

Pedagogic Aids to String Theory

by

Robert D. Klauber, PhD

Maharishi International University

May 2, 2026

www.quantumfieldtheory.info

Summary Overviews of String Theory Topics

By Robert D. Klauber updated April 30, 2026 www.quantumfieldtheory.info

This document contains the following. Pages are not numbered overall. Page numbers begin at 1 for each topic.

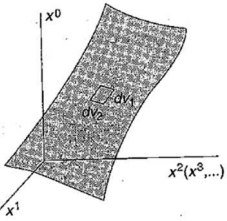
1. **Overview Intro to the Development of String Theory.** A simple, snapshot 2-page preview to quantum string theory.
2. **Dimensions, Branes, and Strings: The Differences.**
3. **Energy in a Superstring.** The way tension energy in a string is calculated in superstring theory differs from that of classical mechanics and so, can be confusing. Authors do not generally address this issue. Also includes T-duality intro.
4. **Simplified In-depth Intro to String Theory.** A more detailed overview of string theory than the first topic above.
5. **From Vibrating Strings to a Unified Theory of All Interactions.** By Barton Zwiebach. A nice simplified overview of string theory as a basis for the standard model of elementary particle theory.
6. **Gravity, Electrodynamics, and Planck Length in Higher Dimensions.**
7. **Gauges and Coordinate Systems in String Theory: A Summary.**
8. **String Gauge Relations Visualized in Spacetime Diagrams.**
9. **Symmetry on the String Worksheet: A Summary.** A wholeness chart of symmetries comparing and contrasting internal vs external symmetries and what is conserved in each case.
10. **Relativistic Particle vs String Solutions.** A wholeness chart summary of point particle vs open string motion in the light-cone gauge. Augments and summarizes much of Chaps. 9, 11, and 12 of Zwiebach (*A First Course in String Theory 2009*).
11. **Generators of Translation, Rotation, and Boost.** A wholeness chart pointing out (as shown in virtually no texts) how the classical generators use Poisson brackets, which are replaced in quantum theory with commutators. Summarizes Sects. 11.5 and 11.6 of Zwiebach.
12. **Deducing $D=26$.** An overview of how the bosonic string can only exist in 26 dimensional spacetime in order to satisfy Lorentz invariance.
13. **Open vs Closed Relativistic String Solutions.** A one-page wholeness chart listing the key parts of the solutions for relativistic strings. It compares and contrasts the solutions and underlying relations for open vs closed strings.
14. **Summary of Lagrangians for Different Systems.** Wholeness chart of different Lagrangians in different areas of physics.
15. **Number Operators in QFT and String Theory.** The number operator in string theory is similar in some respects to, and different in other respects from, that of QFT. This link is a one-page wholeness chart comparing and contrasting them.
16. **Superstrings: Summary of Zwiebach Chapter 14** plus a simplification. Two figures not in Zwiebach are presented that should make the result clearer. Also, a simplified derivation of the last relation in Sect. 14.3 (pgs. 309-312).
17. **Branes and Open Strings: An Overview.** A summary of open strings that can end on different dimension d-branes and of the characteristics (massive, massless, tachyon, etc.) of the resulting fields. Overview of Chap. 15 in Zwiebach.
18. **How Strings Give Rise to Fields Like Maxwell Fields.** A summary of how we identify an oscillating string, governed by its field (wave) equation with a photon (Maxwell field). Essentially showing that the field equation for the string is the same as Maxwell's equation. (This is pretty obscure in texts, but a foundational concept in string theory.)
19. **Kalb-Ramond String Charge Theory vs Electromagnetic Charge Theory.** A wholeness chart comparison of the Maxwell field vs the K-R string field. Covers Sects. 16.1 and 16.2 in Zwiebach.
20. **How Maxwell Charge is Located at Endpoints of Open Strings.** Missing steps in Zwiebach, Sect. 16.3, are supplied.
21. **Electric Charge on Compact D-Branes.** A more detailed treatment than is found in string theory texts for why a compact D-Brane can behave in spacetime external to it like an electric point charge. Parallels Zwiebach, Sect. 16.4.
22. **Moduli.** A simplified overview of moduli and why they need to be stabilized. Parallels first part of Zwiebach, Sect. 21.6.
23. **Polyakov String Action.** Simplified intro to a form of the action useful in the covariant formulation.
24. **String Interactions: An Intro.** Helps with Chap. 25 of Zwiebach (and other string texts dealing with interactions).

