

QED/FIELD THEORY OVERVIEW: PART 1

Wholeness Chart 5-4. From Field Equations to Propagators and Observables

Heisenberg Picture, Free Fields

	<u>Spin 0</u>	<u>Spin 1/2</u>	<u>Spin 1</u>
Classical Lagrangian density, free	$\mathcal{L}_0^0 = K(\partial_\alpha \phi \partial^\alpha \phi - \mu^2 \phi^2)$	None. Macroscopic spinor fields not observed.	$\mathcal{L}_0^1 = \underbrace{\frac{\mu^2}{2} A^\mu A_\mu}_{\mu=0 \text{ for photons}} - \frac{1}{2}(\partial_\nu A_\mu)(\partial^\nu A^\mu)$
2 nd quantization, Postulate #1	Bosons: Quantum field \mathcal{L} (or equivalently, \mathcal{H}) same as classical, fields are complex, and $K=1$. Spinors: Dirac eq from RQM with states \rightarrow fields. Deduce \mathcal{L} from Dirac eq; \mathcal{H} from Legendre transf.		
QFT Lagrangian density, free	$\mathcal{L}_0^0 = (\partial_\alpha \phi^\dagger \partial^\alpha \phi - \mu^2 \phi^\dagger \phi)$	$\mathcal{L}_0^{1/2} = \bar{\psi}(i\partial - m)\psi \quad \partial = \gamma^\alpha \partial_\alpha$	As above for classical.
$\mathcal{L} \uparrow$ into the Euler-Lagrange equation yields \downarrow			
Free field eqs	$(\partial_\alpha \partial^\alpha + \mu^2)\phi = 0$ $(\partial_\alpha \partial^\alpha + \mu^2)\phi^\dagger = 0$	$(i\gamma^\alpha \partial_\alpha - m)\psi = 0$ $(i\partial_\alpha \bar{\psi} \gamma^\alpha + m\bar{\psi}) = 0 \quad \bar{\psi} = \psi^\dagger \gamma^0$	$(\partial_\alpha \partial^\alpha + \mu^2)A^\mu = 0 \quad \text{photon } \mu = 0$ $A^{\mu\dagger} = A^\mu \text{ for chargeless (photon)}$
Conjugate momenta	$\pi_0^0 = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} = \dot{\phi}^\dagger; \pi_0^{0\dagger} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}^\dagger} = \dot{\phi}$	$\pi^{1/2} = i\psi^\dagger; \bar{\pi}^{1/2} = 0$	$\pi_\mu^1 = -\dot{A}_\mu$
Hamiltonian density	$\mathcal{H}_0^0 = \pi_0^0 \dot{\phi} + \pi_0^{0\dagger} \dot{\phi}^\dagger - \mathcal{L}_0^0$ $= (\dot{\phi} \dot{\phi}^\dagger + \nabla \phi^\dagger \cdot \nabla \phi + \mu^2 \phi^\dagger \phi)$	$\mathcal{H}_0^{1/2} = \pi^{1/2} \dot{\psi} - \mathcal{L}_0^{1/2}$	$\mathcal{H}_0^1 = \pi_\mu^1 \dot{A}^\mu - \mathcal{L}_0^1$
Free field solutions	$\phi = \phi^+ + \phi^-$ $\phi^\dagger = \phi^{\dagger+} + \phi^{\dagger-}$	$\psi = \psi^+ + \psi^-$ $\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^-$	$A^\mu = A^{\mu+} + A^{\mu-} \quad (\text{photon})$
Discrete eigenstates (Plane waves, constrained to volume V)	$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} (a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx})$ $\phi^\dagger(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} (b(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx})$	$\psi = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} (c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx})$ $\bar{\psi} = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} (d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx})$	$A^\mu = \sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} (\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx})$
Continuous eigenstates (Plane waves, no volume constraint)	$\phi(x) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} (a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx})$ $\phi^\dagger(x) = \int \frac{d\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}} (b(\mathbf{k})e^{-ikx} + a^\dagger(\mathbf{k})e^{ikx})$	$\psi = \sum_r \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{E_{\mathbf{p}}}} (c_r(\mathbf{p})u_r(\mathbf{p})e^{-ipx} + d_r^\dagger(\mathbf{p})v_r(\mathbf{p})e^{ipx})$ $\bar{\psi} = \sum_r \int \frac{d^3\mathbf{p}}{\sqrt{(2\pi)^3}} \frac{1}{\sqrt{E_{\mathbf{p}}}} (d_r(\mathbf{p})\bar{v}_r(\mathbf{p})e^{-ipx} + c_r^\dagger(\mathbf{p})\bar{u}_r(\mathbf{p})e^{ipx})$ spinor indices on $u_r, v_r,$ and ψ suppressed. $r = 1, 2.$	$A^\mu = \sum_r \int \frac{1}{\sqrt{2(2\pi)^3}} \int \frac{d\mathbf{k}}{\sqrt{\omega_{\mathbf{k}}}} (\varepsilon_r^\mu(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_r^\mu(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx})$ $r = 0, 1, 2, 3 \quad (4 \text{ polarization vectors})$

