

## QED/FIELD THEORY OVERVIEW: PART 1

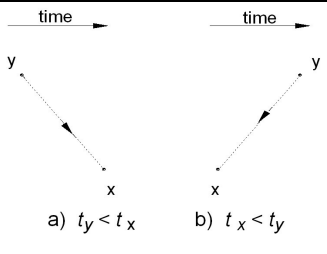
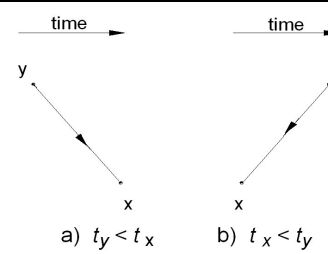
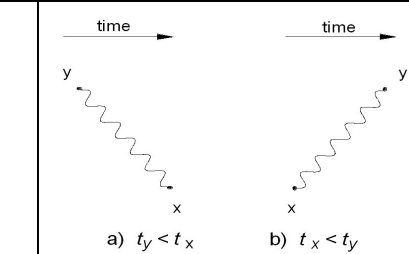
## Wholeness Chart 5-X. From Field Equations to Propagators and Observables

## Heisenberg Picture, Free Fields

	<u>Spin 0</u>	<u>Spin 1/2</u>	<u>Spin 1</u>
Classical Lagrangian density, free	$\mathcal{L}_0^0 = K (\partial_\alpha \phi \partial^\alpha \phi - \mu^2 \phi \phi)$	None. Macroscopic spinor fields not observed.	$\mathcal{L}_0^1 = \frac{\mu^2}{2} A^\mu A_\mu - \frac{1}{2} \partial_\nu A_\mu \partial^\nu A^\mu$ or $= \frac{\mu^2}{2} A^\mu A_\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$ , where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
2 <sup>nd</sup> quantization, Postulate #1	Quantum field Lagrangian density (or equivalently, the Hamiltonian density) same as classical, fields are complex, and $K=1$ . For spinors, Dirac equation from RQM with states $\rightarrow$ fields.		
QFT Lagrangian density, free	$\mathcal{L}_0^0 = (\partial_\alpha \phi^\dagger \partial^\alpha \phi - \mu^2 \phi^\dagger \phi)$	$\mathcal{L}_0^{1/2} = \bar{\psi} (i\partial - m) \psi \quad \partial = \gamma^\alpha \partial_\alpha$	As above for classical.
$\mathcal{L} \uparrow$ into the Euler-Lagrange equation yields $\downarrow$			
Free field eqs	$(\partial_\alpha \partial^\alpha + \mu^2) \phi = 0$ $(\partial_\alpha \partial^\alpha + \mu^2) \phi^\dagger = 0$	$(i\gamma^\alpha \partial_\alpha - m) \psi = 0$ $(i\partial_\alpha \bar{\psi} \gamma^\alpha + m \bar{\psi}) = 0 \quad \bar{\psi} = \psi^\dagger \gamma^0$	$(\partial_\alpha \partial^\alpha + \mu^2) A^\mu = 0$ photon $\mu = 0$ $A^{\mu\dagger} = A^\mu$ for chargeless (photon)
Conjugate momenta	$\pi_0^0 = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} = \dot{\phi}^\dagger; \pi_0^{0\dagger} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}^\dagger} = \dot{\phi}$	$\pi^{1/2} = i\psi^\dagger; \bar{\pi}^{1/2} = 0$	$\pi_\mu^1 = -\dot{A}_\mu$
Hamiltonian density	$\mathcal{H}_0^0 = \pi_0^0 \dot{\phi} + \pi_0^{0\dagger} \dot{\phi}^\dagger - \mathcal{L}_0^0$ $= (\dot{\phi} \dot{\phi}^\dagger + \nabla \phi^\dagger \nabla \phi + \mu^2 \phi^\dagger \phi)$	$\mathcal{H}_0^{1/2} = \pi^{1/2} \dot{\psi} - \mathcal{L}_0^{1/2}$	$\mathcal{H}_0^1 = \pi_\mu^1 \dot{A}^\mu - \mathcal{L}_0^1$
Free field solutions	$\phi = \phi^+ + \phi^-$ $\phi^\dagger = \phi^{\dagger+} + \phi^{\dagger-}$	$\psi = \psi^+ + \psi^-$ $\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^-$	$A^\mu = A^{\mu+} + A^{\mu-}$ (photon)
Discrete eigenstates (Plane waves, constrained to volume $V$ )	$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} (a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx})$ $\phi^\dagger(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} (b(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx})$	$\psi = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} (c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx})$ $\bar{\psi} = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} (d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx})$	$A^\mu = \sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} (\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx})$
Continuous eigenstates (Plane waves, no volume constraint)	$\phi(x) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} (a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx})$ $\phi^\dagger(x) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} (b(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx})$	$\psi = \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3\mathbf{p}}{\sqrt{E_{\mathbf{p}}}} (c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx})$ $\bar{\psi} = \sum_r \sqrt{\frac{m}{(2\pi)^3}} \int \frac{d^3\mathbf{p}}{\sqrt{E_{\mathbf{p}}}} (d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx})$ spinor indices on $u_r, v_r$ , and $\psi$ suppressed. $r = 1, 2$ .	$A^\mu = \sum_r \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d\mathbf{k}}{\sqrt{\omega_{\mathbf{k}}}} (\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx})$ $r = 0, 1, 2, 3$ (4 polarization vectors)

