleration (decelerates)

Wholeness Chart 1. Sources of Spacetime Curvature

5.2 The Sources of Curvature in Our Universe

Note that the cosmological constant in our universe is positive, but we also have mass-energy, so the attractive gravity from mass-energy is opposed by the repulsive gravity from the cosmological constant. Whichever of these is stronger will dictate whether our universal expansion is increasing (accelerating) or decreasing (decelerating).

Note that the effect of mass-energy depends on its density, i.e., the amount of mass-energy per unit 3D volume. But the total mass-energy does not change (where, for simplicity, we are ignoring subtleties of GR). Yet, the volume of the universe does increase as the universe expands. Thus, the mass-energy density decreases over time.

However, the cosmological constant appears (from extant data, so far) to be constant in time. Thus, its effect does not diminish as the universe expands.

In the early universe the mass-energy density effect was larger than the cosmological constant effect, so the universal expansion was continually slowing down. However, at around 6 billion years after the Big Bang, the mass-energy effect became so diluted, the cosmological constant effect became the more dominant of the two. As the universe evolved, the mass-energy effect became less and less, and relative to it, the cosmological constant effect became greater and greater.

So, since that time of the transition in dominance of the two effects, the universal expansion has been accelerating, speeding up.

Data from a number of sources all confirm this effect. These sources include supernovae brightness with distance (redshift) variation, the CMBR, large scale structure of the universe (via early universe baryon acoustic oscillations detection), and predicted abundances of primordial elements (Big Bang nucleosynthesis).

5.3 Quantifying the Effects of the Different Sources of Curvature

The cosmological constant can actually be modeled as a strange type of mass-energy having positive mass but negative pressure. (Mathematically inclined readers can see more detail in the appendix.) That is, instead of a cosmological constant term in the Einstein field equation, we substitute a mass-energy term having these particularly strange properties. This mass-energy term is often referred to as a <u>dark energy</u> term, as it seems to result from some very weird type of mass-energy that is "dark", i.e., we can't see it. Many physicists believe it may arise from the vacuum in some yet to be understood manner. Hence, it is also often referred to as a vacuum energy.

If we model the universal acceleration cause as dark energy, then we find it amounts to about 68% of the mass-energy in the universe, with what we now simply call matter equal to about 32%.

As an aside, we can only see about 5% of all of this mass-energy, i.e., 5% "normal" matter (protons, neutrons, electrons, photons, neutrinos). So, 27% of it all is composed of some kind of something we call <u>dark matter</u>, which like dark energy, we cannot, at present, detect. Wholeness Chart 2 summarizes these components of our universe.

Note that dark energy density does not, as far as present data can detect, appear to change over time, unlike matter density. This would be a characteristic one would expect of the vacuum, which should not change as the universe expands and dilutes the density of material in it. This is another reason dark energy is considered by many to be a vacuum phenomenon.

<u>Source</u>	**<u>Percentage of Mass-Energy of Universe at Current Time</u>**	**<u>Effect on 4D Curvature</u>**	**<u>Effect</u>**	**<u>Do We Know What It Is?</u>**
Ordinary matter density	5%	depends on pressure from the matter	decelerates universe	Yes
Dark matter density	27%	as above	decelerates universe	No
Dark energy density	68%	positive	accelerates universe	No

Wholeness Chart 2. Contributions to Total Mass-Energy of Universe

6 De Sitter vs Anti-de Sitter Space

The main points of this section are summarized in Wholeness Chart 3.

6.1 De Sitter Spaces (dS symbolically)

6.1.1 In 4D Spacetime

De Sitter (dS) spacetime (named after its discoverer) is defined as having no matter (ordinary or dark), but having a positive cosmological constant (or in other terms, dark energy [that does not vary over time]). It is an empty 4D universe, but one that, due to its cosmological constant, has positive curvature (which one can deduce by working through the math, as we do in the appendix). This positive curvature causes the space to expand at an ever-increasing rate, i.e., it accelerates. (See Wholeness Chart 2.) It has nothing in it, no matter of any kind, but it has a distinct, and unusual, spacetime behavior.

