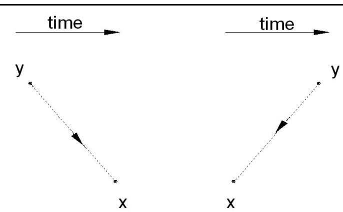
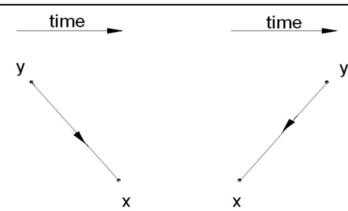
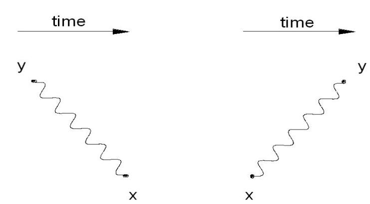
WHOLENESS CHART 3-X: QED/FIELD THEORY OVERVIEW

Part 1. From Field Equations to Propagators and Observables

Heisenberg Picture

	Spin 0	Spin ½	Spin 1
Free field equations	$\left(\partial_\alpha\partial^\alpha + \mu^2\right)\phi = 0$ $\left(\partial_\alpha\partial^\alpha + \mu^2\right)\phi^\dagger = 0$	$(i\gamma^\alpha\partial_\alpha - m)\psi = 0$ $(i\partial_\alpha\bar{\psi}\gamma^\alpha + m\bar{\psi}) = 0$ $\quad \bar{\psi} = \psi^\dagger\gamma^0$	$\left(\partial_\alpha\partial^\alpha + \mu^2\right)A^\mu = 0$ $A^{\mu\dagger} = A^\mu$ for chargeless case
Lagrangian density, free	$\mathcal{L}_0^0 =:\left(\partial_\alpha\phi^\dagger\partial^\alpha\phi - \mu^2\phi^\dagger\phi\right)$	$\mathcal{L}_0^{1/2,1} =:\left(\bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \frac{\mu^2}{2}A^\mu A_\mu - \frac{1}{2}\partial_\nu A_\mu\partial^\nu A^\mu\right)$ or $=:\left(\bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi + \frac{\mu^2}{2}A^\mu A_\mu - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right)$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$	
Minimal substitution	Not treated here	In free Lagrangian (density): $\partial_\alpha \to D_\alpha = \partial_\alpha + ieA_\alpha$	
Lagrangian, free plus interactions	Not treated here	$\mathcal{L}^{1/2,1} = \mathcal{L}_0^{1/2,1} + \mathcal{L}_I^{1/2,1} = \mathcal{L}_0^{1/2} + \mathcal{L}_0^{1} - s^\mu A_\mu$ + higher order $\quad s^\mu = -e\bar{\psi}\gamma^\mu\psi$	
Interacting fields equations	Not treated here	$\left(p\!\!\!/ + m\right)\psi = eA\!\!\!/\psi$ $p\!\!\!/ = \gamma^\mu p_\mu = \gamma^\mu(-i\partial_\mu) = -i\partial\!\!\!/$	$\left(\partial_\alpha\partial^\alpha + \mu^2\right)A^\mu = s^\mu$ + higher order
Conjugate momentum	$\pi_0^0 = \frac{\partial\mathcal{L}_0^0}{\partial\dot{\phi}} = \dot{\phi}^\dagger$; $\pi_0^{0\dagger} = \frac{\partial\mathcal{L}_0^0}{\partial\dot{\phi}^\dagger} = \dot{\phi}$	$\pi^{1/2} = i\psi^\dagger$; $\bar{\pi}^{1/2} = 0$	$\pi_\mu^1 = -\dot{A}_\mu$
Hamiltonian density	$\mathcal{H}_0^0 = \pi_0^0\dot{\phi} + \pi_0^{0\dagger}\dot{\phi}^\dagger - \mathcal{L}_0^0$ $=:\left(\dot{\phi}\dot{\phi}^\dagger + \nabla\phi^\dagger\nabla\phi + \mu^2\phi^\dagger\phi\right)$	$\mathcal{H}^{1/2,1} = \mathcal{H}_0^{1/2,1} + \mathcal{H}_I^{1/2,1} = \pi^{1/2}\dot{\psi} + \pi_\mu^1\dot{A}^\mu - \mathcal{L}_0^{1/2,1} - \mathcal{L}_I^{1/2,1}$	
Free field solutions	$\phi = \phi^+ + \phi^-$ $\phi^\dagger = \phi^{\dagger+} + \phi^{\dagger-}$	$\psi = \psi^+ + \psi^-$ $\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^-$	$A^\mu = A^{\mu+} + A^{\mu-}$
Discrete eigenstates (Plane waves, finite volume or periodic B.C.'s)	$\phi(x) = \sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}(a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx})$ $\phi^\dagger(x) = \sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}(b(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx})$	$\psi = \sum_{r,\mathbf{p}}\left(\frac{m}{VE_{\mathbf{p}}}\right)^{1/2}(c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx})$ $\bar{\psi} = \sum_{r,\mathbf{p}}\left(\frac{m}{VE_{\mathbf{p}}}\right)^{1/2}(d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx})$	$A^\mu = \sum_{r,\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}(\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx})$
