QED/FIELD THEORY OVERVIEW: PART 2

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Wholeness Chart 8-4. From Operators and Propagators to Feynman Rules

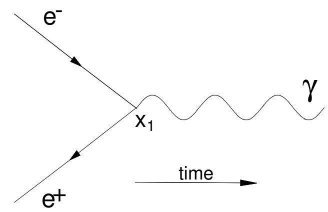
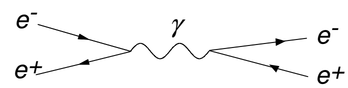
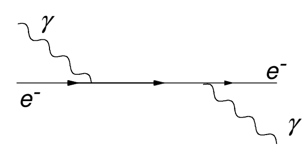
⇩ INTERACTING FIELDS ⇩

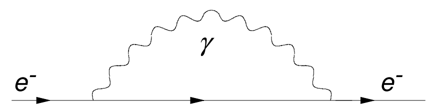
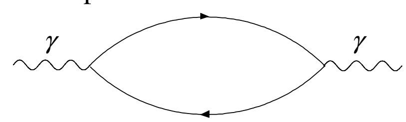
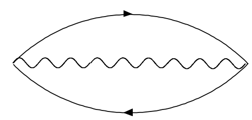
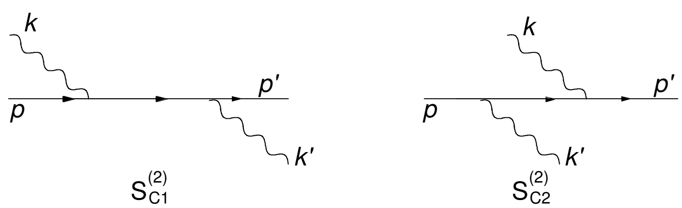
	In theory, the non-linear coupled partial differential interaction fields equations can be solved simultaneously to get interacting fields solutions and hence complete descriptions of all interactions. In practice this has not been possible and the following perturbation scheme has been developed.

Interaction Picture Approach

Interaction Picture $H^I = H_0^I + H_I^I$	Motion of <u>states</u> governed by H_I^I: $\quad i\frac{d}{dt}\|\Psi\rangle_I = H_I^I\|\Psi\rangle_I$ Motion of <u>operators</u> governed by $H_0^I = H_0$ $\quad \frac{dO^I}{dt} = -i\left[O^I, H_0\right]$ ϕ, ψ, A^μ are operators, so depend on H_0 only. Further, for them the above operator equation reduces to the free field equations on the first page of Part 1 of this chart. (See end of Chap 5.)
Results of Interaction Picture	For interactions, if employ I.P., can use 1. free field operator solutions ϕ, ψ, A_μ of Part 1 as I.P. fields solutions 2. free field operator creation and destruction properties 3. free field number operators 4. free field observables operators 5. free field Feynman propagators 6. state equations of motion in H_I^I to determine change in state in time (i.e., interactions)
H_I^I	Spatial integral of $\mathcal{H}_I^I = -\mathcal{L}_I^I$ (with operators taken as free field solutions) = H_I^I. e.g., for QED $H_I^I = \int \mathcal{H}_I^I d^3\mathbf{x}$ with $\mathcal{H}_I^I = -e\bar{\psi}\gamma^\mu\psi A_\mu$
New notation	Drop script "I" on states and operators other than $\mathcal{H}_I^I$ and H_I^I.
S matrix	$\|S_{fi}\|^2$ is probability of ith eigenstate (often multiparticle) transitioning to fth eigenstate S_{fi} is transition amplitude.
S_{oper}	General scattered state: $\quad \|\Psi(t_f)\rangle = S_{oper}\|\Psi(t_i)\rangle \; (= S_{oper}\|i\rangle \text{ typically})$ $\|i\rangle$ = an initial eigenstate. $\|f\rangle$ = a final eigenstate. $\|\Psi(t)\rangle$ = sum of eigenstates usually
Finding S matrix from S_{oper}	$S_{fi} = \langle f\|S_{oper}\|i\rangle = \langle f\|\Psi(t_f)\rangle$ so $\|\Psi(t_f)\rangle = \sum_f \|f\rangle S_{fi}$ Conservation of probability (not particles) is $\sum_f \|S_{fi}\|^2 = 1$
S_{oper} equation of motion	I.P. state eq of motion at top for $\|\Psi(t)\rangle = S_{oper}\|\Psi(t_i)\rangle$ yields $\quad i\frac{dS_{oper}}{dt} = H_I^I S_{oper}$
Solution to S_{oper} eq of motion	$S_{oper} = e^{-i\int_{t_i}^{t_f} H_I^I dt} = e^{-i\int_{t_i}^{t_f}\int_V \mathcal{H}_I^I d^4x}$ so $\|\Psi(t)\rangle = S_{oper}\|\Psi(t_i)\rangle = e^{-i\int_{t_i}^{t} H_I^I dt'}\|\Psi(t_i)\rangle$