2 nd quantization Postulate #2	Bosons: $[\phi^r(\mathbf{x},t), \pi_s(\mathbf{y},t)] = [\phi^r \pi_s - \pi_s \phi^r] = i\delta^r_s \delta(\mathbf{x}-\mathbf{y})$, ϕ^r = any field, other commutators = 0. Spinors: Coefficient anti-commutation relations parallel coefficient commutation relations for bosons.		
	Bosons: using conjugate momenta expressions in \uparrow yields \downarrow		
Equal time commutators (intermediate step only)	$[\phi(\mathbf{x},t), \dot{\phi}^\dagger(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y})$	Not needed for spinor derivation.	$[A^\mu(\mathbf{x},t), \dot{A}^\nu(\mathbf{y},t)] = -ig^{\mu\nu} \delta(\mathbf{x}-\mathbf{y})$
	Bosons: Using free field solutions in \uparrow with 3D Dirac delta function (e.g., for discrete solutions, $\delta(\mathbf{x}-\mathbf{y}) = \frac{1}{V} \sum_{n=-\infty}^{+\infty} e^{-i\mathbf{k}_n \cdot (\mathbf{x}-\mathbf{y})}$), and matching terms, yields the coefficient commutators \downarrow .		
Coefficient commutators	$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = \delta_{\mathbf{k}\mathbf{k}'}$ discrete continuous	$[c_r(\mathbf{p}), c_s^\dagger(\mathbf{p}')] = [d_r(\mathbf{p}), d_s^\dagger(\mathbf{p}')] = \delta_{rs} \delta_{\mathbf{p}\mathbf{p}'}$ continuous	$[a_r(\mathbf{k}), a_s^\dagger(\mathbf{k}')] = \zeta_r \delta_{rs} \delta_{\mathbf{k}\mathbf{k}'}$ $\zeta_0 = -1, \zeta_{1,2,3} = 1$ continuous
Other coeffs	All other commutators = 0	All other anti-commutators = 0	All other commutators = 0
The Hamiltonian Operator			
	Substituting the free field solutions into the free Hamiltonian density \mathcal{H}_0 , integrating $H_0 = \int \mathcal{H}_0 d^3x$, and using the coefficient commutators \uparrow in the result, yields \downarrow . Acting on states with H_0 yields number operators.		
H_0	$\sum_{\mathbf{k}} \omega_{\mathbf{k}} (N_a(\mathbf{k}) + \frac{1}{2} + N_b(\mathbf{k}) + \frac{1}{2})$	$\sum_{\mathbf{p},r} E_{\mathbf{p}} (N_r(\mathbf{p}) - \frac{1}{2} + \bar{N}_r(\mathbf{p}) - \frac{1}{2})$	$\sum_{\mathbf{k},r} \omega_{\mathbf{k}} (N_r(\mathbf{k}) + \frac{1}{2})$
Number operators	$N_a(\mathbf{k}) = a^\dagger(\mathbf{k})a(\mathbf{k})$ $N_b(\mathbf{k}) = b^\dagger(\mathbf{k})b(\mathbf{k})$	$N_r(\mathbf{p}) = c_r^\dagger(\mathbf{p})c_r(\mathbf{p})$ $\bar{N}_r(\mathbf{p}) = d_r^\dagger(\mathbf{p})d_r(\mathbf{p})$	$N_r(\mathbf{k}) = \zeta_r a_r^\dagger(\mathbf{k})a_r(\mathbf{k})$
Creation and Destruction Operators			
	Evaluating $N_a(\mathbf{k}) a(\mathbf{k}) n_{\mathbf{k}}\rangle$ (similar for other particle types) with \uparrow and the coefficient commutators yields \downarrow		
creation	$a^\dagger(\mathbf{k}), b^\dagger(\mathbf{k})$	$c_r^\dagger(\mathbf{p}), d_r^\dagger(\mathbf{p})$	$a_r^\dagger(\mathbf{k})$
destruction	$a(\mathbf{k}), b(\mathbf{k})$	$c_r(\mathbf{p}), d_r(\mathbf{p})$	$a_r(\mathbf{k})$
Normaliz factors			as with scalars
lowering	$a(\mathbf{k}) n_{\mathbf{k}}\rangle = \sqrt{n_{\mathbf{k}}} n_{\mathbf{k}}-1\rangle$	$c_r(\mathbf{p}) \psi_{r,\mathbf{p}}\rangle = 0\rangle$	as with scalars
raising	$a^\dagger(\mathbf{k}) n_{\mathbf{k}}\rangle = \sqrt{n_{\mathbf{k}}+1} n_{\mathbf{k}}+1\rangle$	$c_r^\dagger(\mathbf{p}) 0\rangle = \psi_{r,\mathbf{p}}\rangle$	as with scalars
tot particle num	$N(\phi) = \sum_{\mathbf{k}} (N_a(\mathbf{k}) - N_b(\mathbf{k}))$	$N(\psi) = \sum_{\mathbf{p},r} (N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p}))$	$N(A^\mu) = \sum_{\mathbf{k},r} N_r(\mathbf{k})$
tot particle num: lowering	$\phi = \phi^+ + \phi^-$	$\psi = \psi^+ + \psi^-$	$A^{\mu+}$
raising	$\phi^\dagger = \phi^{\dagger+} + \phi^{\dagger-}$	$\bar{\psi} = \bar{\psi}^+ + \bar{\psi}^-$	$A^{\mu-}$

