Symmetries Summary

Bob Klauber   March 2, 2022, revised & corrected Jan 13, 2023

Field	Symmetry in $\mathcal{L}$	4 Current	Conserved Charge	Conjugate Momentum	Physical Quantities Conserved
$\phi(x^\eta)$	$\phi \to \phi + \varepsilon$ (external)	$j^\alpha = \dfrac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} \underbrace{\dfrac{\partial \phi}{\partial \varepsilon}}_{1}$ (external)	$j^\alpha_{,\alpha} = 0 \to \int j^0 dV$ conserved 1 symmetry parameter ε $\to$ 1 conserved quantity	$\pi = \dfrac{\partial \mathcal{L}}{\partial \phi_{,0}} = j^0$ $\left(\sim \phi_{,0} = \dot\phi \text{ typically} \right)$ Note: π is only time derivative, none spatial	$p_\mu = \pi \dfrac{\partial \phi}{\partial x^\mu} =$ 4-momentum density (Vol. 1, pg. 23, (B2-2.3)) $p_0 = \int p_0 dV = \int \pi \dot\phi dV = \int \dot\phi \dot\phi dV = \int \dfrac{2\omega_\mathbf{k}\omega_\mathbf{k}}{\left(\sqrt{2\omega_\mathbf{k}V}\right)^2} dV = \omega_\mathbf{k} N_\mathbf{k}$ $p_i = \int p_i dV = \int \pi \phi_{,i} dV = \int \dot\phi \phi_{,i}\, dV = \int \dfrac{2k_i\omega_\mathbf{k}}{\left(\sqrt{2\omega_\mathbf{k}V}\right)^2} dV = k_i N_\mathbf{k}$ Since we have only proven $\int \pi dV$ is conserved, it may not be obvious that p_μ is. Also, physically, there are four conserved quantities p_μ, though Noether's theorem only predicts one.
	$\phi \to \phi e^{i\beta}$ (internal)	$j^\alpha = \dfrac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} \underbrace{\dfrac{\partial \phi}{\partial \beta}}_{i\phi}$ (internal)	$j^\alpha_{,\alpha} = 0 \to \int j^0 dV$ conserved 1 symmetry parameter β $\to$ 1 conserved quantity	N/A	$Q = q\int j^0 dV = q\int \dfrac{\partial \mathcal{L}}{\partial \dot\phi} i\phi dV = q\int \dfrac{2\omega_\mathbf{k}}{2\omega_\mathbf{k}V}(N_a - N_b)dV = q(N_a - N_b)$
$A^\rho(x^\eta)$	$A^\mu \to A^\mu + \varepsilon^\mu$ (external)	$j_\mu{}^\alpha = \dfrac{\partial \mathcal{L}}{\partial A^\rho{}_{,\alpha}} \underbrace{\dfrac{\partial A^\rho}{\partial \varepsilon^\mu}}_{\delta^\rho_\mu}$ $= \dfrac{\partial \mathcal{L}}{\partial A^\mu{}_{,\alpha}}$ (external)	$j_\mu{}^\alpha{}_{,\alpha} = 0 \to \int j_\mu{}^0 dV$ conserved for each μ 4 symmetry parameters ε^μ $\to$ 4 conserved quantities	$\pi_\mu = \dfrac{\partial \mathcal{L}}{\partial A^\mu{}_{,0}} = \dfrac{\partial \mathcal{L}}{\partial \dot A^\mu} = j_\mu{}^0$ $\left(\pi_\mu \sim A_{\mu,0} = \dot A_\mu \text{ typically} \right)$ Note: π_0 is conjug to time comp of A_μ; π_i is conjug to space comp. Derivative always wrt time, never space.	$p_\mu = \pi_\rho \dfrac{\partial A^\rho}{\partial x^\mu}$ Each component of the field makes a contribution to the physical 4- momentum density $p_0 = \int p_0 dV = \int \pi_\rho A^\rho{}_{,0} dV = \int \dot A_\rho \dot A^\rho dV = \dfrac{2\omega_\mathbf{k}^2}{2\omega_\mathbf{k}V} V N_\mathbf{k} = \omega_\mathbf{k} N_\mathbf{k}$ $p_i = \int p_i dV = \int \pi_\rho A^\rho{}_{,i} dV = \int \dot A_\rho A^\rho{}_{,i} dV = \dfrac{2\omega_\mathbf{k}k_i}{2\omega_\mathbf{k}V} V N_\mathbf{k} = k_i N_\mathbf{k}$ Note: transformation not on x^η (independent variables), but on A^μ (dependent variables = fields).
	$A^\mu \to A^\mu e^{i\beta}$ (internal)	$j^\alpha = \dfrac{\partial \mathcal{L}}{\partial A^\rho{}_{,\alpha}} \underbrace{\dfrac{\partial A^\rho}{\partial \beta}}_{iA^\rho}$ (internal)	$j^\alpha_{,\alpha} = 0 \to \int j^0 dV$ conserved 1 symmetry parameter β $\to$ 1 conserved quantity	N/A	$Q = q\int j^0 dV = q\int \dfrac{\partial \mathcal{L}}{\partial \dot A^\rho} iA^\rho dV = q\int \dot A_\rho iA^\rho dV$ $= q\int \dfrac{2\omega_\mathbf{k}}{2\omega_\mathbf{k}V} N_\mathbf{k} dV = qN_\mathbf{k}$

String Theory

Indices parallel Zwiebach pg. 158, except using ρ instead of a to label fields in $\mathcal{L}$, so not to confuse a with α (which labels independent variables $\varepsilon^\alpha = \tau, \sigma$). Note also, that Zwiebach uses μ for i, after introducing the meaning of the i index. Also see Notes on next page.