S_{oper} for all space and time = S	$S = S_{\substack{oper\\ V\to\infty}}\left(t_f \to \infty, t_i \to -\infty\right) = e^{-i\int_{-\infty}^{\infty}\mathcal{H}_I^I d^4x}$ S = "S operator" terminology in QFT typically
S matrix for S operator	For ∞ space and time, $S_{fi} = \langle f\lvert S\rvert i\rangle = \langle f\lvert\Psi(t=\infty)\rangle$ so $\lvert\Psi(t=\infty)\rangle = \sum_f \lvert f\rangle S_{fi}$
Dyson expansion	Below is Dyson expansion of S solution above
exact: 2nd order:	$S = \underbrace{I}_{S^{(0)}} \underbrace{-i\int_{-\infty}^{\infty}\mathcal{H}_I^I(x_1)d^4x_1}_{S^{(1)}} \underbrace{-\frac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\left\{\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2)\right\}d^4x_1d^4x_2}_{S^{(2)}} + ... = \sum_{n=0}^{\infty}S^{(n)}$ $= \sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty}d^4x_1d^4x_2...d^4x_n T\left\{\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2)...\mathcal{H}_I^I(x_n)\right\}$ To this point treatment is exact. Perturbation arises from using fewer than ∞ terms in the above. $S \cong I + (-i)\int_{-\infty}^{\infty}d^4x_1\mathcal{H}_I^I(x_1) + \frac{(-i)^2}{2!}\int_{-\infty}^{\infty}d^4x_1d^4x_2 T\left\{\mathcal{H}_I^I(x_1)\mathcal{H}_I^I(x_2)\right\}$
Contractions of operators	Definition: $\underbrace{AB} = \langle 0\lvert T\{AB\}\rvert 0\rangle$ = Feynman propagators if A and B are certain fields. Special (only non-zero) cases: $\underbrace{\phi(x_1)\phi^\dagger(x_2)} = \underbrace{\phi^\dagger(x_2)\phi(x_1)} = i\Delta_F(x_1 - x_2)$ $\underbrace{\psi_a(x_1)\overline{\psi}_\beta(x_2)} = -\underbrace{\overline{\psi}_\beta(x_2)\psi_a(x_1)} = iS_{F\alpha\beta}(x_1 - x_2)$ $\underbrace{A^\mu(x_1)A^\nu(x_2)} = iD_F^{\mu\nu}(x_1 - x_2)$
Wick's Theorem	$T\left\{(AB...)_{x_1}.......(AB...)_{x_n}\right\} = N\left\{(AB...)_{x_1}.......(AB...)_{x_n}\right\}$ $+N\left\{\underbrace{(AB...)_{x_1}(AB...)_{x_2}}...\right\} + N\left\{\underbrace{(AB...)_{x_1}(AB...)_{x_2}}...\right\} + \left(\text{all other normal ordered non equal time contractions}\right)$ $+N\left\{\underbrace{\underbrace{(AB...)_{x_1}(AB...)_{x_2}}}...\right\} + \left(\text{all other normal ordered non equal time double contractions}\right)$ $+\left(\text{all normal ordered non equal time triple contractions}\right)$ + etc.
QED interactions	Above basic principles of interaction theory applied to QED below
$\mathcal{H}_I^I$ for QED	$\mathcal{H}_I^I = -\mathcal{L}_I^I = -e\overline{\psi}A^\mu\gamma_\mu\psi = -e\left(\overline{\psi}^+ + \overline{\psi}^-\right)\left(\not{A}^+ + \not{A}^-\right)\left(\psi^+ + \psi^-\right)$ (for electrons and positrons)
Dyson expansion for QED	Using above in Dyson expansion of S operator and using extended Wick's theorem to evaluate the time ordered normal ordered integrand yields the QED S operator $S = \sum_{n=0}^{\infty}S^{(n)} = S^{(0)} + S^{(1)} + S^{(2)} + S^{(3)} + ...$ (higher order terms) where $S^{(0)}$, $S^{(0)}$, $S^{(0)}$ shown below