2 <sup>nd</sup> quantization Postulate #2	$[\phi^r(\mathbf{x},t), \pi_s(\mathbf{y},t)] = [\phi^r \pi_s \mp \pi_s \phi^r] = i\delta^r_s \delta(\mathbf{x}-\mathbf{y}) \quad \phi^r = \text{any quantized field}$ <p>All other commutators = 0. Anti-commutator for spin 1/2 fields (use plus sign.)</p>		
	Using conjugate momenta expressions in $\uparrow$ yields $\downarrow$		
Equal time commutators (intermediate step only)	$[\phi(\mathbf{x},t), \dot{\phi}^\dagger(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y})$	$[\psi_\alpha(\mathbf{x},t), \bar{\psi}_\beta(\mathbf{y},t)]_+ = \gamma_{\alpha\beta}^0 \delta(\mathbf{x}-\mathbf{y})$ <p>outer product, spinor indices <math>\alpha, \beta</math> shown with <math>\alpha, \beta = 1, 2, 3, 4</math>.</p>	$[A^\mu(\mathbf{x},t), \dot{A}^\nu(\mathbf{y},t)] = -ig^{\mu\nu} \delta(\mathbf{x}-\mathbf{y})$
	Using free field solutions in $\uparrow$ and the 3D Dirac delta function (e.g., for discrete solutions, $\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{V} \sum_{n=-\infty}^{+\infty} e^{-i\mathbf{k}_n \cdot (\mathbf{x}-\mathbf{y})}$ ), and matching terms, yields the coefficient commutators $\downarrow$ .		
Coefficient commutators	$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'}$	$[c_r(\mathbf{p}), c_s^\dagger(\mathbf{p}')] = [d_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')] = \delta_{rs} \delta_{\mathbf{p}\mathbf{p}'}$	$[a_r(\mathbf{k}), a_s^\dagger(\mathbf{k}')] = \zeta_r \delta_{rs} \delta_{\mathbf{k}\mathbf{k}'}$ <p><math>\zeta_0 = -1, \zeta_{1,2,3} = 1</math></p>
discrete	$= \delta_{\mathbf{k}\mathbf{k}'}$	$= \delta_{rs} \delta_{\mathbf{p}\mathbf{p}'}$	$= \zeta_r \delta_{rs} \delta_{\mathbf{k}\mathbf{k}'}$
continuous	$= \delta(\mathbf{k}-\mathbf{k}')$	$= \delta_{rs} \delta(\mathbf{p}-\mathbf{p}')$	$= \zeta_r \delta_{rs} \delta(\mathbf{k}-\mathbf{k}')$
<b>The Hamiltonian Operator</b>			
	Substituting the free field solutions into the free Hamiltonian density $\mathcal{H}_0$ , integrating $H_0 = \int \mathcal{H}_0 d^3x$ , using the coefficient commutators $\uparrow$ in the result, and acting on states with $H_0$ yields $\downarrow$		
$H_0$	$\sum_{\mathbf{k}} \omega_{\mathbf{k}} (N_a(\mathbf{k}) + \frac{1}{2} + N_b(\mathbf{k}) + \frac{1}{2})$	$\sum_{\mathbf{p}, r} E_{\mathbf{p}} (N_r(\mathbf{p}) - \frac{1}{2} + \bar{N}_r(\mathbf{p}) - \frac{1}{2})$	$\sum_{\mathbf{k}, r} \omega_{\mathbf{k}} (\zeta_r N_r(\mathbf{k}) + \frac{1}{2})$
Number operators	$N_a(\mathbf{k}) = a^\dagger(\mathbf{k}) a(\mathbf{k})$ $N_b(\mathbf{k}) = b^\dagger(\mathbf{k}) b(\mathbf{k})$	$N_r(\mathbf{p}) = c_r^\dagger(\mathbf{p}) c_r(\mathbf{p})$ $\bar{N}_r(\mathbf{p}) = d_r^\dagger(\mathbf{p}) d_r(\mathbf{p})$	$N_r(\mathbf{k}) = \zeta_r a_r^\dagger(\mathbf{k}) a_r(\mathbf{k})$
<b>Creation and Destruction Operators</b>			
	Evaluating $N_a(\mathbf{k}) a(\mathbf{k})  n_{\mathbf{k}}\rangle$ (similar for other particle types) with $\uparrow$ and the coefficient commutators yields $\downarrow$		
creation	$a^\dagger(\mathbf{k}), b^\dagger(\mathbf{k})$	$c_r^\dagger(\mathbf{p}), d_r^\dagger(\mathbf{p})$	$a_r^\dagger(\mathbf{k})$
destruction	$a(\mathbf{k}), b(\mathbf{k})$	$c_r(\mathbf{p}), d_r(\mathbf{p})$	$a_r(\mathbf{k})$
Normaliz factors lowering	$a(\mathbf{k})  n_{\mathbf{k}}\rangle = \sqrt{n_{\mathbf{k}}}  n_{\mathbf{k}} - 1\rangle$	$c(\mathbf{p})  1\rangle =  0\rangle$	as with scalars
raising	$a^\dagger(\mathbf{k})  n_{\mathbf{k}}\rangle = \sqrt{n_{\mathbf{k}} + 1}  n_{\mathbf{k}} + 1\rangle$	$c^\dagger(\mathbf{p})  0\rangle =  1\rangle$	as with scalars
tot particle num	$N(\phi) = \sum_{\mathbf{k}} (N_a(\mathbf{k}) - N_b(\mathbf{k}))$	$N(\psi) = \sum_{\mathbf{p}, r} (N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p}))$	$N(A^\mu) = \sum_{\mathbf{k}, r} \zeta_r N_r(\mathbf{k})$
tot particle num: lowering	$\phi = \phi^+ + \phi^-$	$\psi = \psi^+ + \psi^-$	$A^\mu +$
raising	$\phi^\dagger = \phi^{\dagger+} + \phi^{\dagger-}$	$\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^-$	$A^\mu -$

