Strong Force Gauge Invariance

Bob Klauber March 2, 2015 © 2015

Mandl and Shaw (and other authors) state, but do not actually show, that the quark Lagrangian is gauge invariant. In this document, it is shown explicitly, step-by-step.

Note: This is not a complete document, in the sense that it does not teach every developmental step of all material contained herein. It is intended as an auxiliary aid to other texts, particularly Mandl & Shaw. I may produce a more complete rendition of this material at some point in the future.

Color Operators (Matrices in SU(3))

The color matrices $\hat{F}_i$ are defined in terms of a specific (what turns out later on to be useful) set of linearly independent Hermitian matrices λ_i in 3 dimensions (color = dimension here, i = 1,2,… 8 are color indices) as

$$\hat{F}_i = \frac{\lambda_i}{2} \qquad (\lambda_i \text{ defined in Mandl and Shaw, (11.11b) pg. 223}). \tag{1}$$

They have the following commutation relations.

$$\left[\hat{F}_i, \hat{F}_j\right] = if_{ijk}\hat{F}_k \quad \rightarrow \quad \left[\lambda_i, \lambda_j\right] = 2if_{ijk}\lambda_k \tag{2}$$

where f_{ijk} are called the structure constants of the commutation relations. For the particular λ_i employed herein,

$$f_{ijk} = -f_{jik} = f_{jki} = -f_{kji} = f_{kij} = f_{ikj} \ \left(\text{totally antisymmetric}\right). \tag{3}$$

Free Gluon Lagrangian

The Lagrangian for free gluons $A_i^\mu(x)$,where i = 1,2,… 8 are the same color indices corresponding to the 8 color operators of SU(3) for QCD above, is

$$\mathcal{L}_G = -\tfrac{1}{4} G_{i\mu\nu} G_i^{\mu\nu} \tag{4}$$

where

$$G_i^{\mu\nu} = F_i^{\mu\nu} + g_s f_{ijk} A_j^\mu(x) A_k^\nu(x) \tag{5}$$

$$F_i^{\mu\nu} = \partial^\nu A_i^\mu - \partial^\mu A_i^\nu \tag{6}$$

Thus,

$$\mathcal{L}_G = -\tfrac{1}{4} F_{i\mu\nu} F_i^{\mu\nu} + g_s f_{ijk} A_{i\mu}(x) A_{j\nu}(x) A_k^\nu(x) - \tfrac{1}{4} g_s^2 f_{ijk} f_{ilm} A_j^\mu(x) A_{l\mu}(x) A_{m\nu}(x) \tag{7}$$

Using (6) in (7), we have

$$\begin{aligned}
\mathcal{L}_G &= -\tfrac{1}{4}\left(\partial_\nu A_\mu - \partial_\mu A_\nu\right)\left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) + g_s f_{ijk} A_{i\mu}(x) A_{j\nu}(x) A_k^\nu(x) - \tfrac{1}{4} g_s^2 f_{ijk} f_{ilm} A_j^\mu(x) A_{l\mu}(x) A_{m\nu}(x) \\
&= -\tfrac{1}{4}\Bigg(\underbrace{\left(\partial_\nu A_\mu\right)\left(\partial^\nu A^\mu\right) + \left(-\partial_\mu A_\nu\right)\left(-\partial^\mu A^\nu\right)}_{2\left(\partial_\nu A_\mu\right)\left(\partial^\nu A^\mu\right)} + \underbrace{\left(-\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right) + \left(\partial_\nu A_\mu\right)\left(-\partial^\mu A^\nu\right)}_{-2\left(\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right)} \Bigg) \\
&\qquad + g_s f_{ijk} A_{i\mu}(x) A_{j\nu}(x) A_k^\nu(x) - \tfrac{1}{4} g_s^2 f_{ijk} f_{ilm} A_j^\mu(x) A_{l\mu}(x) A_{m\nu}(x) \\
&= -\tfrac{1}{2}\left(\partial_\nu A_\mu\right)\left(\partial^\nu A^\mu\right) - \tfrac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\nu A^\mu\right) + g_s f_{ijk} A_{i\mu}(x) A_{j\nu}(x) A_k^\nu(x) - \tfrac{1}{4} g_s^2 f_{ijk} f_{ilm} A_j^\mu(x) A_{l\mu}(x) A_{m\nu}(x)
\end{aligned} \tag{8}$$

