Symmetries Summary

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Field	**Symmetry in $\mathcal{L}$**	**4 Current**	**Conserved Charge**	**Conjugate Momentum**	**Physical Quantities Conserved**
$\phi(x^\eta)$	$\phi \to \phi + \varepsilon$	$j^\alpha = \frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} \underbrace{\frac{\partial \phi}{\partial \varepsilon}}_{1}$	$j^\alpha{}_{,\alpha} = 0 \to \int j^0 dV$ conserved 1 symmetry parameter ε $\to$ 1 conserved quantity	$\pi = \frac{\partial \mathcal{L}}{\partial \phi_{,0}} = j^0$ $\left(\sim \phi_{,0} = \dot{\phi}\right.$ typically$\left.\right)$ Note: π is only time derivative, none spatial	$\mathscr{p}_\mu = \pi \frac{\partial \phi}{\partial x^\mu}$ = 4-momentum density (Vol. 1, pg. 23, (B2-2.3)) $p_0 = \int \mathscr{p}_0 dV = \int \pi \dot{\phi} dV = \int \dot{\phi}\dot{\phi} dV = \int \frac{2\omega_{\mathbf{k}}\omega_{\mathbf{k}}}{\left(\sqrt{2\omega_{\mathbf{k}} V}\right)^2} dV = \omega_{\mathbf{k}} N_{\mathbf{k}}$ $\mathscr{p}_i = \int \mathscr{p}_i dV = \int \pi \phi_{,i} dV = \int \dot{\phi}\phi_{,i} dV = \int \frac{2k_i\omega_{\mathbf{k}}}{\left(\sqrt{2\omega_{\mathbf{k}} V}\right)^2} dV = k_i N_{\mathbf{k}}$ Since we have only proven $\int \pi dV$ is conserved, it may not be obvious that p_μ is. Also, physically, there are four conserved quantities p_μ, though Noether's theorem only predicts one.
	$\phi \to \phi e^{i\beta}$	$j^\alpha = \frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} \underbrace{\frac{\partial \phi}{\partial \beta}}_{i\phi}$	$j^\alpha{}_{,\alpha} = 0 \to \int j^0 dV$ conserved 1 symmetry parameter β $\to$ 1 conserved quantity	N/A	$Q = q\int j^0 dV = q\int \frac{\partial \mathcal{L}}{\partial \dot{\phi}} i\phi dV = q\int \frac{2\omega_{\mathbf{k}}}{2\omega_{\mathbf{k}} V}\left(N_a - N_b\right) dV = q\left(N_a - N_b\right)$
$A^\rho(x^\eta)$	$A^\mu \to A^\mu + \varepsilon^\mu$	$j_\mu{}^\alpha = \frac{\partial \mathcal{L}}{\partial A^\rho{}_{,\alpha}} \underbrace{\frac{\partial A^\rho}{\partial \varepsilon^\mu}}_{\delta^\rho_\mu}$ $= \frac{\partial \mathcal{L}}{\partial A^\mu{}_{,\alpha}}$	$j_\mu{}^\alpha{}_{,\alpha} = 0 \to \int j_\mu{}^0 dV$ conserved for each μ 4 symmetry parameters ε^μ $\to$ 4 conserved quantities	$\pi_\mu = \frac{\partial \mathcal{L}}{\partial A^\mu{}_{,0}} = \frac{\partial \mathcal{L}}{\partial \dot{A}^\mu} = j_\mu{}^0$ $\left(\pi_\mu \sim A_{\mu,0} = \dot{A}_\mu\right.$ typically$\left.\right)$ Note: π_0 is conjug to time comp of A_μ; π_i is conjug to space comp. Derivative always wrt time, never space.	$\mathscr{p}_\mu = \pi_\rho \frac{\partial A^\rho}{\partial x^\mu}$ Each component of the field makes a contribution to the physical 4- momentum density $p_0 = \int \mathscr{p}_0 dV = \int \pi_\rho A^\rho{}_{,0} dV = \int \dot{A}_\rho \dot{A}^\rho dV = \frac{2\omega_{\mathbf{k}}^2}{2\omega_{\mathbf{k}} V} V N_{\mathbf{k}} = \omega_{\mathbf{k}} N_{\mathbf{k}}$ $p_i = \int \mathscr{p}_i dV = \int \pi_\rho A^\rho{}_{,i} dV = \int \dot{A}_\rho A^\rho{}_{,i} dV = \frac{2\omega_{\mathbf{k}} k_i}{2\omega_{\mathbf{k}} V} V N_{\mathbf{k}} = k_i N_{\mathbf{k}}$ Note: transformation not on x^η (independent variables), but on A^μ (dependent variables = fields).
	$A^\mu \to A^\mu e^{i\beta}$	$j^\alpha = \frac{\partial \mathcal{L}}{\partial A^\rho{}_{,\alpha}} \underbrace{\frac{\partial A^\rho}{\partial \beta}}_{iA^\rho}$	$j^\alpha{}_{,\alpha} = 0 \to \int j^0 dV$ conserved 1 symmetry parameter β $\to$ 1 conserved quantity	N/A	$Q = q\int j^0 dV = q\int \frac{\partial \mathcal{L}}{\partial \dot{A}^\rho} iA^\rho dV = q\int \dot{A}_\rho iA^\rho dV$ $= q\int \frac{2\omega_{\mathbf{k}}}{2\omega_{\mathbf{k}} V} N_{\mathbf{k}} dV = qN_{\mathbf{k}}$

