Student Friendly Quantum Field Theory Volume 2 – The Standard Model
Proofs of Gamma Matrices Trace Relations (5-87), Page 160
by Luc Longtin

From (5-49) and (5-51), we have: $\gamma^5\gamma^5 = I$ and $\gamma^5\gamma^\alpha = -\gamma^\alpha\gamma^5$

We also recall the cyclic property of traces, namely Tr (*AB*) = Tr (*BA*).

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu\right) = \mathrm{Tr}\,(\underbrace{\gamma^5\gamma^5}_{=I}\,\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu)$$ …then, using the cyclic property

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu\right) = \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu\gamma^5\right)$$

Using the anticommutation of γ^5 with any of the gamma matrices, we can then move, say, the γ^5 matrix at the right end, to the left, right before the γ^α matrix. For an initial *odd* number of gamma matrices, such a moving of the γ^5 will pick up an *odd* number of (-1) factors. Therefore:

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu\right) = -\mathrm{Tr}\,(\underbrace{\gamma^5\gamma^5}_{=I}\,\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu)$$ (odd number of initial γ's)

$$\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu\right) = -\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu\right)$$ (*odd* number of γ's)

So: $\boxed{\mathrm{Tr}\left(\gamma^\alpha\gamma^\beta \ldots \gamma^\mu\gamma^\nu\right) = 0}$ (*odd* number of γ's)

Now, using the cyclic property of traces, we have that:

$$\mathrm{Tr}\,\gamma^5 = \mathrm{Tr}\,(i\underbrace{\gamma^0\gamma^1\gamma^2}_{A}\underbrace{\gamma^3}_{B}) = \mathrm{Tr}\,(i\gamma^3\gamma^0\gamma^1\gamma^2)$$ …but $\gamma^\alpha\gamma^\beta = -\gamma^\beta\gamma^\alpha$ $(\alpha \neq \beta)$…

$$\mathrm{Tr}\,\gamma^5 = (-1)^3\mathrm{Tr}\,(i\gamma^0\gamma^1\gamma^2\gamma^3) = -\mathrm{Tr}\,\gamma^5$$ so $\boxed{\mathrm{Tr}\,\gamma^5 = 0}$

Next: $\mathrm{Tr}\,(\gamma^5\gamma^\alpha) = \mathrm{Tr}\,(\gamma^\alpha\gamma^5) = \mathrm{Tr}\,(-\gamma^5\gamma^\alpha)$ (using (5-51), $\gamma^5\gamma^\alpha = -\gamma^\alpha\gamma^5$)

$$\mathrm{Tr}\,(\gamma^5\gamma^\alpha) = -\mathrm{Tr}\,(\gamma^5\gamma^\alpha)$$ so $\boxed{\mathrm{Tr}\,(\gamma^5\gamma^\alpha) = 0}$

Or, using our first result above, since γ^5 consists of 4 gamma matrices, and so $\gamma^5\gamma^\alpha$ consists of 5 (an *odd* number of) gamma matrices, it follows immediately that $\mathrm{Tr}\,(\gamma^5\gamma^\alpha) = 0$.

Next: $\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right)$ Using the cyclic property of the trace, then anticommutation, we get…

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right) = \mathrm{Tr}\left(\gamma^\beta\gamma^5\gamma^\alpha\right) = -\mathrm{Tr}\left(\gamma^5\gamma^\beta\gamma^\alpha\right)$$ (moving γ^5 to left one place)

Therefore, the trace must be *totally antisymmetric* in the indices $\alpha\beta$.