Because the cosmological constant term in the Einstein field equation is constant everywhere in space and time (by definition for de Sitter spacetime), the curvature it causes is the same everywhere. It acts like the 4D hypersurface of a 5D hypersphere, in a sense, where the curvature everywhere on the hypersurface is the same.* Thus, de Sitter space is maximally symmetric.

Because de Sitter space has no mass-energy in it, but our universe does, one should not say ours is a de Sitter universe. One says our vacuum (which has no mass) is a de Sitter vacuum. However, folks often take liberties by calling our universe a de Sitter space, but strictly speaking that statement is not true.

Note that as the universe ages, after some time, its matter will get so diluted, the effect on curvature from it will approach zero, and the cosmological constant effect will be virtually the whole show. In effect, we are moving toward a universe that will, for all intents and purposes, be effectively a de Sitter universe. So, we approach a de Sitter universe, as time $t \to \infty$. One says our universe is *asymptotically de Sitter*.

6.1.2 In Other Dimensional Spaces

Although when we talk of de Sitter space in cosmology, we generally mean the 4D spacetime of our universe, mathematically, we are not limited to four dimensions. That is, we can consider de Sitter space of 3D, or 5D, or 6D, or any number of dimensions we like. In these cases, we typically define the space not in terms of a cosmological constant, but in terms of its symmetry.

That is, a de Sitter space of any arbitrary dimension has constant, positive curvature everywhere in that space. It is like the hypersurface of a hypersphere, where in some abstract way, we can think of the hypersurface as everywhere being the same distance from a center point, always being a radius-like distance from that point, much like the 3-sphere surface of our experience is everywhere equidistant from the sphere's center. That radius-like distance is the "r" value of (1), which works for spaces of any dimension.

6.2 Anti-de Sitter Spaces (AdS symbolically)

6.2.1 In 4D Spacetime

Like dS, anti-de Sitter (AdS) spacetime has no matter in it. However, in other regards, anti-de Sitter (AdS) spacetime is the opposite of de Sitter spacetime. It has a negative cosmological constant, which (a lot of math can prove) results in negative curvature of spacetime. This would cause an expanding AdS universe to slow down, i.e., decelerate.

AdS spacetime, like dS spacetime, has constant curvature everywhere, though that curvature is negative. It is difficult to visualize a 4D saddle shape, of the same shape at every point in spacetime, as our 3D oriented minds are very limited in this regard. But mathematically, there is no problem. The math demonstrates this.

So, similar to dS spacetime, AdS is maximally symmetric, just in a different way.

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* Note for physicists and mathematicians. We are ignoring subtleties with regard to the metric signature being Lorentzian, for which the spacetime surface at constant spacetime interval Δs would be hyperbolic.

AdS spacetime is not the vacuum of our particular universe, but in the multi-verse scenario, there are probably many universes that do have AdS vacuums.

6.2.2 In Other Dimensions

As with dS spaces, the concept of constant curvature everywhere in a space of any number of dimensions, where here that curvature is negative, serves as a general, mathematical definition of an AdS space. Mathematically, the "r" value of (2) defines the negative curvature, and this works for spaces of any dimension.

Space	**Matter?**	**4D Spacetime Curvature**	**3D Space Acceleration**	**Other Dimension Spaces Curvature**
De Sitter (dS)	no	positive, constant	accelerates	positive, constant
Anti de Sitter (AdS)	no	negative, constant	decelerates	negative, constant
Our Universe:				
up to 6 billion yrs (mass dominates)	yes	positive, changing	decelerating	N/A
after 6 billion yrs (c.c. dominates)	yes	positive, changing	accelerating	N/A
As $t \to \infty$	effectively, no	positive, effectively constant	acceleration approaching ∞	N/A