	Operator	Matrix Elements
$S^{(0)}$	$= I$ no transition of particles $\lvert f\rangle = \lvert i\rangle$	$S^{(0)}_{fi} = \langle f \lvert S^{(0)} \rvert i\rangle$ typical process: $\lvert e^-,\gamma\rangle \to \lvert e^-,\gamma\rangle$ no virtual particles, no 4 momentum change
$S^{(1)}$	$= (-i)\int d^4x_1 N\left\{-e\bar{\psi}_l A^\mu \gamma_\mu \psi_l\right\}_{x_1}$ = all three external particles interaction terms = 8 terms in all, but these processes are <u>not real physical processes</u> (because resultant particle(s) off shell)	Typical non-physical process: 
$S^{(2)}$		
$S_A^{(2)}$	$= \frac{-e^2}{2!}\int d^4x_1 d^4x_2 N\left\{(\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi)_{x_2}\right\}$ No real physical processes.	Two processes like $S^{(1)}$ above going on independently.
$S_B^{(2)}$	$= \frac{-e^2}{2!}\int d^4x_1 d^4x_2 N\left\{(\bar{\psi}\not{A}\underbrace{\psi)_{x_1}(\bar{\psi}}\not{A}\psi)_{x_2}\right\}$ = all four external leptons interaction terms	$S^{(2)}_{Bfi} = \langle f \lvert S_B^{(2)} \rvert i\rangle$ typical process (Bhabbha scattering): $\lvert e^-,e^+\rangle \to \lvert e^-,e^+\rangle$ 
$S_C^{(2)}$	$= \frac{-e^2}{2!}\int d^4x_1 d^4x_2 N\left\{(\bar{\psi}\not{A}\underbrace{\psi)_{x_1}(\bar{\psi}}\not{A}\psi)_{x_2}\right\}$ $+ N\left\{(\underbrace{\bar{\psi}\not{A}\psi)_{x_1}(\bar{\psi}\not{A}\psi}_{x_2})\right\}$ $= -e^2\int d^4x_1 d^4x_2 N\left\{(\bar{\psi}\not{A}\underbrace{\psi)_{x_1}(\bar{\psi}}\not{A}\psi)_{x_2}\right\}$ = all two external leptons, two external photons interaction terms.	$S^{(2)}_{Cfi} = \langle f \lvert S_C^{(2)} \rvert i\rangle$ typical process (Compton scattering): $\lvert i\rangle = \lvert e^-,\gamma\rangle \to \lvert f\rangle = \lvert e^-,\gamma\rangle$ with virtual electron mediating scatter. 