Field	Symmetry in $\mathcal{L}$	4 Current	Conserved Charge	Conjugate Momentum	Physical Quantities Conserved
$X^\rho(\xi^\alpha)$ $= X^\rho(\xi^1, \xi^2)$ (ξ spatial ↑) ($\alpha = \tau, \sigma$ ↓) $= X^\rho(\tau, \sigma)$ $= X^\rho(\xi^0, \xi^1)$ $\rho = 0,1,2,3$ $\alpha = 0,1$	4D translation $X^\mu \to X^\mu + \varepsilon^\mu$ $\mu = 0,1,2,3$ in Zwiebach, μ sometimes as i (external)	$j_\mu^{\ \alpha} = \left(j_\mu^\tau, j_\mu^\sigma\right)$ $= \left(j_\mu^0, j_\mu^1\right)$ $= \dfrac{\partial \mathcal{L}}{\partial X^\rho{}_{,\alpha}} \underbrace{\dfrac{\partial X^\rho}{\partial \varepsilon^\mu}}_{\delta_\mu^\rho}$ $= \dfrac{\partial \mathcal{L}}{\partial X^\mu{}_{,\alpha}} = \underbrace{\left(\mathcal{P}_\mu^\tau, \mathcal{P}_\mu^\sigma\right)}_{\text{Zwiebach notation}}$ $= \mathcal{P}_\mu^\alpha$	$j_\mu^{\ \alpha}{}_{,\alpha} = 0 \to \int j_\mu^0 d\sigma$ conserved for each μ 4 symmetry parameters ε^μ → 4 conserved quantities Aside 1st eq above → eq of motion $\underbrace{\dfrac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \dfrac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} = 0}_{\substack{\text{Zwiebach} \\ \text{notation}}}$	$\pi_\mu = \dfrac{\partial \mathcal{L}}{\partial \left(\dfrac{\partial X^\mu}{\partial \xi^0}\right)} = \dfrac{\partial \mathcal{L}}{\partial \left(\dfrac{\partial X^\mu}{\partial \tau}\right)}$ $= \dfrac{\partial \mathcal{L}}{\partial X^\mu{}_{,0}} = \dfrac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = j_\mu^\tau = j_\mu^0 = \mathcal{P}_\mu^\tau$ $\left(\pi_\mu \sim X_{\mu,0} = \dot{X}_\mu \text{ typically?}\right)$ Note: π_0 is conjugate to time comp X^0; $\pi_{1 \text{ or } 2 \text{ or } 3}$ is conjugate to space comp $X^{1 \text{ or } 2 \text{ or } 3}$. Derivative always wrt τ, not σ.	Note that x^η are the 4D coordinates in the frame of the observer and are independent (whereas X^ρ are 4D coordinates dependent on τ, σ). 4-momentum density $\left(\text{per unit } \sigma\right)$ $p_\mu = \pi_\rho \underbrace{\dfrac{\partial X^\rho}{\partial x^\mu}}_{\delta_\mu^\rho} = \pi_\mu = j_\mu^\tau = j_\mu^0 = \mathcal{P}_\mu^\tau(\tau,\sigma)$ total string 4-momentum $p_\mu = \int j_\mu^\tau d\sigma = \int j_\mu^0 d\sigma = \int \mathcal{P}_\mu^\tau(\tau,\sigma) d\sigma$ p_μ conserved (for each μ) via 1st eq, 4th column
As above	$X^i \to X^i e^{i\beta}$	Not in Zwiebach as of pg 177 (internal)			
As above	4D Lorentz $X'^\mu = \Lambda_\nu^\mu X^\nu$ $= \Lambda^{\mu\nu} X_\nu$ Infinitesimal $\varepsilon^{\mu\nu} << 1$ $X'^\mu = X^\mu + \varepsilon^{\mu\nu} X_\nu$ $\delta X'^\mu = \varepsilon^{\mu\nu} X_\nu$ ($\varepsilon^{\mu\nu}$ antisym with 3 indep boosts + 3 indep rotations → 6 indep variables) (external)	$j_{\mu\nu}^\alpha = \left(j_{\mu\nu}^\tau, j_{\mu\nu}^\sigma\right)$ $= \left(j_{\mu\nu}^0, j_{\mu\nu}^1\right)$ $= \dfrac{\partial \mathcal{L}}{\partial X^\rho{}_{,\alpha}} \dfrac{\partial X^\rho}{\partial \varepsilon^{\mu\nu}}$ $= j_\mu^\alpha \left(\dfrac{\partial \varepsilon^{\rho\xi}}{\partial \varepsilon^{\mu\nu}}\right) X_\xi$ $= j_\mu^\alpha \left(\begin{matrix} \delta_\mu^\rho \delta_\nu^\xi \\ -\delta_\mu^\xi \delta_\nu^\rho \end{matrix}\right) X_\xi$ $= j_\mu^\alpha X_\nu - j_\nu^\alpha X_\mu$ $= \underbrace{\mathcal{P}_\mu^\alpha X_\nu - \mathcal{P}_\nu^\alpha X_\mu}_{\substack{\text{Zwiebach} \\ \text{notation}}}$ $=$ Lorentz current	$j_{\mu\nu}^\alpha{}_{,\alpha} = 0 \to \int j_{\mu\nu}^0 d\sigma = M_{\mu\nu}$ = Lorentz charge Conserved for each indep set of μ and ν 6 symmetry parameters = six indep comps of $\varepsilon^{\mu\nu}$ → 6 conserved quantities $\mu = 0, \nu = 1,2,3$, 3 boost charges Other off diagonal terms, 3 rotation charges. Note the term "charge" is usually reserved for internal symmetries, but here used for Lorentzian.	N/A	For $j, k \neq 0$, $L_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$ = angular momentum, which is conserved because M_{jk} are. Boost M_{0j} is related to initial position, i.e., it is conserved during string motion.