<b>Four Currents and Probability</b>			
Four currents (operators) $j^\mu_{,\mu} = 0$	$j^\mu = (\rho, \mathbf{j})$ $= -i(\phi^\dagger \cdot \mu \phi - \phi^\mu \phi^\dagger)$	$j^\mu = (\rho, \mathbf{j}) = -i(\bar{\psi} \gamma^\mu \psi)$	$j^\mu = -i(A_\alpha^{\mu\dagger} A^\alpha - A_\alpha^\mu A^{\alpha\dagger})$ $= 0$ for photons ( $A_\alpha^\dagger = A_\alpha$ )
	Emphasis in field theory is usually on the number of particles ( $N(\mathbf{k})$ operator), and particle probability densities are rarely used. For completeness, however, and to make the connection with quantum mechanics, they are included below. (Antiparticles would have negative values of those below!)		
Single particle probability density (not operator)	$\rho(\mathbf{x}, t) = \langle \phi(\mathbf{x}', t)   j^0(\mathbf{x}, t)   \phi(\mathbf{x}', t) \rangle$ Note integration over $\mathbf{x}'$ , not $\mathbf{x}$ For type $a$ plane wave, $\rho = \frac{1}{V}$	As at left, but with Dirac $j^0$ above.	$= 0$ for chargeless particles.
Charge, not probability	Scalar type $b$ particle $\rightarrow$ negative $\rho$ . Photons $\rightarrow \rho = 0$ . Led to conclusion that $j^0$ is really proportional to <i>charge</i> probability density.		
<b>Observables</b>			
	Observable operators like total energy, three momentum, and charge are found by integrating corresponding density operators over all 3-space. (For spin $\frac{1}{2}$ , electrons assumed below with $q = -e$ )		
$H$	$P_0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} (N_a(\mathbf{k}) + N_b(\mathbf{k}))$	$P_0 = \sum_{\mathbf{p}, r} E_{\mathbf{p}} (N_r(\mathbf{p}) + \bar{N}_r(\mathbf{p}))$	$P_0 = \sum_{\mathbf{k}, r} \omega_{\mathbf{k}} \zeta_r N_r(\mathbf{k})$
$P_i = 3$ -momentum	$\mathbf{P} = \sum_{\mathbf{k}} \mathbf{k} (N_a(\mathbf{k}) + N_b(\mathbf{k}))$	$\mathbf{P} = \sum_{\mathbf{p}, r} \mathbf{p} (N_r(\mathbf{p}) + \bar{N}_r(\mathbf{p}))$	$\mathbf{P} = \sum_{\mathbf{k}, r} \mathbf{k} \zeta_r N_r(\mathbf{k})$
$s^\mu$	$q j^\mu = q(\rho, \mathbf{j})$	$q j^\mu = -e(\rho, \mathbf{j})$	0 for photons
$Q$	$\int s^0 d^3x = q \sum_{\mathbf{k}} (N_a(\mathbf{k}) - N_b(\mathbf{k}))$	$\int s^0 d^3x = -e \sum_{\mathbf{p}, r} (N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p}))$	0 for photons
Spin operator for RQM states and QFT fields	N/A	$\boldsymbol{\Sigma} = \Sigma_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \quad i=1,2,3$ $\sigma_i = 2\text{D Pauli matrices}$	magnitude = 1 for photons,
Helicity operator for RQM states and QFT fields	N/A	$\frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{ \mathbf{p} }$	helicity eigenstates
Spin operator for QFT states	N/A	$\int \psi^\dagger \boldsymbol{\Sigma} \psi d^3x$	magnitude = 1 for photons,
Helicity operator for QFT states	N/A	$\int \psi^\dagger \left( \frac{\boldsymbol{\Sigma} \cdot \mathbf{p}}{ \mathbf{p} } \right) \psi d^3x$	helicity eigenstates
		$\uparrow$ shows $c_r^\dagger(\mathbf{p})$ , $d_r^\dagger(\mathbf{p})$ create helicity eigenstates	