1. Overview Intro to the Development of String Theory

Robert D. Klauber January 26, 2024

www.quantumfieldtheory.info

Strings in Spacetime



Whereas a particle (of zero spatial dimensions) in spacetime traces out a world path (1D, a line, generally curved, but possibly straight), a string (of 1 spatial dimension) traces out a world sheet (a 2D surface, generally curved, but possibly flat.)

In 3D spatial space, a surface like a soap bubble automatically takes the smallest surface area. A free-floating bubble assumes a spherical shape, which has the minimum surface area for its volume. A soap bubble on a planar wire loop forms a flat surface, the minimum area that touches the loop all around its edges.

This is the principle of least action, where the action is the surface area. This principle is extended to 4D spacetime (and higher D spacetimes, as well) in the formulation of string theory.

Basic Principles upon Which the Theory is Built

1. For classical relativistic strings, minimize the area of the string world sheet (which is equivalent to minimizing the action, as the action in this case is the world sheet area).
2. Quantize classical relativistic strings (parallel to QFT quantization).
 - i) Use the same form for the action (which is equivalent to using the same form for \mathcal{L} , which is equivalent to using the same form for \mathcal{H} , which is equivalent to using the same equation of motion) as in classical relativistic strings.
 - ii) Take the numerical expansion coefficients in the solution to the equation of motion as creation and destruction operators (which is equivalent to converting classical Poisson brackets to commutators).

Things to Note

Quantization

A classical field could be a scalar or a vector. For a scalar, the field ϕ could be temperature, pressure, displacement in only one direction, or something else. It depends on spatial location and time. When we quantize to get QFT, the scalar field becomes an operator field that creates and destroys states (particles) which are scalars (like the Higgs particle).

For a vector, the classical field could represent the displacement from equilibrium in three directions in space, such as that of a wave moving through a medium. The displacement varies with spatial location and time. Such a wave would, in general, have oscillating motion in two spatial directions transverse to the wave velocity plus motion in the longitudinal direction. In relativity, we can think of the wave as a 4D vector, where each of the four components of the vector is a function of space and time as observed in a particular reference frame. When we quantize in QFT, the vector field becomes an operator field that creates and destroys states (particles) which are vectors (like the photon, W, Z, and gluon particles).

In summary, where X^μ represents the (dependent) displacement components and x^α represents the location in space and time that has displacement X^μ .

$$\underbrace{X^\mu(t, x^i)}_{\text{e.g., displacement of field}} \xrightarrow{\text{quantization}} \underbrace{A^\mu(x^\alpha)}_{\text{operator field}} \quad (1)$$

Of course, the classical vector could be something other than displacement of a fluid or solid, like the classical e/m 4-vector potential.

$$\underbrace{A^\mu(x^\alpha)}_{\text{e.g., classical e/m 4-potential}} \xrightarrow{\text{quantization}} \underbrace{A^\mu(x^\alpha)}_{\text{e/m operator field}} \quad (2)$$

In classical string theory, we find the displacement of a string X^μ (visualize a taut string attached along the x axis to a wall at either end, vibrating in the y [transverse] direction). It could, of course, also vibrate in the z direction, as well as longitudinally in the x direction. Taking x as x^1 in more general notation,

$$\underbrace{X^\mu(t, x^1)}_{\text{string displacement}} \xrightarrow{\text{quantization}} \underbrace{X^\mu(t, x^1)}_{\text{string operator field}} \quad \text{generalized parameters } \tau \text{ and } \sigma \text{ used in place of } t \text{ and } x^1. \quad (3)$$

Note that in (3), like we had in (1), what we have classically as a displacement field changes in quality when we quantize, i.e., changes to something that no longer represents displacement, but becomes an operator. In this way, quantum string theory finds that vibrating strings can, in principle, represent fields found in QFT that create and destroy the particles (which are wavelike) found in our universe.

Fermions vs Bosons

Fermions do not manifest as classical fields (whereas bosons do), so we can't quantize them in the same way as described above. Generally, string theory is developed as a theory of bosons (bosonic string theory) and after one gets acclimated to that, supersymmetry is invoked to bring in fermions.

String Theory's Place in Physics

In Klauber, *SFQFT*, Vol. 1, pg. 5, the overall structure of physics is displayed, but as noted there, string theory is not included in Wholeness Chart 1-1. The chart below adds string theory to that earlier chart.