Four Currents and Probability			
Four currents (operators) $j^\mu, \mu = 0$	$j^\mu = (\rho, \mathbf{j}) = -i(\phi^\dagger \cdot^\mu \phi - \phi \cdot^\mu \phi^\dagger)$	$j^\mu = (\rho, \mathbf{j}) = \bar{\psi} \gamma^\mu \psi$	$j^\mu = -i(A_\alpha \cdot^\mu \dagger A^\alpha - A_\alpha \cdot^\mu A \cdot^\alpha \dagger)$ = 0 for photons ($A_\alpha^\dagger = A_\alpha$)
Emphasis in field theory is usually on the number of particles ($N(\mathbf{k})$ operator), and particle probability densities are rarely used. For completeness, however, and to make the connection with quantum mechanics, they are included below. (Antiparticles would have negative values of those below!)			
Single particle probability density (not operator)	$\rho(\mathbf{x}, t) = \langle \phi(\mathbf{x}', t) j^0(\mathbf{x}, t) \phi(\mathbf{x}', t) \rangle$ Note integration over \mathbf{x}' , not \mathbf{x} For type a plane wave, $\rho = \frac{1}{V}$	As at left, but with Dirac j^0 above.	= 0 for chargeless particles.
Charge, not probability	Scalar type b particle \rightarrow negative ρ . Photons $\rightarrow \rho = 0$. Led to conclusion that j^0 is really proportional to <i>charge</i> probability density.		
Observables			
Observable operators like total energy, three momentum, and charge are found by integrating corresponding density operators over all 3-space. (For spin 1/2, electrons assumed below with $q = -e$)			
H	$P_0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} (N_a(\mathbf{k}) + N_b(\mathbf{k}))$	$P_0 = \sum_{\mathbf{p}, r} E_{\mathbf{p}} (N_r(\mathbf{p}) + \bar{N}_r(\mathbf{p}))$	$P_0 = \sum_{\mathbf{k}, r} \omega_{\mathbf{k}} N_r(\mathbf{k})$
$P_i = 3$ -momentum	$\mathbf{P} = \sum_{\mathbf{k}} \mathbf{k} (N_a(\mathbf{k}) + N_b(\mathbf{k}))$	$\mathbf{P} = \sum_{\mathbf{p}, r} \mathbf{p} (N_r(\mathbf{p}) + \bar{N}_r(\mathbf{p}))$	$\mathbf{P} = \sum_{\mathbf{k}, r} \mathbf{k} N_r(\mathbf{k})$
s^μ	$q j^\mu = q(\rho, \mathbf{j})$	$q(j^\mu - (\text{constant})) \rightarrow \partial_\mu s^\mu = 0$	0 for photons
Q	$\int s^0 d^3x = q \sum_{\mathbf{k}} (N_a(\mathbf{k}) - N_b(\mathbf{k}))$	$\int s^0 d^3x = -e \sum_{\mathbf{p}, r} (N_r(\mathbf{p}) - \bar{N}_r(\mathbf{p}))$	0 for photons
Spin operator for RQM states and QFT fields	N/A	$\Sigma = \Sigma_i = \frac{1}{2} \begin{bmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{bmatrix} \quad i = 1, 2, 3$ $\sigma_i = 2D$ Pauli matrices	magnitude = 1 for photons,
Helicity operator for RQM states and QFT fields	N/A	$\frac{\Sigma \cdot \mathbf{p}}{ \mathbf{p} }$	helicity eigenstates
Spin operator for QFT states	N/A	$\int \psi^\dagger \Sigma \psi d^3x$	magnitude = 1 for photons,
Helicity operator for QFT states	N/A	$\int \psi^\dagger \left(\frac{\Sigma \cdot \mathbf{p}}{ \mathbf{p} } \right) \psi d^3x$	helicity eigenstates