The Gauge Transformations

Generalizing from QED, and being prescient in choosing the right transformation ahead of time, we presume (to be proven) the full Lagrangian will be invariant under transformations of

<u>Finite transformation form</u>

$$\begin{aligned}\Psi^f(x) &\to \Psi'^f(x) = e^{ig_s\hat{F}_j\omega_j(x)}\Psi^f(x) = e^{ig_s\lambda_j\omega_j(x)/2}\Psi^f(x)\\ \bar{\Psi}^f(x) &\to \bar{\Psi}'^f(x) = e^{-ig_s\lambda_j\omega_j(x)/2}\bar{\Psi}^f(x)\end{aligned} \tag{9}$$

$$A_i^\mu(x) \to A_i'^\mu(x) = \text{ complicated (see infinitesimal transformation)} \tag{10}$$

As noted in (10), the finite transformation is very complicated. It is easier to deal with infinitesimal transformation variations, i.e., very small ω_i,. If the infinitesimal variation is invariant, then if we were to integrate that to get a finite transformation, the finite transformation would be invariant. So, in that limit, doing a Taylor expansion of the exponential factor, (9) reduces to (11) and (10) becomes (take my word for it) (12).

<u>Infinitesimal transformation form (small ω_i)</u>

$$\begin{aligned}\Psi^f(x) &\to \Psi'^f(x) = \left(1+\tfrac{i}{2}g_s\lambda_j\omega_j(x)\right)\Psi^f(x)\\ \bar{\Psi}^f(x) &\to \bar{\Psi}'^f(x) = \bar{\Psi}^f(x)\left(1-\tfrac{i}{2}g_s\lambda_j\omega_j(x)\right)\end{aligned} \tag{11}$$

$$A_i^\mu(x) \to A_i'^\mu(x) = A_i^\mu(x) - \partial^\mu\omega_i(x) - g_s f_{ijk}\omega_j(x)A_k^\mu(x) \tag{12}$$

BELOW NOT NEEDED IN DERIVATION TO FOLLOW:

Definitions M&S and Others Use to Simplify (Presumably, but probably not)[1] Things

The 3X3 matrices of M&S (11.51) pg. 236,

$$\omega(x) = \lambda_i\omega_i(x) \qquad A^\mu(x) = \lambda_i A_i^\mu(x) \tag{13}$$

with summation implied by repeated indices. Thus, from (11.11) and (11.12) of M&S,

$$\begin{aligned}\left[\omega, A^\mu\right] &= \left[\lambda_j\omega_j, \lambda_k A_k^\mu\right] = \left[\lambda_j, \lambda_k\right]\omega_j A_k^\mu = i\underbrace{\left(\lambda_j\lambda_k - \lambda_k\lambda_j\right)}_{\text{use M\&S (11.11a)}\to}\omega_j A_k^\mu = i\left(2\hat{F}_j 2\hat{F}_k - 2\hat{F}_k 2\hat{F}_j\right)\omega_j A_k^\mu\\ &= i4\left(\hat{F}_j\hat{F}_k - \hat{F}_k\hat{F}_j\right)\omega_j A_k^\mu = i4\left[\hat{F}_j\hat{F}_k\right]\omega_j A_k^\mu = i4\underbrace{\left[\hat{F}_j\hat{F}_k\right]}_{\substack{\text{use M\&S}\\ \text{(11.12a)}\to}}\omega_j A_k^\mu = i4 f_{ijk}\underbrace{\hat{F}_i}_{\substack{\text{M\&S}\\ \text{(11.11a)}}}\omega_j A_k^\mu = i4 f_{ijk}\frac{\lambda_i}{2}\omega_j A_k^\mu\\ &= i2 f_{ijk}\lambda_i\omega_j A_k^\mu\ .\end{aligned} \tag{14}$$

BELOW NOT NEEDED IN DERIVATION TO FOLLOW:

Relation of A^μ transformation to commutation relations

Start with the relation below and show it equals (12). Use (1) in last step of (15).

