Scalar Quantum Fields in General Relativity

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Ref: Mukhanov, V.F., and Winitzki, S., *Introduction to Quantum Effects in Gravity*, Cambridge (2007), Chap. 5

	Minimally Coupled		**Non-Minimally Coupled Example**		**Comment**
	Real Scalar	**Gravity**	**Real Scalar**	**Gravity**	
Field	ϕ	$g_{\mu\nu}$	ϕ	$g_{\mu\nu}$	
Back-ground	Physical length (measured in meters with meter sticks) in x_1 direction is $d\underset{\sim}{x}_1 = \sqrt{-g_{11}}\,dx_1$ Coordinate length = dx_1 (The length of this distance in meters measured with meter sticks depends on the particular generalized coordinate grid chosen, i.e., on g_{11}, which represents that grid in the x_1 direction.) Physical 4D volume in an orthogonal (i.e., $g_{\mu\nu}$ is diagonal) spacetime coordinate system is $d\underset{\sim}{V} = d\underset{\sim}{x}_1 d\underset{\sim}{x}_2 d\underset{\sim}{x}_3 d\underset{\sim}{x}_0$ $= \sqrt{-g_{11}}\,dx_1\sqrt{-g_{22}}\,dx_2\sqrt{-g_{33}}\,dx_3\sqrt{g_{00}}\,dx_0 = \sqrt{-g}\,dx_1dx_2dx_3dx_0 = \sqrt{-g}\,dV = \sqrt{-g}\,d^4x$				g_{11} is negative; phys value has "~" underneath g is Det $g_{\mu\nu}$; dV is coord volume
Procedure	In standard QFT 1) $\eta_{\mu\nu} \rightarrow g_{\mu\nu}$ 2) " , " $\rightarrow$ " ; " 3) $d^4x \rightarrow \sqrt{-g}\,d^4x$	Choose $\mathcal{L}_{grav}$ as in classical GR	In this example, as in 2nd column, but add extra term.	In this example, $\mathcal{L}_{grav}$ of 3rd column.	In QFT we change to natural units, where $c=\hbar=1$
Action S	$S_m = \int \underbrace{\underset{\sim}{\mathcal{L}}_m}_{\text{phys Lagr}} \underbrace{\sqrt{-g}\,d^4x}_{\text{phys } d\underset{\sim}{V}}$ $= \int \underbrace{\mathcal{L}_m}_{\text{coord Lagr}} \underbrace{d^4x}_{\text{coord } dV}$	$S_{grav} = \int \underbrace{\underset{\sim}{\mathcal{L}}_{grav}}_{\text{phys Lagr}} \underbrace{\sqrt{-g}\,d^4x}_{\text{phys } d\underset{\sim}{V}}$ $= \int \mathcal{L}_{grav}\,d^4x$	As in 2nd column	As in 3rd column	$\sqrt{-g}$ of 3) above included in $\mathcal{L}_m$ and $\mathcal{L}_{grav}$
Lagran-gian $\mathcal{L}$	$\mathcal{L}_m = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi)\right)$	$\mathcal{L}_{grav} = -\frac{\sqrt{-g}}{8\pi G}(R + 2\Lambda)$	$\mathcal{L}_m = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - V(\phi) - \frac{\xi}{2}R\phi^2\right)$	In this example, $\mathcal{L}_{grav}$ of 3rd column	Other non-min $\mathcal{L}_m$, $\mathcal{L}_{grav}$ could have other terms in $R_{\alpha\beta\gamma\delta}$
Free field V	$V(\phi) = \frac{1}{2}m^2\phi^2$		As in 2nd column		"Free", but ϕ, grav coupled
Total $\mathcal{L}$	$\mathcal{L} = \mathcal{L}_m + \mathcal{L}_{grav}$	$\mathcal{L} = \mathcal{L}_m + \mathcal{L}_{grav}$	As in 2nd column	As in 3rd column	
Variation of S	From $\delta S = 0$, where $\mathcal{L}_{grav}$ has terms in $g_{\alpha\beta,\mu\nu}$ due to R, one gets the Euler-Lagrange eqs below. Gravity eq has an extra term beyond the more familiar Euler-Lagrange eq due to 2nd derivative terms in $\mathcal{L}_{grav}$. Note the math derivation is based on the integrand used with d^4x, so it uses $\mathcal{L} = \underset{\sim}{\mathcal{L}}\sqrt{-g}$.				
Euler-Lagrange equation	$\partial_\mu \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$	$-\partial_\mu\partial_\nu \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu\nu}} + \partial_\mu \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\mu}} - \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} = 0$	As in 2nd column	As in 3rd column	From $\delta S = 0$
Note:	Derivation to get gravity result below is laborious. See Mukhanov & Minitzki pgs. 229-232.				