Bosons, Fermions, and Commutators

	Operations on states with creation, destruction, and number operators above yield the properties below.		
Properties of states:	$n_a(\mathbf{k}) = 0, 1, 2, \dots, \infty$ So spin 0 states bosonic.	$n_r(\mathbf{p}) = 0, 1$ only So spin $\frac{1}{2}$ states fermionic.	$n_r(\mathbf{k}) = 0, 1, 2, \dots, \infty$ So spin 1 states bosonic.
Bosons can only employ commutators Fermions can only employ anti-commutators	If anti-commutators used instead of commutators with Klein-Gordon equation solutions, then observable (not counting vacuum energy) Hamiltonian operator would have form $H_0^0 = 0$ and $H_0^0 \phi_{\mathbf{k}}\rangle = 0$, i.e., all scalar particles would have zero energy. Hence, we cannot use anticommutators with spin 0 bosons.	Commutators lead to 2 or more identical particle states co-existing in same multiparticle state. Anti-commutators lead to only one given single particle state per multi-particle state. Therefore, commutators cannot be used with spin $\frac{1}{2}$ fermions. This is further proof that we need commutators with bosons.	Same as spin 0.

The Feynman Propagator

	Creation and destruction of free particles (& antiparticles) and their propagation visualized below.		
Feynman diagrams	<p style="font-size: small;">a) $t_y < t_x$ b) $t_x < t_y$</p>	<p style="font-size: small;">a) $t_y < t_x$ b) $t_x < t_y$</p>	<p style="font-size: small;">a) $t_y < t_x$ b) $t_x < t_y$</p>
Step 1 Time ordered operator T	If $t_y < t_x$, $T\{\phi(x)\phi^\dagger(y)\} = \phi(x)\phi^\dagger(y)$, i.e., the $\phi^\dagger(y)$ operates first, and should be placed on the right. If $t_x < t_y$, $T\{\phi(x)\phi^\dagger(y)\} = \phi^\dagger(y)\phi(x)$, i.e., the $\phi(x)$ operates first, and should be placed on the right. Note that $\phi(x)$ commutes with $\phi^\dagger(y)$ for $x \neq y$. [Fermion fields anti-commute.]		
Transition amplitude (double density in x and y)	$\langle 0 T\{\phi(x)\phi^\dagger(y)\} 0\rangle = i\Delta_F$ $\langle 0 T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\} 0\rangle = iS_{F\alpha\beta}$ $\langle 0 T\{A^\mu(y)A^\nu(x)\} 0\rangle = iD_F^{\mu\nu}$ The above vacuum expectation values (transition amplitudes) represent both 1) creation of a particle at y , destruction at x , and 2) creation of an antiparticle at x , destruction at y } transition amplitude = Feynman propagator		
Step 2 Propagator in terms of two commutators	By adding a term equal to zero to the Feynman propagator above, it can be expressed as vacuum expectation values (VEVs) of two commutators		
	$i\Delta_F(x-y) =$ $\langle 0 [\phi^+(x), \phi^{\dagger-}(y)] 0\rangle_{t_y < t_x}$ $\langle 0 [\phi^{\dagger+}(y), \phi^-(x)] 0\rangle_{t_x < t_y}$	$iS_{F\alpha\beta}(x-y) =$ $\langle 0 [\psi_\alpha^+(x), \bar{\psi}_\beta^-(y)]_+ 0\rangle_{t_y < t_x}$ $-\langle 0 [\bar{\psi}_\beta^+(y), \psi_\alpha^-(x)]_+ 0\rangle_{t_x < t_y}$	$iD_F^{\mu\nu}(x-y) =$ $\langle 0 [A^{\mu+}(x), A^{\nu-}(y)] 0\rangle_{t_y < t_x}$ $\langle 0 [A^{\nu+}(y), A^{\mu-}(x)] 0\rangle_{t_x < t_y}$