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) - \partial^\mu\omega(x) - \tfrac{1}{2}g_s\underbrace{\left[\omega, A^\mu\right]}_{\left[\lambda_j\omega_j(x), \lambda_k A_k^\mu(x)\right]} \overbrace{= A^\mu - \partial^\mu\omega - \tfrac{1}{2}g_s\omega_j A_k^\mu 4\underbrace{\left[\hat{F}_j, \hat{F}_k\right]}_{\substack{\text{in QED } =0\\ \text{in QCD } \neq 0}}}^{\text{Drop } x \text{ dependency for simplicity}} \tag{15}$$

Writing out with $A^\mu = \lambda_i A_i^\mu$ and using (2), we have

[1] It simplifies the notation, but makes learning what is going on harder (i.e., less simple.)

$$\begin{aligned}\lambda_i A_i^\mu \to \lambda_i A_i'^\mu &= \lambda_i A_i^\mu - \lambda_i \partial^\mu \omega_i + 2 g_s \lambda_i \omega_j A_k^\mu f_{ijk} \hat{F}_i \\ &= \lambda_i A_i^\mu - \lambda_i \partial^\mu \omega_i - g_s \lambda_i f_{ijk} \omega_j A_k^\mu\end{aligned} \tag{16}$$

Or without the λ_i,

$$A_i^\mu \to A_i^\mu - \partial^\mu \omega_i - g f_{ijk} \omega_j A_k^\mu \,, \tag{17}$$

which is (12). Therefore the LHS of (15) and (12) are the same thing

Covariant Derivative

The covariant derivative of M&S (11.27), pg. 227, (deduced by analogy with QED) is

$$D^\mu \Psi^f(x) = \left(\partial^\mu + i g_s \hat{F}_j A_j^\mu(x)\right)\Psi^f(x) = \left(\partial^\mu + i g_s \frac{\lambda_j}{2} A_j^\mu(x)\right)\Psi^f(x) \tag{18}$$

(18) can be re-written, in the (unneeded here) notation of (13), as

$$D^\mu \Psi^f = \left(\partial^\mu + i g_s \frac{A^\mu}{2}\right)\Psi^f \,. \tag{19}$$

The Full QCD Lagrangian

The full Lagrangian for color (for a single flavor of quark) is (using (18) in the second line below)

$$\begin{aligned}\mathcal{L}\text{ (here)} &= \mathcal{L}_{QCD} = \bar{\Psi}\left(i\not{D} - m\right)\Psi - \tfrac{1}{4} G_{i\,\mu\nu} G_i^{\mu\nu} = i\bar{\Psi}\not{D}\Psi - \bar{\Psi} m \Psi - \tfrac{1}{4} G_{i\,\mu\nu} G_i^{\mu\nu} \\ &= \underbrace{i\bar{\Psi}\gamma_\mu \underbrace{\left(\partial^\mu + i g_s \frac{\lambda_i A_i^\mu}{2}\right)}_{D^\mu}\Psi - \bar{\Psi} m \Psi}_{\mathcal{L}_{quark}} - \tfrac{1}{4} G_{i\,\mu\nu} G_i^{\mu\nu} \\ &= i\bar{\Psi}\not{D}\Psi - \bar{\Psi} m \Psi - \tfrac{1}{4} G_{i\,\mu\nu} G_i^{\mu\nu} = i\bar{\Psi}\not{\partial}\Psi - \frac{g_s}{2}\bar{\Psi}\not{A}\Psi - \bar{\Psi} m \Psi - \tfrac{1}{4} G_{i\,\mu\nu} G_i^{\mu\nu}.\end{aligned} \tag{20}$$

Gauge Invariance of the QCD Lagrangian

As promised, (20) should be invariant under (11) and (12). Mandl and Shaw (Sect. 11.4.2, pgs. 237-238) show that the $-\tfrac{1}{4} G_{i\,\mu\nu} G_i^{\mu\nu}$ term is invariant under the given infinitesimal transformation (just (12) needed for that).