$S_D^{(2)}$	$= -e^2 \int d^4x_1 d^4x_2 N\left\{ (\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2} \right\}$ = two external leptons terms (lepton self energy)	$S_{Dfi}^{(2)} = \langle f \mid S_D{}^{(2)} \mid i \rangle$ electron and positron loops 
$S_E^{(2)}$	$= \frac{-e^2}{2!} \int d^4x_1 d^4x_2 N\left\{ (\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2} \right\}$ = two external photons term (photon self energy)	$S_{Efi}^{(2)} = \langle f \mid S_E{}^{(2)} \mid i \rangle$ photon loop 
$S_F^{(2)}$	$= \frac{-e^2}{2!} \int d^4x_1 d^4x_2 N\left\{ (\bar{\psi} \not{A} \psi)_{x_1} (\bar{\psi} \not{A} \psi)_{x_2} \right\}$ = no external particles term	$S_{Ffi}^{(2)} = \langle f \mid S_F{}^{(2)} \mid i \rangle$ vacuum bubble 
$S^{(3)}$, $S^{(4)}$, etc.	Higher order terms. Ignored for now.	
Sample probability determination	Compton scattering, two ways: (Assumption: Particles are plane waves in box of volume *V*.)  See Chap. 8, XXX Sect. 8.4.3, pg. 226 XXX (online at http://www.quantumfieldtheory.info, click on Compton Scattering Transition Amplitude link) for derivation of the following Compton's $S_{fi} = S_{Compton} = \langle f \mid S \mid i \rangle_{Comp} = \langle f \mid S_{C1}{}^{(2)} + S_{C2}{}^{(2)} \mid i \rangle$ $= \left(\prod_{\mathbf{p}''}^{\text{all external fermions}} \sqrt{\frac{m}{VE_{\mathbf{p}''}}} \right) \left(\prod_{\mathbf{k}''}^{\text{all external bosons}} \sqrt{\frac{1}{2V\omega_{\mathbf{k}''}}} \right) (2\pi)^4 \delta(p' + k' - p - k) \left(\mathcal{M}_{C1}^{(2)} + \mathcal{M}_{C2}^{(2)} \right).$ where $\mathcal{M}_{C1}^{(2)} = -e^2 \bar{u}_{s'\alpha}(\mathbf{p}') \varepsilon_{r'}^{\mu}(\mathbf{k}') \gamma_{\mu}^{\alpha\beta} i S_{F\beta\delta}(q = p + k) \varepsilon_r^{\nu}(\mathbf{k}) \gamma_{\nu}^{\delta\eta} u_{s\eta}(\mathbf{p})$ $\mathcal{M}_{C2}^{(2)} = -e^2 \bar{u}_{s'\alpha}(\mathbf{p}') \varepsilon_r^{\mu}(\mathbf{k}) \gamma_{\mu}^{\alpha\beta} i S_{F\beta\delta}(q = p - k') \varepsilon_{r'}^{\nu}(\mathbf{k}') \gamma_{\nu}^{\delta\eta} u_{s\eta}(\mathbf{p})$ Probability of Compton scattering = $\left| \langle f \mid S \mid i \rangle_{Comp} \right|^2$	

Adding amplitudes	When two or more diagrams have the same external states in and out, add amplitudes for each contributing diagram, then square the absolute value of the result to get probability. For probability that any of two or more outcomes (different external states out) may occur from the same external states in, square absolute value of individual amplitudes first and then add.
2 ways to calculate probability	1) Go through tedious derivation like Chap. 8, Sect. 8.4.3 for each interaction 2) Use short cut of Feynman rules (listed in Chap. 8, XXX Sect. 8.5.2, pg. 236 XXX)
	All three lepton types treated below.
Mixed lepton $\mathcal{H}_I^l$	$\mathcal{H}_I^l = -\mathcal{L}_I^l = -e\sum_l \bar{\psi}_l A^\mu \gamma_\mu \psi_l = -e\sum_l \left(\bar{\psi}_l^+ + \bar{\psi}_l^-\right)\left(\not{A}^+ + \not{A}^-\right)\left(\psi_l^+ + \psi_l^-\right)$
Mixed lepton S operator	Each $\bar{\psi}\not{A}\psi$ term in S expression above for single lepton type replaced by $\sum_l \bar{\psi}_l \not{A}\psi_l$ term. $S = \sum_{n=0}^{\infty} \frac{(ie)^n}{n!}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty} d^4x_1...d^4x_n \sum_{l_1=1}^{3}....\sum_{l_n=1}^{3} T\left\{\left(\bar{\psi}_{l_1}\not{A}\psi_{l_1}\right)_{x_1}...\left(\bar{\psi}_{l_n}\not{A}\psi_{l_n}\right)_{x_n}\right\}$ = terms like previous blocks for e^-, e^+ + " " " " " muons + " " " " " taus + terms mixing lepton types (but not mixed at a vertex).
Typical interaction	$e^- + e^+ \rightarrow \mu^- + \mu^+$ (with photon mediating.)
Mixed lepton summary	1) Obtain the Feynman amplitude assuming all leptons are electrons/positrons. 2) For lines in the Feynman diagram representing other lepton flavors, replace spinors and/or propagators with those representing the other flavors.