Notes

$j_\mu{}^\alpha(\tau,\sigma) = \left(j_\mu^\tau(\tau,\sigma), j_\mu^\tau(\tau,\sigma)\right) = \left(j_\mu^0(\tau,\sigma), j_\mu^1(\tau,\sigma)\right) = \left(\mathcal{P}_\mu^\tau(\tau,\sigma), \mathcal{P}_\mu^\sigma(\tau,\sigma)\right)$ is a vector dependent on τ and σ, so, visually, we can imagine a field of vectors whose base points lie on the world sheet (because they are functions of world sheet coordinates τ and σ), though the tips of the vectors would, in general, be off of the world sheet. There would be two component vectors composing $j_\mu{}^\alpha(\tau,\sigma)$ at every point on the world sheet, $j_\mu^\tau(\tau,\sigma) = j_\mu^0(\tau,\sigma) = \mathcal{P}_\mu^\tau(\tau,\sigma)$ and $j_\mu^\sigma(\tau,\sigma) = j_\mu^1(\tau,\sigma) = \mathcal{P}_\mu^\sigma(\tau,\sigma)$. Each of those component vectors has 4 components μ as seen in the 4D spacetime Minkowski space, and they add vectorially to give $j_\mu{}^\alpha(\tau,\sigma)$, which itself has 4 components for $\mu = 0,1,2,3$.

We show the conserved quantity $\int p_\mu d\sigma = \int j_\mu^\tau d\sigma = \int j_\mu^0 d\sigma = \int \mathcal{P}_\mu^\tau d\sigma = p_\mu$ is actually physical 4-momentum using the general formula in the last column above that relates conjugate momentum to physical momentum. (See Klauber, Vol. 1, pg. 23, (B2-2.3).) Generally, $\pi_\mu \neq p_\mu$, but in this particular case, they are equal.

Zwiebach shows p_μ is physical 4-momentum in a different way, via 3 steps, as follows. 1) Finding conserved quantity p_μ (without knowing what it is physically) in the static gauge, 2) showing p_μ is the same in any gauge, and 3) showing p_μ is actually physical 4-momentum in the static gauge, so therefore p_μ is physical 4-momentum in any gauge (and is gauge invariant.)

Note that the physical 4-momentum density of the string equals $p_\mu = \pi_\mu = j_\mu^\tau = j_\mu^0 = \mathcal{P}_\mu^\tau$, and has nothing to do with $j_\mu^\sigma = j_\mu^1 = \mathcal{P}_\mu^\sigma$. Thus, the total 4-momentum p_μ has nothing to do with $j_\mu^\sigma = j_\mu^1 = \mathcal{P}_\mu^\sigma$.

We know that the 4-momentum density has form $p_\mu = \rho_m \dfrac{\partial X_\mu}{\partial \tau_{proper}}$, where ρ_m is the rest mass per unit σ length parameter. The derivative of any 4D position vector coordinates of a point with respect to proper time is tangent to the world line of the point. For a string, it would be tangent to the world sheet at each point on the world sheet. Thus, the $p_\mu \left(= \pi_\mu = j_\mu^\tau = j_\mu^0 = \mathcal{P}_\mu^\tau\right)$ vectors at each point are tangent to the world sheet. Unless the world sheet happens to be flat, they will point in directions off of the world sheet, though they are actually vectors confined to the world sheet. One can also show the $j_\mu^\sigma = j_\mu^1 = \mathcal{P}_\mu^\sigma$ vector is everywhere tangent to the world sheet, since it ends up having form X'_μ, where the prime indicates derivative with respect to σ. Such a derivative is tangent to constant τ line and that line lies in the world sheet.