So, showing gauge invariance of $\mathcal{L}$ reduces to showing the other three terms in the last or middle line of (20) are invariant. (Note M&S do not show this explicitly.) Using the middle line of (20), we have

$$\begin{aligned}\mathcal{L}_{quark} &= i\bar{\Psi}\gamma_\mu \partial^\mu \Psi - \bar{\Psi} g_s \gamma_\mu \frac{\lambda_i A_i^\mu}{2}\Psi - \bar{\Psi} m \Psi \\ \xrightarrow[A_i^\mu \to A_i^\mu - \partial^\mu \omega_i - g_s f_{ijk}\omega_j A_k^\mu]{\Psi^f \to \left(1 + \frac{i}{2} g_s \lambda_j \omega_j\right)\Psi^f} \quad \mathcal{L}'_{quark} &= i\bar{\Psi}\left(1 - \tfrac{i}{2} g_s \lambda_j \omega_j\right)\gamma_\mu \partial^\mu \left(1 + \tfrac{i}{2} g_s \lambda_k \omega_k\right)\Psi \\ &\quad - \bar{\Psi}\left(1 - \tfrac{i}{2} g_s \lambda_j \omega_j\right) g_s \gamma_\mu \frac{\lambda_i}{2}\left(A_i^\mu - \partial^\mu \omega_i - g_s f_{ijk}\omega_j A_k^\mu\right)\left(1 + \tfrac{i}{2} g_s \lambda_k \omega_k\right)\Psi \\ &\quad - \bar{\Psi}\left(1 - \tfrac{i}{2} g_s \lambda_j \omega_j\right) m \left(1 + \tfrac{1}{2} g_s \lambda_k \omega_k\right)\Psi\end{aligned} \tag{21}$$

$$
\left.
\begin{aligned}
&= i\bar{\Psi}\gamma_\mu\partial^\mu\Psi \quad \underbrace{+ \quad i\bar{\Psi}\gamma_\mu\,\partial^\mu\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\underbrace{i\bar{\Psi}\gamma_\mu\left(\frac{i}{2}g_s\lambda_j\partial^\mu\omega_j\right)\Psi}_{\text{Call this X}}\underbrace{+i\bar{\Psi}\gamma_\mu\left(\frac{i}{2}g_s\lambda_j\omega_j\right)\partial^\mu\Psi}_{\text{Call this Y}}} \quad \underbrace{+\,i\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)\gamma_\mu\partial^\mu\Psi}_{-Y} \\
&\qquad\qquad + \quad \underbrace{i\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_i\omega_i\right)\gamma_\mu\partial^\mu\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\text{order }\omega_i^2\approx 0}
\end{aligned}
\right\}\text{1st line of prior eq}
$$

$$
\left.
\begin{aligned}
&-\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(A_i^\mu\right)\Psi-\underbrace{\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(-\partial^\mu\omega_i\right)\Psi}_{-X}-\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(-g_s f_{ijk}\omega_j A_k^\mu\right)\Psi \\
&-\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)g_s\gamma_\mu\frac{\lambda_i}{2}\left(A_i^\mu\right)\Psi-\underbrace{\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)g_s\gamma_\mu\frac{\lambda_i}{2}\left(-\partial^\mu\omega_i\right)\Psi}_{\text{order }\omega_i^2\approx 0} \\
&\qquad\qquad -\underbrace{\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)g_s\gamma_\mu\frac{\lambda_i}{2}\left(-g_s f_{ijk}\omega_j A_k^\mu\right)\Psi}_{\text{order }\omega_i^2\approx 0} \\
&-\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(A_i^\mu\right)\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi-\underbrace{\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(-\partial^\mu\omega_i\right)\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\text{order }\omega_i^2\approx 0} \\
&\qquad\qquad -\underbrace{\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(-g_s f_{ijk}\omega_j A_k^\mu\right)\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\text{order }\omega_i^2\approx 0} \\
&\underbrace{-\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)g_s\gamma_\mu\frac{\lambda_i}{2}\left(A_i^\mu\right)\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\text{order }\omega_i^3\approx 0} \\
&\qquad\qquad -\underbrace{\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)g_s\gamma_\mu\frac{\lambda_i}{2}\left(-\partial^\mu\omega_i\right)\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\text{order }\omega_i^3\approx 0} \\
&\qquad\qquad\qquad -\underbrace{\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)g_s\gamma_\mu\frac{\lambda_i}{2}\left(-g_s f_{ijk}\omega_j A_k^\mu\right)\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\text{order }\omega_i^3\approx 0}
\end{aligned}
\right\}\text{2nd line}
\tag{22}
$$

$$
\left.
+i\bar{\Psi}m\Psi+\underbrace{i\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)m\Psi}_{\text{cancels with next term}}+\underbrace{i\bar{\Psi}m\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\text{cancels with prior term}}+\underbrace{i\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_i\omega_i\right)m\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{\text{order }\omega_i^2\approx 0}
\right\}\text{3rd line}
$$