Equation of motion	$\left(\sqrt{-g}\, g^{\alpha\beta}\phi_{,\beta}\right)_{,\alpha} + \sqrt{-g}\,\frac{\partial V}{\partial\phi}=0$	$G_{\alpha\beta}+g_{\alpha\beta}\Lambda = R_{\alpha\beta}-\frac{1}{2} g_{\alpha\beta}R+g_{\alpha\beta}\Lambda = 8\pi G T_{\alpha\beta}$	$\left(\sqrt{-g}\, g^{\alpha\beta}\phi_{,\beta}\right)_{,\alpha}+\sqrt{-g}\left(\frac{\partial V}{\partial\phi}+\xi R\phi\right)=0$	As in 3rd column w different $T_{\alpha\beta}$	From Euler-Lagrange equation
Stress-energy part of above		$T_{\alpha\beta}=\frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}} = \phi_{,\alpha}\phi_{,\beta}-g_{\alpha\beta}\left(\frac{1}{2} g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-V(\phi)\right)$		$T_{\alpha\beta}=\frac{2}{\sqrt{-g}}\times\left(\partial_\mu\partial_\nu\frac{\partial\mathcal{L}_m}{\partial g_{\alpha\beta,\mu\nu}}-\partial_\mu\frac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}{}_{,\mu}}+\frac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}}\right)$	←$\mathcal{L}_m$ in 5th column from "Lagrangian $\mathcal{L}$" row above, 4th column
Covariant form of eq of motion	$\phi_{;\alpha}^{;\alpha}+\frac{\partial V}{\partial\phi}=0$	$G_{\alpha\beta}+g_{\alpha\beta}\Lambda = R_{\alpha\beta}-\frac{1}{2} g_{\alpha\beta}R+g_{\alpha\beta}\Lambda = 8\pi G T_{\alpha\beta}$	$\phi_{;\alpha}^{;\alpha}+\frac{\partial V}{\partial\phi}+\xi R\phi=0$	As in 3rd column w different $T_{\alpha\beta}$	Gravity field eq of motion above already covariant
Above are	GR version of Klein-Gordon eq	Einstein's field equation			
	<u>E/m Interactions with (Complex) Scalar Field</u>				
Procedure	Initially as above with free field, but now need complex scalar field, as real scalar field is charge neutral.				
Action S	$S_m=\int\mathcal{L}_m\, d^4x$	All gravity as above in free scalar field			
Free field Lagran-gian $\mathcal{L}$	$\mathcal{L}_m=\sqrt{-g}\times\left(\frac{1}{2} g^{\mu\nu}\phi^*{}_{,\mu}\phi_{,\nu}-\frac{1}{2} m^2\phi^*\phi\right)$				For free field $V\left(\phi^*,\phi\right)=\frac{1}{2}m^2\phi^*\phi$
Interaction $\mathcal{L}_m$	with $\phi_{,\mu}\to D_\mu\phi$ $\mathcal{L}_m=\sqrt{-g}\times\left(\frac{1}{2} g^{\mu\nu}\left(D_\mu\phi\right)^* D_\nu\phi-\frac{1}{2} m^2\phi^*\phi\right)$				$D_\mu=\partial_\mu+iA_\mu$
Re-write interact $\mathcal{L}_m$	$\mathcal{L}_m=\sqrt{-g}\left(\frac{1}{2} g^{\mu\nu}\phi^*{}_{,\mu}\phi_{,\nu}-\frac{1}{2}m^2\phi^*\phi-\frac{1}{2}g^{\alpha\beta}iA_\alpha\left(\phi^*\phi_{,\beta}-\phi\phi^*_{,\beta}\right)+\frac{1}{2}g^{\alpha\beta}A_\alpha A_\beta\phi^*\phi\right)$				
Interaction $\mathcal{L}_{e/m}$	$F_{\mu\nu}=A_{\nu;\mu}-A_{\mu;\nu}=A_{\nu,\mu}-A_{\mu,\nu}$ Standard QFT $\mathcal{L}_{e/m}=-\frac{1}{16\pi}\eta^{\alpha\beta}\eta^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}\ \rightarrow\ -\frac{1}{16\pi}g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}$				$\Gamma^{\alpha}_{\beta\gamma}$ drop out of $F_{\mu\nu}$
Total $\mathcal{L}$	$\mathcal{L}=\mathcal{L}_m+\mathcal{L}_{grav}+\mathcal{L}_{e/m}$				
Equations of Motion	Use above $\mathcal{L}$ in Euler-Lagrange eq for 3 fields, ϕ, $g_{\mu\nu}$, and A_μ. Get 3 coupled eqs.				