<b>Bosons, Fermions, and Commutators</b>			
	Operations on states with creation, destruction, and number operators above yield the properties below.		
Properties of states:	$n_a(\mathbf{k}) = 0, 1, 2, \dots, \infty$ So spin 0 states bosonic.	$n_r(\mathbf{p}) = 0, 1$ only So spin $\frac{1}{2}$ states fermionic.	$n_r(\mathbf{k}) = 0, 1, 2, \dots, \infty$ So spin 1 states bosonic.
Bosons can only employ commutators Fermions can only employ anti-commutators	If anti-commutators used instead of commutators with Klein-Gordon equation solutions, then observable (not counting vacuum energy) Hamiltonian operator would have form $H_0^0 = 0$ and $H_0^0  \phi_{\mathbf{k}}\rangle = 0$ , i.e., all scalar particles would have zero energy. Hence, we cannot use anticommutators with spin 0 bosons.	If commutators used with Dirac equation, then $H = \sum_{\mathbf{p}, r} E_{\mathbf{p}} (N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p})).$ The minus sign means negative energy antiparticles, and since antiparticles and particles interact, no stable lowest energy (i.e., no stable ground state.) Therefore, commutators cannot be used with spin $\frac{1}{2}$ fermions. XX check that above is not more like left box for bosons XX	Same as spin 0.
<b>The Feynman Propagator</b>			
Step 1	Using the free field solutions in the LHS below (covariant field commutators) and the coefficient commutators yields RHS of below. (Note that different authors define the terms below slightly differently.)		
Covariant commutators, continuous only	$[\phi^\pm(x), \phi^\mp(y)] = i\Delta^\pm(x-y)$	$[\psi^\pm(x), \bar{\psi}^\mp(y)]_{+\alpha\beta} = iS_{\alpha\beta}^\pm(x-y)$	$[A^{\mu\pm}(x), A^{\nu\mp}(y)] = iD^{\mu\nu\pm}(x-y)$
3-momentum space form	$i\Delta^\pm = \frac{\pm 1}{2(2\pi)^3} \int d^3\mathbf{k} \frac{e^{\mp ik(x-y)}}{\omega_{\mathbf{k}}}$	$iS^\pm = (i\not{p} + m)i\Delta^\pm$	$iD^{\mu\nu\pm} = -g^{\mu\nu}i\Delta^\pm$
Step 2 contour integral form of above	$= \frac{-i}{(2\pi)^4} \int_{C^\pm} \frac{d^4k e^{-ik(x-y)}}{k^2 - \mu^2}$	$= \frac{-i}{(2\pi)^4} \int_{C^\pm} \frac{d^4p e^{-ip(x-y)} (\not{p} + m)}{p^2 - m^2}$	$= \frac{-ig^{\mu\nu}}{(2\pi)^4} \int_{C^\pm} \frac{d^4k e^{-ik(x-y)}}{k^2 - \underbrace{\mu^2}_{\text{photon}=0}}$
	Creation and destruction of free particles (& antiparticles) and their propagation visualized below.		
Feynman diagrams (aid for Step 3 below)			
Step 3 Time ordered operator $T$	If $t_y < t_x$ , $T\{\phi(x)\phi^\dagger(y)\} = \phi(x)\phi^\dagger(y)$ , i.e., the $\phi^\dagger(y)$ operates first, and should be placed on the right. If $t_x < t_y$ , $T\{\phi(x)\phi^\dagger(y)\} = \phi^\dagger(y)\phi(x)$ , i.e., the $\phi(x)$ operates first, and should be placed on the right. Note that $\phi(x)$ commutes with $\phi^\dagger(y)$ . [Fermions would anti-commute.]		