Continuous eigenstates (Plane waves, infinite volume, no B.C.'s)	$\phi(x) = \int\frac{dk}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}(a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx})$ $\phi^\dagger(x) = \int\frac{dk}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}(b(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx})$	$\psi = \sum_r\left(\frac{m}{(2\pi)^3}\right)^{1/2}\int\frac{d^3\mathbf{p}}{\sqrt{E_{\mathbf{p}}}}(c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx})$ $\bar{\psi} = \sum_r\left(\frac{m}{(2\pi)^3}\right)^{1/2}\int\frac{d^3\mathbf{p}}{\sqrt{E_{\mathbf{p}}}}(d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx})$ spinor indices on u_r, v_r, and ψ suppressed. $r = 1,2$.	$A^\mu = \sum_r\frac{1}{\sqrt{2(2\pi)^3}}\int\frac{d\mathbf{k}}{\sqrt{\omega_{\mathbf{k}}}}(\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx})$ $r = 0,1,2,3$ (4 polarization vectors)

Second quantization	$\left[\Psi_r(\mathbf{x},t),\Pi_s(\mathbf{y},t)\right] = \left[\Psi_r\Pi_s \mp \Pi_s\Psi_r\right] = i\delta_{rs}\delta(\mathbf{x}-\mathbf{y})$ Ψ_r = any quantized field All other commutators = 0. Anti-commutator for spin ½ fields (use plus sign.)		
	⇩ **FREE FIELDS ONLY** ⇩		
	Using conjugate momenta expressions in the second quantization relation above yields		
Equal time commutators	$\left[\phi(\mathbf{x},t),\dot{\phi}^\dagger(\mathbf{y},t)\right] = i\delta(\mathbf{x}-\mathbf{y})$	$\left[\psi_\alpha(\mathbf{x},t),\bar{\psi}_\beta(\mathbf{y},t)\right]_+ = \gamma^0_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})$ outer product, spinor indices α, β shown with $\alpha,\beta = 1,2,3,4$.	$\left[A^\mu(\mathbf{x},t),\dot{A}^\nu(\mathbf{y},t)\right] = -ig^{\mu\nu}\delta(\mathbf{x}-\mathbf{y})$
	Using free field solutions in the above and the 3D Dirac delta function (e.g., for discrete solutions, $\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{V}\sum_{n=-\infty}^{+\infty} e^{-i\mathbf{k}_n\cdot(\mathbf{x}-\mathbf{y})}$) yields the coefficient commutators (below).		
Coefficient commutators	$\left[a(\mathbf{k}),a^\dagger(\mathbf{k}')\right] = \left[b(\mathbf{k}),b^\dagger(\mathbf{k}')\right]$	$\left[c_r(\mathbf{p}),c_s^\dagger(\mathbf{p}')\right]_+ = \left[d_r(\mathbf{p}),d_s^\dagger(\mathbf{p}')\right]_+$	$\left[a_r(\mathbf{k}),a_s^\dagger(\mathbf{k}')\right]$
discrete	$=\delta_{\mathbf{kk}'}$	$=\delta_{rs}\delta_{\mathbf{pp}'}$	$=\zeta_r\delta_{rs}\delta_{\mathbf{kk}'}$ r not summed $\zeta_0 = -1$, $\zeta_{1,2,3} = 1$
continuous	$=\delta(\mathbf{k}-\mathbf{k}')$	$=\delta_{rs}\delta(\mathbf{p}-\mathbf{p}')$	$=\zeta_r\delta_{rs}\delta(\mathbf{k}-\mathbf{k}')$
	Using the free field solutions in the LHS below (covariant field commutators) and the relations above yields RHS of below. (Note that different authors define the terms below slightly differently.)		
