How Strings Give Rise to Fields Like Maxwell Fields

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1 Quantization: The Big Picture

Recall from QFT that when we quantize a classical field, the classical field is typically a displacement vector. Think of an elastic solid or a fluid continuum, where the classical field is comprised of the vector displacement in space of the continuum at every point, and that displacement is generally a function of time. We can symbolize the displacement by X^μ, so

$$X^\mu = X^\mu(t,\mathbf{x}) = X^\mu(x^\alpha), \quad \text{a classical displacement field.} \tag{1}$$

Upon quantization, i.e., upon invoking the canonical commutation relations, the classical field becomes a quantum field, which is not a displacement typically, but an operator (that creates and destroys states). We can symbolize the quantum field by A^μ, so

$$X^\mu = X^\mu(x^\alpha) \xrightarrow{\text{quantization}} A^\mu = A^\mu(x^\alpha), \text{ a quantum creation/destruction operator field.} \tag{2}$$

2 QFT

2.1 Maxwell Fields in QFT

In 4D, if A^μ has two independent components (two transverse fields in 3D space) plus is massless, it can be a Maxwell field, i.e., a photon field. That is, for a Maxwell field, the number of independent components in 4D is 4–2=2, or $D-2$.

We can generalize to higher dimensions D, where d is the number of spatial dimensions.

$$\text{independent components } = D-2 = d-1 \text{ necessary for a Maxwell field.} \tag{3}$$

Thus, it is necessary that a Maxwell field must be i) massless plus have ii) $d-1$ independent components, but that is not enough (not sufficient). A candidate field must also satisfy Maxwell's equation.

Bottom line #1: A Maxwell field i) is massless, ii) has $d-1$ independent components, iii) has D total components, and iv) satisfies Maxwell's equation. These are necessary and sufficient conditions for (2) to be a Maxwell field.

2.2 Massive Fields in QFT

In 4D, massive vector fields in QFT, such as the Ws and Z of electroweak theory, have three independent components, not two. (See Klauber, *Student Friendly QFT, Vol. 2, The Standard Model*, Sect. 5.4, pgs. 154-157.) That is, massive 4D boson vector fields have 4–1= $D-1 = d = 3$ independent components.

We can generalize to higher D, again.

$$\text{independent components } = D-1 = d \text{ necessary for a massive boson field.} \tag{4}$$

Bottom line #2: A massive vector field has i) mass, ii) has d independent components, iii) has D total components, and iv) satisfies the Proca equation [(5-80), pg. 157 in above reference.] These are necessary and sufficient conditions for (2) to be a massive vector field.

2.3 Scalar Fields in QFT

In 4D, a scalar field has a single component. It is not a vector with D components of any mix of independent and dependent components.

Generalizing to any D, we have

$$\text{independent components } = 1 \text{ necessary for a scalar field for any } D. \tag{5}$$

Bottom line #3: A scalar field, whether massive or massless, has i) one independent component, and ii) satisfies the Klein-Gordon equation. These are necessary and sufficient conditions for (2) to be a scalar field (for no component index μ) or a collection of scalar fields where each value of μ represents a different independent scalar field (not a component of a vector field).

3 String Theory in Light-Cone Gauge and Maxwell Equation

3.1 Classical Relativistic String Field Equation of Motion in Light-Cone Gauge

In classical relativistic string theory, we start with a one-dimensional relativistic string existing in D dimensions and having $D = d + 1$ displacement components X^{μ}, as in (1), where $\mu = 0,1,\ldots,d$. Upon minimizing the world sheet area (the action), we end up with the classical equation of motion for the string, which in the chosen parametrization and gauge (light-cone), has the form, in natural units, of

$$\ddot{X}^{\mu} - X^{\mu''} = \left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2}\right)X^{\mu} = 0 \quad \rightarrow \quad \partial_{\alpha}\partial^{\alpha}X^{\mu} = 0 \quad \alpha = \tau,\sigma \qquad \text{Zwiebach (9.39) [183].} \quad (6)$$

The solution of (6) in the light-cone gauge for open strings in terms of world-sheet parameters τ and σ is

$$X^{\mu} = x_0^{\mu} + \underbrace{\sqrt{2\alpha'}\alpha_0^{\mu}}_{2\alpha' p^{\mu}}\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{\mu}e^{-in\tau}\cos n\sigma \qquad \text{Zwiebach (9.56) [186], } p^{\mu} \text{ from (9.52) [185],} \quad (7)$$

where

$$X^{\mu} = X^{+}, X^{-}, X^{I} \quad \text{all functions of } \tau \text{ and } \sigma, \text{ i.e.,} \quad X^{\mu} = X^{\mu}(\tau,\sigma). \qquad (8)$$

In the light-cone gauge, X^+ is determined by p^+, X^- is determined by all of the X^I (or more commonly expressed in terms of the coefficients α_n^I in the X^I solution of (7)). So, the independent parameters are τ, p^+, and α_n^I, but since the α_n^I dependence can be expressed in terms of the transverse 3-momenta $\vec{p}_T$, the independent parameters can be considered τ, p^+, and $\vec{p}_T$.