Wholeness Chart 3. Comparing dS Space, AdS Space, and Our Universe

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The Following is for Readers with a Physics Background

7 Appendix: Relevant General Relativistic Cosmology

7.1 Einstein's Field Equation, the C.C., and the Stress-Energy Tensor

Einstein's field equation for general relativity (where $g_{\mu\nu}$ is the metric of the (generally curved) spacetime, Λ is the cosmological constant, and $G_{\mu\nu}$ is different from G, Newton's gravitational constant) is

$$\underbrace{G_{\mu\nu}}_{\substack{\text{4D}\\\text{curvature}\\\text{term}}} + \underbrace{g_{\mu\nu}\Lambda}_{\substack{\text{cosmological}\\\text{constant term}}} = \underbrace{8\pi G}_{\text{constant}}\ \underbrace{T_{\mu\nu}}_{\substack{\text{mass-}\\\text{energy}\\\text{term}}} \,. \qquad (3)$$

$G_{\mu\nu}$, called the Einstein tensor, is (where $R_{\mu\nu}$ is the Ricci curvature tensor and R is the scalar curvature)

$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R \,. \qquad (4)$$

So, without going into all the gory math, one can see that the tensor $G_{\mu\nu}$ is related to the curvature.

The stress-energy tensor $T_{\mu\nu}$ contains all the information regarding the mass-energy density, including the stresses, within the system under consideration (the universe, in our case) and it, along with Λ in (3), gives us $G_{\mu\nu}$, i.e., $T_{\mu\nu}$ and Λ tell the spacetime how to curve. Input $T_{\mu\nu}$ and Λ into (3), and you can solve (in principle) for $G_{\mu\nu}$. And from $G_{\mu\nu}$, using (4), you can solve for the curvature $R_{\mu\nu}$ tensor and the scalar curvature (where $R = R^{\mu}{}_{\mu}$, and repeated indices indicate summation.) Relatively easy in principle. Usually, brutally hard in practice.

For small curvature, $g_{\mu\nu}$ in (3) and (4) can be approximated by the Minkowski metric*,

$$g_{\mu\nu} \approx \eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.^{\dagger} \tag{5}$$

For an ideal gas at rest (which the matter in the universe can be approximated by at large scales), $T_{\mu\nu}$ is (ρ is mass-energy density in the rest frame of the mass, and p is the pressure [which is a form of stress] in the same rest frame)

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \tag{6}$$

Using (6) in (3), we find Einstein's field equation becomes (where we don't write out the components of the 4X4 matrix $G_{\mu\nu}$ to keep things simple, and because they are unknowns to be solved for anyway).

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \eta_{\mu\nu}\Lambda \;\rightarrow\; G_{\mu\nu} = 8\pi G \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} - \begin{bmatrix} -\Lambda & 0 & 0 & 0 \\ 0 & \Lambda & 0 & 0 \\ 0 & 0 & \Lambda & 0 \\ 0 & 0 & 0 & \Lambda \end{bmatrix}$$

$$= \underbrace{8\pi G \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}}_{\substack{\text{Influence of mass-energy} \\ \text{in the universe}}} + \underbrace{\begin{bmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & -\Lambda & 0 \\ 0 & 0 & 0 & -\Lambda \end{bmatrix}}_{\substack{\text{Influence of something else,} \\ \text{perhaps the vacuum}}} \tag{7}$$

$$= \underbrace{8\pi G T_{\mu\nu}^{(\text{mass-energy of a gas})}}_{\substack{\text{curvature} \\ \text{generating,} \\ \text{depends on } p}} + \underbrace{8\pi G T_{\mu\nu}^{(\text{vac or something})}}_{\substack{\text{positive} \\ \text{curvature} \\ \text{generating}}}.$$

Note that the cosmological constant term behaves as if it were a mass-energy distribution with negative pressure equal in magnitude to its positive mass-energy density. If the vacuum were to give rise to the cosmological constant term, then it would behave like an ideal gas with a constant mass-energy density $\rho = \Lambda/8\pi G$ and pressure $p = -\Lambda/8\pi G$.