Covariant commutators, continuous only	$\left[\phi^\pm(x),\phi^{\mp}(y)\right] = i\Delta^\pm(x-y)$	$\left[\psi^\pm(x),\bar{\psi}^\mp(y)\right]_{+\alpha\beta} = iS^\pm_{\alpha\beta}(x-y)$	$\left[A^{\mu\pm}(x),A^{\nu\mp}(y)\right] = iD^{\mu\nu\pm}(x-y)$
3-momentum space form	$i\Delta^\pm = \frac{\pm 1}{2(2\pi)^3}\int\frac{d^3\mathbf{k}\, e^{\mp ik(x-y)}}{\omega_k}$	$iS^\pm = \left(i\not{\partial}+m\right)i\Delta^\pm$	$iD^{\mu\nu\pm} = -g^{\mu\nu}i\Delta^\pm$
contour integral form	$=\frac{-i}{(2\pi)^4}\int_{C^\pm}\frac{d^4k\, e^{-ik(x-y)}}{k^2-\mu^2}$	$=\frac{-i}{(2\pi)^4}\int_{C^\pm}\frac{d^4p\, e^{-ip(x-y)}\left(\not{p}+m\right)}{p^2-m^2}$	$=\frac{-ig^{\mu\nu}}{(2\pi)^4}\int_{C^\pm}\frac{d^4k\, e^{-ik(x-y)}}{k^2}$ $\left(\mu_\gamma = 0\right)$
	The coefficient commutation relations used with the Hamiltonian acting on states lead to creation and destruction operator interpretation of the coefficients and to the number operators below.		
Operators:			
creation	$a^\dagger(\mathbf{k})$, $b^\dagger(\mathbf{k})$	$c_r^\dagger(\mathbf{p})$, $d_r^\dagger(\mathbf{p})$	$a_r^\dagger(\mathbf{k})$
destruction	$a(\mathbf{k})$, $b(\mathbf{k})$	$c_r(\mathbf{p})$, $d_r(\mathbf{p})$	$a_r(\mathbf{k})$
number	$N_a(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k})$ $N_b(\mathbf{k}) = b^\dagger(\mathbf{k})b(\mathbf{k})$	$N_r(\mathbf{p}) = c_r^\dagger(\mathbf{p})c_r(\mathbf{p})$ $\bar{N}_r(\mathbf{p}) = d_r^\dagger(\mathbf{p})d_r(\mathbf{p})$	$N_r(\mathbf{k}) = \zeta_r\, a_r^\dagger(\mathbf{k})a_r(\mathbf{k})$
tot partic num	$N(\phi) = \sum_{\mathbf{k}}\left(N_a(\mathbf{k})-N_b(\mathbf{k})\right)$	$N(\psi) = \sum_{\mathbf{p},r}\left(N_r(\mathbf{p})-\bar{N}_r(\mathbf{p})\right)$	$N(A^\mu) = \sum_{\mathbf{k},r}\zeta_r N_r(\mathbf{k})$
particle num: lowering	$\phi = \phi^+ + \phi^-$	$\psi = \psi^+ + \psi^-$	$A^{\mu+}$
raising	$\phi^\dagger = \phi^{\dagger+} + \phi^{\dagger-}$	$\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^-$	$A^{\mu-}$

Normaliz factors lowering raising	$a(\mathbf{k})\,|\,n_k\rangle = \left(n_k\right)^{1/2}\,|\,n_k - 1\rangle$ $a^\dagger(\mathbf{k})\,|\,n_k\rangle = \left(n_k + 1\right)^{1/2}\,|\,n_k + 1\rangle$	$c(\mathbf{p})\,|1\rangle = |\,0\rangle$ $c^\dagger(\mathbf{p})\,|\,0\rangle = |1\rangle$	as with scalars as with scalars
	Operations on states with creation, destruction, and number operators above yield the properties below.		