3.2 Quantum String Field Equation of Motion

In Sect. 12.7 [268-270], Zwiebach shows the quantum string equation of motion, in the light-cone gauge, parallels the form of the Schrödinger equation. For $N^{\perp} = 1$ (a single string in the lowest mode), this is

$$i\frac{\partial X^J}{\partial\tau} = \alpha' p^I p^I X^J \qquad I,J = \text{transverse modes} \quad (\psi \text{ in Zwiebach} = X \text{ here}). \quad \text{Zwiebach (12.190) [269]} \quad (9)$$

3.3 Classical Maxwell Field Equation of Motion

The Maxwell equation, in its general form, is

$$\partial_{\alpha}\partial^{\alpha}A^{\mu} - \partial^{\mu}\left(\partial_{\beta}A^{\beta}\right) = 0, \qquad \text{Zwiebach (10.70) [206]} \qquad (10)$$

which, for the Lorenz gauge condition, $\partial_{\beta}A^{\beta} = 0$, becomes

$$\partial_{\alpha}\partial^{\alpha}A^{\mu} = 0 \qquad \alpha,\mu = 0,1,2,3 \quad \text{or} \quad \alpha,\mu = +,-,2,\ldots,d. \qquad (11)$$

Fourier transforming (11), we have

$$p_{\alpha}p^{\alpha}A^{\mu} = p^2A^{\mu} = 0 \qquad \alpha,\mu = 0,1,2,3 \quad \text{or} \quad \alpha,\mu = +,-,2,\ldots,d. \qquad \text{Zwiebach (10.82) [208]} \quad (12)$$

3.4 Quantum Maxwell Field Equation

In Sect. 12.7, the quantum Maxwell equation, in the light-cone gauge, is shown to be expressible as

$$i\frac{\partial A^J}{\partial\tau} = \alpha' p^I p^I A^J \qquad A^J = A^J\left(\tau, p^+, \vec{p}_T\right). \qquad \text{Zwiebach (12.194) [270].} \qquad (13)$$

3.5 Bottom Line: Light-Cone Gauge

The equivalence of (9) and (13) tells us that the string oscillation modes, for massless strings, can be considered equivalent to photons.

4 String Theory in More Traditional Form and Maxwell Equation

The equation of motion of a relativistic string is

$$\frac{\partial^2 \vec{X}}{\partial \sigma^2} - \frac{1}{c^2}\frac{\partial^2 \vec{X}}{\partial t^2} = 0\,, \qquad \text{(7.18) [135] and (7.26) [136]}$$

where

$$d\sigma = \frac{ds}{\sqrt{1 - v_T^2 / c^2}} = \frac{1}{T_0} dE \qquad \left(= \frac{1}{T_0}\frac{T_0 ds}{\sqrt{1 - v_T^2 / c^2}} \quad \text{since } dE = \frac{T_0 ds}{\sqrt{1 - v_T^2 / c^2}} \right). \qquad \text{(7.19)[135]}$$

Strictly speaking, *ds* varies in time and does not equal distance along the non-oscillating string. However, for small oscillations it is effectively that, i.e., *s* is effectively the distance along the non-oscillating string.

(7.18)/(7.26) means the speed of the wave along the string *relative to* σ, is *c*. But, σ is not a measure of distance in space. Relative to the actual distance in space (= *s* for small oscillations), the wave is *not* traveling at speed *c*, but something less.

More specifically, (7.18)/(7.26) with (the LHS of (7.19) becomes

$$\left(1 - \frac{v_T^2}{c^2}\right)\frac{\partial^2 \vec{X}}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 \vec{X}}{\partial t^2} = 0 \qquad \rightarrow \qquad v = c\sqrt{1 - \frac{v_T^2}{c^2}}\,. \qquad (14)$$

This same result can be obtained by using the effective tension and mass density of (7.15) [134].

$$v = \sqrt{\frac{T_{eff}}{\mu_{eff}}} = \sqrt{\frac{T_0\sqrt{1 - \frac{v_T^2}{c^2}}}{T_0 / c^2 \sqrt{1 - \frac{v_T^2}{c^2}}}} = c\sqrt{1 - \frac{v_T^2}{c^2}} \qquad (15)$$

Maxwell's equation for a wave propagating along the *x* axis, where *x* measure actual distance in space, is

$$\partial_\mu \partial^\mu A^\alpha = 0 \qquad \rightarrow \qquad \frac{\partial^2 \vec{A}}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = 0 \qquad (16)$$

which is different from (14).

Bottom line: The relativistic string equation (14) is NOT the same as Maxwell's equation. So, it doesn't seem that it can be quantized and taken as a Maxwell field.

I don't know how to resolve this issue and welcome any insights by readers. (Click on feedback link on home page of website under title above.)