The bottom line: We can model the cosmological constant term in Einstein's field equation as a stress-energy term arising from the vacuum where the vacuum has a constant (in time and over all 3D space) positive mass-energy density ρ with constant pressure $p = -\rho$. Or we can model it as simply the cosmological constant term without any vacuum type effect. We can model it either way.

Physicists consider the relationship between ρ and p for an ideal gas to be its equation of state. This can be expressed as a ratio of p to ρ, and this ratio is labeled w. So, for a vacuum that mimics a cosmological constant, we would have an equation of state

* Note the sign on the metric is conventional. Here we use the convention of Misner, Thorne, and Wheeler [*Gravitation* (Freeman, NY 1973) pg. 53] which is generally preferred by relativists. In quantum field theory, the metric usually employed has the opposite sign of that here. Either one gives the right answers, as long as one is consistent throughout.

† And in this case, $g^{\mu\nu} = g_{\mu\nu}$, since $g^{\mu\nu}$ is defined by $g^{\mu\nu}g_{\nu\alpha} = \delta^{\mu}{}_{\alpha}$ (4X4 identity matrix).

$$w = \frac{p}{\rho} = -1\,. \tag{8}$$

<u>The problem:</u> We know of nothing that has such properties, except as Zel'dovich [1] and Martin[2] have shown, the zero-point energy (ZPE) of the vacuum (which has a negative pressure). However, the magnitude of the ZPE is some 10^{55} times greater than what we observe in our universe for vacuum energy (i.e., dark energy). Not even close. A major conundrum.

7.2 Curvature of Spacetime by the Cosmological Constant

7.2.1 De Sitter Spacetime (Positive Cosmological Constant)

Consider Einstein's field equation (3) for the case of a universe that has no mass-energy, i.e., a universe composed of a vacuum and nothing else. For this, the stress-energy tensor $T_{\mu\nu} = 0$.

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu} = 0\,. \tag{9}$$

With (4), this becomes

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\, g_{\mu\nu}\Lambda\,. \tag{10}$$

Pre-multiplying (10) by $g^{\nu\mu}$, where repeated indices indicate summation, we get

$$g^{\nu\mu} R_{\mu\nu} - \tfrac{1}{2} g^{\nu\mu} g_{\mu\nu} R = -\, g^{\nu\mu} g_{\mu\nu}\Lambda \qquad \rightarrow \qquad R^{\nu}{}_{\nu} - \tfrac{1}{2}\delta^{\nu}{}_{\nu} R = -\,\delta^{\nu}{}_{\nu}\Lambda\,. \tag{11}$$

In 4D, the trace of the Kronecker delta equals 4. Knowing that the curvature scalar R is related to the Ricci curvature tensor $R_{\mu\nu}$ by $R = R^{\nu}{}_{\nu}$, we find (11) becomes

$$R - \tfrac{1}{2} 4R = -\,4\Lambda \qquad \rightarrow \qquad R = 2\Lambda\,. \tag{12}$$

The curvature, represented by the scalar R (see (1)), is twice the cosmological constant Λ. Note curvature is positive, and it is constant (at all points in 4D space and time).

Thus, a positive cosmological constant (with no mass-energy) means spacetime has positive curvature, and that curvature is the same everywhere, at all times. In essence, such a universe behaves like the 4D surface of a 5-sphere. This 4D space is called a <u>de Sitter spacetime (dS spaetime)</u>.

From (1) and (12), we can conclude that the 5D radius of curvature, the radius r of the 5-sphere, is inversely proportional to the square root of the cosmological constant Λ.

7.2.2 Anti- De Sitter Spacetime (Negative Cosmological Constant)

If the cosmological constant is negative (with no mass-energy), all steps above are the same, except that in our final result (12), Λ is negative. Thus, the curvature is negative and constant throughout all spacetime. This is called <u>anti-de Sitter spacetime (AdS spacetime)</u>.