	The operator fields in the $T$ operator ↓ will create and destroy kets on RHS. In the wave mechanics formulation, bracket integration is over the dummy $\mathbf{x}'$ variable in the bra and ket, not $\mathbf{x}, \mathbf{y}$ of $T$ operator.		
Transition amplitude (double density in $x$ and $y$ )	$\langle 0 T\{\phi(x)\phi^\dagger(y)\} 0\rangle$	$\langle 0 T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} 0\rangle$	$\langle 0 T\{A^\mu(y)A^\nu(y)\} 0\rangle$
	The above represent both 1) creation of a particle at $y$ , destruction at $x$ , and 2) creation of an antiparticle at $x$ , destruction at $y$ } equals Feynman propagator		
Step 4	Fields such as $\phi$ represent integration over all momenta. The transition amplitude (density) above for fixed $x$ and $y$ equals the Feynman propagator between $x$ and $y$ . Note this is a number, not an operator. Only the continuous field solutions are relevant as no boundary conditions exist in this case.		
Feynman propagators	$\Delta_F(x-y) =$ $+ \Delta^+(x-y)$ if $t_y < t_x$ $- \Delta^-(x-y)$ if $t_x < t_y$	$S_{F\alpha\beta} = +S_{\alpha\beta}^+(x-y)$ if $t_y < t_x$ $= -S_{\alpha\beta}^-(x-y)$ if $t_x < t_y$	$D_F^{\mu\nu} = +D^{+\mu\nu}(x-y)$ if $t_y < t_x$ $= -D^{-\mu\nu}(x-y)$ if $t_x < t_y$
Step 5	Evaluating the above in complex space and taking certain limits with contour integrals yields a form for Feynman propagators which works for any time ordering and will prove more convenient.		
in physical space	$\Delta_F(x-y) =$ $\frac{1}{(2\pi)^4} \int d^4k e^{-ik(x-y)} \frac{1}{k^2 - \mu^2 + i\epsilon}$	$S_{F\alpha\beta} =$ $\frac{1}{(2\pi)^4} \int d^4p e^{-ip(x-y)} \frac{(\not{p} + m)}{p^2 - m^2 + i\epsilon}$	$D_F^{\mu\nu}(x-y) =$ $\frac{-g^{\mu\nu}}{(2\pi)^4} \int d^4k e^{-ik(x-y)} \frac{1}{k^2 + i\epsilon}$
in momentum space	$\Delta_F(k) = \frac{1}{k^2 - \mu^2 + i\epsilon}$	$S_{F\alpha\beta}(p) = \frac{\not{p} + m}{p^2 - m^2 + i\epsilon}$	$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\epsilon}$

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