String Theory

Indices parallel Zwiebach pg. 158, except using ρ instead of a to label fields in $\mathcal{L}$, so not to confuse a with α (which labels independent variables x^{α}). Note also, that Zwiebach uses μ for i, after introducing the meaning of the i index.

Field	**Symmetry in $\mathcal{L}$**	**4 Current**	**Conserved Charge**	**Conjugate Momentum**	**Physical Quantities Conserved**
$X^{\rho}(\xi^{\alpha})$ $= X^{\rho}(\xi^1, \xi^2)$ $= X^{\rho}(\tau,\sigma)$ $\rho = 0,1,2,3$ $\alpha = 1, 2$	$X^i \to X^i + \varepsilon^i$ $i = 0,1,2,3$ in Zwiebach $i \to \mu$ $X^{\mu} \to X^{\mu} + \varepsilon^{\mu}$ $\mu = 0,1,2,3$	$j_i^{\ \alpha} = j_{\mu}^{\ \alpha} = \left(j_{\mu}^0, j_{\mu}^1\right)$ $= \dfrac{\partial \mathcal{L}}{\partial X^{\rho}{}_{,\alpha}} \underbrace{\dfrac{\partial X^{\rho}}{\partial \varepsilon^{\mu}}}_{\delta_{\mu}^{\rho}}$ $= \dfrac{\partial \mathcal{L}}{\partial X^{\mu}{}_{,\alpha}} = \underbrace{\left(\mathcal{P}_{\mu}^{\tau}, \mathcal{P}_{\mu}^{\sigma}\right)}_{\text{Zwiebach notation}}$	$j_i^{\ \alpha}{}_{,\alpha} = j_{\mu}^{\ \alpha}{}_{,\alpha} = 0 \to \int j_{\mu}^{\ 0} dV$ conserved for each μ $(=i)$ 4 symmetry parameters ε^{μ} $\to$ 4 conserved quantities	$\pi_{\mu} = \dfrac{\partial \mathcal{L}}{\partial \left(\dfrac{\partial X^{\mu}}{\partial \xi^0}\right)} = \dfrac{\partial \mathcal{L}}{\partial \left(\dfrac{\partial X^{\mu}}{\partial \tau}\right)}$ $= \dfrac{\partial \mathcal{L}}{\partial X^{\mu}{}_{,0}} = \dfrac{\partial \mathcal{L}}{\partial \dot{X}^{\mu}} = j_{\mu}^{\ 0}$ $\left(\pi_{\mu} \sim X_{\mu,0} = \dot{X}_{\mu} \text{ typically?}\right)$ Note: π_0 is conjugate to time comp X^0; $\pi_{1 \text{ or } 2 \text{ or } 3}$ is conjugate to space comp $X^{1 \text{ or } 2 \text{ or } 3}$. Derivative always wrt τ, not σ.	Note that x^{η} are the 4D coordinates in the frame of the observer and are independent (whereas X^{ρ} are 4D coordinates dependent on τ, σ). $\mathcal{p}_{\mu} = \pi_{\rho} \underbrace{\dfrac{\partial X^{\rho}}{\partial x^{\mu}}}_{\delta_{\mu}^{\rho}} = \pi_{\mu} = j_{\mu}^0 = \mathcal{P}_{\mu}^{\tau}(\tau,\sigma)$ = 4-momentum density wrt σ $p_{\mu} = \int j_{\mu}^0 d\sigma = \int \mathcal{P}_{\mu}^{\tau}(\tau,\sigma) d\sigma$ = total string 4-momentum p_{μ} conserved (for each μ) via 4th column
	$X^i \to X^i e^{i\beta}$	Not in Zwiebach as of pg 177			