Step 3 As 3-momentum integrals	With the coefficient commutation relations, the above two commutators (for each spin type) can be expressed as two integrals over 3-momentum space		
Definition of symbols for commutators	$[\phi^+(x), \phi^{\dagger-}(y)] = i\Delta^+(x-y)$ $[\phi^{\dagger+}(y), \phi^-(x)] = i\Delta^-(x-y)$	$[\psi_\alpha^+(x), \bar{\psi}_\beta^-(y)]_+ = iS_{\alpha\beta}^+(x-y)$ $-\left[\bar{\psi}_\beta^+(y), \psi_\alpha^-(x)\right]_+ = iS_{\alpha\beta}^-(x-y)$	$[A^{\mu+}(x), A^{\nu-}(y)] = iD^{\mu\nu+}(x-y)$ $[A^{\nu+}(y), A^{\mu-}(x)] = iD^{\mu\nu-}(x-y)$
	$i\Delta^\pm = \frac{1}{2(2\pi)^3} \int \frac{e^{\mp ik(x-y)}}{\omega_k} d^3\mathbf{k}$	$iS^\pm = \frac{\pm 1}{2(2\pi)^3} \int \frac{(\not{\epsilon} \pm m) e^{\mp ip(x-y)}}{E_p} d^3\mathbf{p}$	$iD^{\mu\nu\pm} = -g^{\mu\nu} i\Delta^\pm$
	<p>$\Delta^+, S^+, D^{\mu\nu+}$ represent particles; $\Delta^-, S^-, D^{\mu\nu-}$ represent anti-particles. Symbols $S^\pm = S^\pm_{\alpha\beta}$</p> <p>Although fields such as ϕ are operators, because of their coefficient commutation relations, each integral above is a number, not an operator. The expectation value of a number X is simply the same number X. ($\langle 0 X 0\rangle = X\langle 0 0\rangle = X$. So, the Feynman propagator will also be simply a number (no brackets needed.)</p>		
Step 4 As contour integrals	Contour integral theory (integration in the complex plane) permits the above two integrals (for each spin type) over real 3-momentum space to be expressed as contour integrals.		
	$i\Delta^\pm = \frac{\mp i}{(2\pi)^4} \int_{C^\pm} \frac{e^{-ik(x-y)}}{k^2 - \mu^2} d^4k$	$iS^\pm = \frac{\mp i}{(2\pi)^4} \int_{C^\pm} \frac{(\not{\epsilon} + m) e^{-ip(x-y)}}{p^2 - m^2} d^4p$	$iD^{\mu\nu\pm} = \frac{\mp i g^{\mu\nu}}{(2\pi)^4} \int_{C^\pm} \frac{e^{-ik(x-y)}}{k^2 - \underbrace{\mu^2}_{\text{photon}=0}} d^4k$
Step 5 As one integral	Taking certain limits with contour integrals in the complex plane yields a single form for the Feynman propagator that works for any time ordering and will prove more convenient.		
in physical space	$\Delta_F(x-y) = \frac{1}{(2\pi)^4} \int \frac{e^{-ik(x-y)}}{k^2 - \mu^2 + i\epsilon} d^4k$	$S_{F\alpha\beta}(x-y) = \frac{1}{(2\pi)^4} \int \frac{(\not{\epsilon} + m) e^{-ip(x-y)}}{p^2 - m^2 + i\epsilon} d^4p$	$D_F^{\mu\nu}(x-y) = \frac{-g^{\mu\nu}}{(2\pi)^4} \int_{C^\pm} \frac{e^{-ik(x-y)}}{k^2 + i\epsilon} d^4k$
in momentum space	$\Delta_F(k) = \frac{1}{k^2 - \mu^2 + i\epsilon}$	$S_{F\alpha\beta}(p) = \frac{\not{\epsilon} + m}{p^2 - m^2 + i\epsilon}$	$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\epsilon}$