Properties of states:	$n_a(\mathbf{k}) = 0,1,2,...\infty$ state symmetric under particle exchange, Bose-Einstein stats	$n_r(\mathbf{p}) = 0,1$ only state changes sign under particle exchange, Fermi-Dirac statistics	$n_r(\mathbf{k}) = 0,1,2,...\infty$ state symmetric under particle exchange, Bose-Einstein stats
Four currents (operators)	$j^\mu = (\rho, \mathbf{j}) = -i : \left(\phi^{\dagger,\mu}\phi - \phi^{,\mu}\phi^\dagger\right)$	$j^\mu = (\rho, \mathbf{j}) = -i : \left(\bar{\psi}\gamma^\mu\psi\right)$	$j^\mu = -i : \left(A_\alpha^{\ ,\mu\dagger}A^\alpha - A_\alpha^{\ ,\mu}A^{,\alpha\dagger}\right)$ $= 0$ for photons $\left(A_\alpha^{\ \dagger} = A_\alpha\right)$
	Emphasis in field theory is usually on the number of particles (*N*(**k**) operator), and particle probability densities are rarely used. For completeness, however, and to make the connection with quantum mechanics, they are included below. (Antiparticles would have negative values of those below!)		
Single particle probability density (not operator)	$\rho(\mathbf{x},t) = \left\langle \phi(\mathbf{x}',t)\,|\,j^0(\mathbf{x},t)\,|\,\phi(\mathbf{x}',t)\right\rangle$ Note integration over $\mathbf{x}'$, not **x** For plane wave, $\rho = \frac{1}{V}$	As at left, but with Dirac j^0 above.	= 0 for chargeless particles. Led to conclusion that j^0 is really proportional to *charge* probability density.
	Creation and destruction of free particles (& antiparticles) and their propagation visualized below.		
Feynman diagrams	 a) $t_y < t_x$ b) $t_x < t_y$	 a) $t_y < t_x$ b) $t_x < t_y$	 a) $t_y < t_x$ b) $t_x < t_y$
Time ordered operator *T*	If $t_y < t_x$, $T\left\{\phi(x)\phi^\dagger(y)\right\} = \phi(x)\phi^\dagger(y)$, i.e., the $\phi^\dagger(y)$ operates first, and should be placed on the right. If $t_x < t_y$, $T\left\{\phi(x)\phi^\dagger(y)\right\} = \phi^\dagger(y)\phi(x)$, i.e, the $\phi(x)$ operates first, and should be placed on the right. Note that $\phi(x)$ commutes with $\phi^\dagger(y)$. [Fermions would anti-commute.]		
	The operator fields in the *T* operator below will create and destroy kets on RHS. In the wave mechanics formulation, bracket integration is over the dummy $\mathbf{x}'$ variable in the bra and ket, not **x,y** of *T* operator.		
Transition amplitude density	$\left\langle 0\,|\,T\left\{\phi(x)\phi^\dagger(y)\right\}|\,0\right\rangle$	$\left\langle 0\,|\,T\left\{\psi_\alpha(x)\bar{\psi}_\beta(y)\right\}|\,0\right\rangle$	$\left\langle 0\,|\,T\left\{A^\mu(x)A^\nu(y)\right\}|\,0\right\rangle$
	The above represent both 1) creation of a particle at y, destruction at x, and 2) creation of an antiparticle at x, destruction at y		
	Fields such as ϕ represent integration over all momenta. The transition amplitude (density) above for fixed *x* and *y* equals the Feynman propagator $\Delta_F(x-y)$ between *x* and *y*. Note this is a number, not an operator. Only the continuous field solutions are relevant as no boundary conditions exist in this case.		