Notes

$j_i^{\ \alpha}(\tau,\sigma) = j_{\mu}^{\ \alpha}(\tau,\sigma) = \left(j_{\mu}^0(\tau,\sigma), j_{\mu}^1(\tau,\sigma)\right) = \left(\mathcal{P}_{\mu}^{\tau}(\tau,\sigma), \mathcal{P}_{\mu}^{\sigma}(\tau,\sigma)\right)$ is a vector dependent on τ and σ, so, visually, we can imagine a field of vectors whose base points lie on the world sheet (because they are functions of world sheet coordinates τ and σ), though the tips of the vectors would, in general, be off of the world sheet. There would be two such vectors at every point on the world sheet, $j_{\mu}^0(\tau,\sigma) = \mathcal{P}_{\mu}^{\tau}(\tau,\sigma)$ and $j_{\mu}^1(\tau,\sigma) = \mathcal{P}_{\mu}^{\sigma}(\tau,\sigma)$. Each of those vectors has 4 components μ as seen in the 4D spacetime Minkowski space.

Note that the physical 4-momentum density of the string equals $\pi_{\mu} = \mathcal{p}_{\mu} = j_{\mu}^0 = \mathcal{P}_{\mu}^{\sigma}$, and has nothing to do with $j_{\mu}^1 = \mathcal{P}_{\mu}^{\sigma}$. Thus, the total 4-momentum p_{μ} has nothing to do with $j_{\mu}^1 = \mathcal{P}_{\mu}^{\sigma}$. Generally, $\pi_{\mu} \neq \mathcal{p}_{\mu}$, but in this particular case, they are equal.

We know that the 4-momentum density has form $\mathcal{p}_{\mu} = \rho_m \dfrac{\partial X_{\mu}}{\partial \tau_{proper}}$, where ρ_m is the rest mass per unit σ length parameter. The derivative of any 4D position vector coordinates of a point with respect to proper time is tangent to the world line of the point. For a string, it would be tangent to the world sheet at each point on the world sheet. Thus, the $\mathcal{p}_{\mu}$ vectors at each point are tangent to the world sheet. Unless the world sheet happens to be flat, they will point in directions off of the world sheet, though they are actually vectors confined to the world sheet. One can also show the $j_{\mu}^1 = \mathcal{P}_{\mu}^{\sigma}$ vector is everywhere tangent to the world sheet, since it ends up having form X'_{μ}, where the prime indicates derivative with respect to σ. Such a derivative is tangent to constant τ line and that line lies in the world sheet.

We show the conserved quantity $\int j_{\mu}^{\ 0} dV = p_{\mu}$ is actually physical 4-momentum using the general formula in the last column above that relates conjugate momentum to physical momentum. (See Klauber, Vol. 1, pg. 23, (B2-2.3).) Zwiebach shows it in a different way, via 3 steps: 1) finding conserved quantity p_{μ} (without knowing what it is physically) in the static gauge, 2) showing p_{μ} is the same in any gauge, and 3) showing p_{μ} is actually physical 4-momentum in the static gauge, so therefore p_{μ} is physical 4-momentum in any gauge (and is gauge invariant.)