7.3 Curvature Caused by Ordinary Mass-Energy

Consider a universe with mass spread evenly throughout it and no cosmological constant, where the mass can be modeled as an ideal gas. Einstein's field equation (3), with (4), becomes

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \qquad \rightarrow \qquad R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}\,. \tag{13}$$

Using the approximation (5) and pre-multiplying by $g^{\nu\mu}$, we get

$$g^{\nu\mu} R_{\mu\nu} - \tfrac{1}{2} g^{\nu\mu} g_{\mu\nu} R = \qquad \rightarrow \qquad \underbrace{R^{\nu}{}_{\nu}}_{R} - \tfrac{1}{2}\underbrace{\delta^{\nu}{}_{\nu}}_{4} R = 8\pi G\, g^{\nu\mu} T_{\mu\nu}\,, \tag{14}$$

where, with (5) and (6),

$$g^{\nu\mu}T_{\mu\nu} \approx \text{Trace}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} = \text{Trace}\begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} = -\rho + 3p . \tag{15}$$

Plugging (15) into (14)

$$-R = 8\pi G\left(-\rho + 3p\right) \qquad \rightarrow \qquad R = 8\pi G\left(\rho - 3p\right) \tag{16}$$

Note that curvature is positive if $\rho > 3p$, but negative if $3p > \rho$.

The mass-energy in our universe has little pressure. Specifically, p is on the order of 10^{-6} times ρ, so the result is a positively curved universe for most of its history (post the very early universe when pressure was significant).

7.4 Expansion and Deceleration/Acceleration

7.4.1 Universe Acceleration from General Relativity Relations

Nevertheless, via general relativistic calculations we won't get into here[3], the effect of mass-energy on universal deceleration is proportional to $\rho + 3p$,

$$\text{universe acceleration} \propto -\left(\rho + 3p\right), \tag{17}$$

Thus, even when $p \approx 0$, the effect of mass-energy will still be negative acceleration, i.e. deceleration, slowing down expansion.

Note that for the cosmological constant modeled as an ideal fluid, $p = -\rho$, so (17) will yield positive acceleration, which is what we have heard so often the cosmological constant leads to.

7.4.2 Caution on How to Think About Pressure in General Relativity

One should be careful not to think of pressure as pushing outward to cause the expansion, as one would consider a classical system to behave. Negative pressure is actually a tension, which in a classical system would tend to pull parts of the system together, not to push them apart.

Pressure makes a contribution in GR to system dynamics that is not found in classical systems. It contributes in a way like mass-energy to 4D curvature and to system behavior. The effect comes in via the stress-energy tensor, though only the mass part of which manifests to any measurable degree in the most familiar cases. That is, the GR pressure contribution is zero, or miniscule, for typical gravitational systems. (Such as the mass-energy distribution of the universe (6) where $p \approx 10^{-6}\rho$, or around a celestial body, where $p \approx 0$). So, in typical gravitational systems analysis, pressure doesn't play any role. But for the cosmological constant, modeled as vacuum mass-energy and pressure, from (8), (16), and (17), it plays a significant role. Not because of the force that pressure exerts, but from its role in the stress-energy tensor of Einstein's field equation.

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[1] Ya. B. Zel'dovich, "The Cosmological Constant and the Theory of Elementary Particles", Institute of Applied Mathematics, USSR Academy of Sciences, Usp. Fiz. Nauk, 95, 209 – 230 (May 1968). users.camk.edu.pl/akr/zeldoedibiopap.grg.pdf

2 J. Martin, "Everything you always wanted to know about the cosmological constant problem (but were afraid to ask)", C. R. Physique, 13, 566–665 (2012).

3 For details, see www.quantumfieldtheory.info/Brief_Summary_of_Cosmology.pdf .