$$
\begin{aligned}
&= i\bar{\Psi}\gamma_\mu\partial^\mu\Psi-\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(A_i^\mu\right)\Psi-\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(-g_s f_{ijk}\omega_j A_k^\mu\right)\Psi \\
&\qquad\qquad \underbrace{-\bar{\Psi}\left(-\tfrac{i}{2}g_s\lambda_j\omega_j\right)g_s\gamma_\mu\frac{\lambda_i}{2}\left(A_i^\mu\right)\Psi-\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(A_i^\mu\right)\left(\tfrac{i}{2}g_s\lambda_j\omega_j\right)\Psi}_{-\frac{i}{4}\bar{\Psi}g_s^2\gamma_\mu A_i^\mu\omega_j\left(\lambda_i\lambda_j-\lambda_j\lambda_i\right)\Psi=-\frac{i}{4}\bar{\Psi}g_s^2\gamma_\mu A_i^\mu\omega_j\left[\lambda_i\lambda_j\right]\Psi}+i\bar{\Psi}m\Psi
\end{aligned}
\tag{23}
$$

$$= i\bar{\Psi}\gamma_\mu\partial^\mu\Psi - \bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(A_i^\mu\right)\Psi + i\bar{\Psi}m \underbrace{\underbrace{-\bar{\Psi}g_s\gamma_\mu\frac{\lambda_i}{2}\left(-g_s f_{ijk}\omega_j A_k^\mu\right)\Psi}_{\frac{1}{2}\bar{\Psi}g_s^2\gamma_\mu f_{ijk}\lambda_i\omega_j A_k^\mu\Psi} \underbrace{-\tfrac{i}{4}\bar{\Psi}g_s^2\gamma_\mu A_i^\mu\omega_j\underbrace{\left[\lambda_i\lambda_j\right]}_{i2 f_{ijk}\lambda_k}\Psi}_{\frac{1}{2}\bar{\Psi}g_s^2\gamma_\mu f_{ijk}A_i^\mu\omega_j\lambda_k\Psi}}_{\frac{1}{2}\bar{\Psi}g_s^2\gamma_\mu f_{kji}\lambda_k\omega_j A_i^\mu\Psi \;\; \frac{1}{2}\bar{\Psi}g_s^2\gamma_\mu\left(-f_{ikj}\right)\lambda_k\omega_j A_i^\mu\Psi}_{\text{These cancel}} \tag{24}$$

Or,

$$\mathcal{L}'_{quark} \approx i\bar{\Psi}\underbrace{\gamma_\mu\left(\partial^\mu + ig_s\frac{\lambda_i A_i^\mu}{2}\right)}_{\not{D}}\Psi + i\bar{\Psi}m\Psi = \mathcal{L}_{quark} \tag{25}$$

And the quark part of the color Lagrangian is symmetric under the coupled transformation (11) & (12).

Note that the ∂^μ in the transformation (12) acts only on $\omega(x)$, and not on the Ψ that follows. (See term labeled $-X$ above.)

Transformation of Covariant Derivative

As an exercise, one can show the covariant derivative transforms like the spinor field itself. That is,

$$\begin{aligned} D^\mu\Psi = \left(\partial^\mu + ig_s\frac{\lambda_j}{2}A_j^\mu\right)\Psi &\xrightarrow[A_i^\mu \to A_i^\mu - \partial^\mu\omega_i - g_s f_{ijk}\omega_j A_k^\mu]{\Psi \to \left(1+\frac{i}{2}g_s\lambda_j\omega_j\right)\Psi} \left(1+\tfrac{i}{2}g_s\lambda_j\omega_j\right)\left(\partial^\mu + ig_s\frac{\lambda_j}{2}A_j^\mu\right)\Psi \\ &= \left(1+\tfrac{i}{2}g_s\lambda_j\omega_j\right)D^\mu\Psi \end{aligned} \tag{26}$$

Note that Mandl and Shaw (Sect. 11.4.1, pg. 236) assume (26) holds true and use it to determine the transformation on the gluon fields (12). This seems decidedly more cumbersome and harder to understand that assuming (12) and determining (26).

By assuming (12) (along with (11)) first and showing the QCD Lagrangian is invariant under that assumption, we have justification for (12). There is little justification for starting by assuming (26). There is the justification of symmetry by starting with (12).