We recall the gamma anticommutation relations: $\left[\gamma^\alpha, \gamma^\beta\right]_+ = 2g^{\alpha\beta}I$

Case $\alpha = \beta$: $\mathrm{Tr}\,(\gamma^5\gamma^\alpha\gamma^\alpha) = \mathrm{Tr}\,(\gamma^5 g^{\alpha\alpha} I) = g^{\alpha\alpha}\underbrace{\mathrm{Tr}\,(\gamma^5)}_{=0} = 0$ (no sum over α)

Case $\alpha \neq \beta$: $\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right)$ …using $\gamma^\mu\gamma^\mu = g^{\mu\mu}I = \pm I$ (no sum over μ)

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right) = \pm\mathrm{Tr}\left(\gamma^\mu\gamma^\mu\gamma^5\gamma^\alpha\gamma^\beta\right)$$ …choose $\mu \neq \alpha$ and $\mu \neq \beta$

Because $\mu \neq \alpha$ and $\mu \neq \beta$, γ^μ will anticommute with γ^5, γ^α, and γ^β, so we can write:

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right) = \pm(-1)^3\,\mathrm{Tr}\left(\gamma^\mu\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\right) = \mp\mathrm{Tr}\left(\gamma^\mu\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\right)$$

Then, using the cyclic property of the trace, we can write:

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right) = \mp\mathrm{Tr}\,(\underbrace{\gamma^\mu\gamma^\mu}_{=\pm 1}\gamma^5\gamma^\alpha\gamma^\beta) = -\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right)$$

So: $\boxed{\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right) = 0}$

Next: $\boxed{\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\right) = 0}$ (we have an *odd* number (4+3) of gamma matrices)

Next: $\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta[2g^{\mu\nu}I - \gamma^\nu\gamma^\mu]\right)$

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = 2g^{\mu\nu}\,\underbrace{\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\right)}_{=0} - \mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\mu\right)$$

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = -\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\nu\gamma^\mu\right)$$

We could thus switch any two adjacent gammas among $\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu$ and get a minus sign. Now if any two of these gammas were to be the same, we could switch gammas until we would get these same gammas next to each other. Then, their product would be $\pm I$, and we would end up with the trace of γ^5 multiplied with 2 other gammas, which we have shown above is zero.

Therefore, in order for the trace to be different from zero, $\gamma^\alpha, \gamma^\beta, \gamma^\mu, \gamma^\nu$ must all be different. Since $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ contains all the different gammas, and $\gamma^\alpha, \gamma^\beta, \gamma^\mu, \gamma^\nu$ are all different, it follows that $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ will each appear *twice* in the product of the eight gammas. We saw that switching any two adjacent gammas yields a minus sign. So, we can keep switching gammas until all the four different gammas stand together in pairs. Each such pair then equals $\pm I$.

So, the end result is: $\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = i\,\mathrm{Tr}\left(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = \pm i\,\mathrm{Tr}\,I = \pm i4$

Say the order of $\alpha\beta\mu\nu$ is the same as that of 0123. In that case, we get:

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = i\,\mathrm{Tr}\,(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3) = -i\,\mathrm{Tr}\,(\underbrace{\gamma^0\gamma^0}_{=I}\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3)$$

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = -i\,\mathrm{Tr}\,(\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3) = -i\,\mathrm{Tr}\,(\underbrace{\gamma^1\gamma^1}_{=-I}\gamma^2\gamma^3\gamma^2\gamma^3)$$

$$\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = i\,\mathrm{Tr}\,(\gamma^2\gamma^3\gamma^2\gamma^3) = -i\,\mathrm{Tr}\left(\underbrace{\gamma^2\gamma^2}_{=-I}\,\underbrace{\gamma^3\gamma^3}_{=-I}\right) = -i\,\mathrm{Tr}\,I = -i4$$

Now, if the order of $\alpha\beta\mu\nu$ is an *even* permutation of 0123, we will get the same result, $-i4$, as we would need an *even* number of switches to return the order to that of 0123, which implies an even number of factors of (-1). And, if the order of $\alpha\beta\mu\nu$ is an *odd* permutation of 0123, we will get the result $i4$. We can combine these results compactly in the following form:

So: $\boxed{\mathrm{Tr}\left(\gamma^5\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = -i4e^{\alpha\beta\mu\nu}}$ …where $e^{\alpha\beta\mu\nu}$ is the Levi-Civita tensor.