Feynman propagators	$\Delta_F(x-y) = +\Delta^+(x-y)$ if $t_y < t_x$ $= -\Delta^-(x-y)$ if $t_x < t_y$	$S_{F\alpha\beta} = +S^+_{\alpha\beta}(x-y)$ if $t_y < t_x$ $= -S^-_{\alpha\beta}(x-y)$ if $t_x < t_y$	$D_F^{\mu\nu} = +D^{+\mu\nu}(x-y)$ if $t_y < t_x$ $= -D^{-\mu\nu}(x-y)$ if $t_x < t_y$
	Evaluating the above in complex space and taking certain limits with contour integrals yields a form for Feynman propagators which works for any time ordering and will prove more convenient.		
in physical space	$\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int \frac{d^4k\, e^{-ik(x-y)}}{k^2-\mu^2+i\varepsilon}$	$S_{F\alpha\beta} = \frac{1}{(2\pi)^4}\int \frac{d^4p\, e^{-ip(x-y)}\left(\not{p}+m\right)}{p^2-m^2+i\varepsilon}$	$D_F^{\mu\nu}(x-y) = \frac{-g^{\mu\nu}}{(2\pi)^4}\int_{C^\pm} \frac{d^4k\, e^{-ik(x-y)}}{k^2+i\varepsilon}$
in momentum space	$\Delta_F(k) = \frac{1}{k^2-\mu^2+i\varepsilon}$	$S_{F\alpha\beta}(p) = \frac{\not{p}+m}{p^2-m^2+i\varepsilon}$	$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2+i\varepsilon}$
	Observable operators like total energy, three momentum, and charge found by integrating corresponding density operators over all 3-space.		
Observables:		Electrons assumed below (q = - e)	
H	$P_0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}}\left(N_a(\mathbf{k})+N_b(\mathbf{k})\right)$	$P_0 = \sum_{\mathbf{p},r} E_{\mathbf{p}}\left(N_r(\mathbf{p})+\bar{N}_r(\mathbf{p})\right)$	$P_0 = \sum_{\mathbf{k},r} \omega_{\mathbf{k}}\zeta_r N_r(\mathbf{k})$
P_i = 3-momentum	$\mathbf{P} = \sum_{\mathbf{k}} \mathbf{k}\left(N_a(\mathbf{k})+N_b(\mathbf{k})\right)$	$\mathbf{P} = \sum_{\mathbf{p},r} \mathbf{p}\left(N_r(\mathbf{p})+\bar{N}_r(\mathbf{p})\right)$	$\mathbf{P} = \sum_{\mathbf{k},r} \mathbf{k}\,\zeta_r N_r(\mathbf{k})$
s^μ	$qj^\mu = q(\rho,\mathbf{j})$	$qj^\mu = -e(\rho,\mathbf{j})$	0 for photons
Q	$\int s^0 d^3x = q\sum_{\mathbf{k}}\left(N_a(\mathbf{k})-N_b(\mathbf{k})\right)$	$\int s^0 d^3x = -e\sum_{\mathbf{p},r}\left(N_r(\mathbf{p})-\bar{N}_r(\mathbf{p})\right)$	0 for photons
$\boldsymbol{\sigma}\ (=\sigma_i)$	N/A	$\sigma_1 = \sigma^{23};\ \sigma_2 = \sigma^{31};\ \sigma_3 = \sigma^{12}$ $\sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$	N/A
Spin $\mathbf{S}$ $(= S_i)$	0	$\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}\quad \left(S_i = \frac{1}{2}\sigma_i\right)$	1 for photons
Proof that integer spin = bosons, ½ integer = fermions	If <u>anticommutators</u> used instead of commutators with Klein-Gordon equation solutions, it can be shown that for two observables A,B (such as energy density, momentum density, etc) $\left[A(x),B(y)\right] \neq 0$ and $(x-y)$ spacelike But by microcausality, $A(x)$ and $B(y)$ must not interfere for $(x-y)$ spacelike. Hence, we cannot use anticommutators with K-G equation and K-G particles must be bosons.	If <u>commutators</u> used with Dirac equation, then $H = \sum_{\mathbf{p},r} E_{\mathbf{p}}\left(N_r(\mathbf{p})-\bar{N}_r(\mathbf{p})\right)$. The minus sign means no stable lowest energy (i.e., no stable ground state.) Therefore, commutators cannot be used with Dirac equation, and Dirac particles can only be fermions.	Same as spin 0.

(next) Part 2. From Operators and Propagators to Feynman Rules