Wholeness Chart. The Overall Structure of Physics Including Strings

	<u>Non-relativistic</u>			<u>Relativistic</u>		
	Particle	Field	String: Special Case of Field	Particle	Field	String: Special Case of Field
Classical mechanics (non-quantum)	Newtonian	Newtonian	Newtonian	Relativistic	Relativistic	Relativistic
Properties (Dynamical variables) ↓ Operators	↓ 1 st quantization ↓	↓ 2 nd quantization ↓	N/A	↓ 1 st quantization ↓	↓ 2 nd quantization ↓	↓ 2 nd quantization ↓
Quantum mechanics	NRQM	NRQFT rarely taught.	N/A	RQM	QFT (not gravity)	Quantum string theory

The Usual Route to Study Quantum Strings

In studying modern string theory, one proceeds in steps, as Zwiebach does. (Some steps, one typically has covered before studying strings, like 1, 2, 3, 6, and 8.)

1. Classical non-relativistic (Newtonian) strings (which are a certain kind of field)
2. Classical relativistic theory of particles
- ↓
3. Classical relativistic theory of fields
- ↓
4. Classical relativistic theory of strings (which are a certain kind of field)
5. Quantum relativistic theory of particles (RQM, quantization of #2)
- ↓
6. Quantum relativistic theory of fields (QFT, quantization of #3)
- ↓
7. Quantum relativistic theory of strings (which are a certain kind of field; quantization of #4)
- ↓
8. Quantum relativistic theory of supersymmetry (SUSY, an extension of QFT)
- ↓
9. SUSY applied to quantum relativistic strings.
- ↓
10. Assorted topics like branes, how charge arises in string theory, dualities, and more.

2. Dimensions, Branes, and Strings: The Differences

Robert D. Klauber March 10, 2024 www.quantumfieldtheory.info

Three concepts are fundamental to string theory, those of dimensions, strings, and branes. These, particularly branes vs dimensions, can be confused, so we define them, with examples, in the following.

1 Definitions

1.1 Dimension

This is nothing new, except that in string theory, we have more than the 3 spatial and 1 time dimensions we are familiar with. In string theory without supersymmetry (SUSY), it turns out that requiring Lorentz invariance constrains the number of dimensions to exactly 26 (25 spatial, 1 time). With SUSY, the constraint changes to exactly 10 (9 spatial, 1 time). Our 4D world is a subspace of a 26 (in elementary string theory) or 10 (in superstring theory) dimensional space.

In string theory, the spatial dimensions beyond the three of our world are considered to be compactified, i.e., curled up and closed upon themselves. In analyses, each such dimension is commonly taken as circular and very small, on the order of the Planck length in circumference.

The symbol d is used to denote the number of spatial dimensions; the symbol D , the total number of dimensions. The world we are accustomed to has $d = 3$ and $D = 4$. 10D superstring theory has $d = 9$ and $D = 10$. $D = d + 1$, in general.

1.2 String

A string has one spatial dimension and traces out a world sheet in spacetime. The spacetime in string theory is 26D, or 10D for superstrings. The strings oscillate in all spatial dimensions, the 3D of our familiar world plus in the higher dimensions.

For strings as elementary particles, the oscillations in 3 spatial dimensions ($d = 3$) are those we associate with any (wavelike) particle. The string oscillates and moves, i.e., it manifests as a wave. The oscillations and boundary conditions of the string in higher dimensions determine what kind of elementary particle it is.

1.3 Brane

A brane is an object of any number of spatial dimensions, up to 25 (elementary string theory) or 9 (superstring theory). Although we talk of branes as having a certain number of dimensions, they themselves are *not* dimensions, *per se*. They are objects. The number of spatial dimension over which a brane extends is symbolized by p .

What they are made of is not addressed. They are characterized by their impact, in string theory, on the behavior of strings. Strings end on branes, as shown in Fig. 1 below, where, in this example, $D = 4$, $d = 3$, and $p = 2$, though higher dimensions are analogous (branes can be of more than the two dimensions shown).

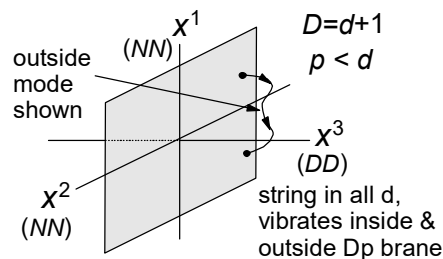


Figure 1. A String with Ends on a $2d$ Brane in a $3d$ World (of $4D$ spacetime)

Unlike Fig. 1, for which we show only part of the brane, branes generally extend throughout the p dimensional space they are in. In the figure, the two-dimensional brane really extends to infinity in the x^1 and x^2 directions.

Boundary conditions exist for the string ends. In this example, the B.C.s allow the ends to slide sideways (without friction) on the $2d$ brane, but are fixed in the x^3 direction. A sliding B.C. is called a Neumann B.C.; a fixed one, a Dirichlet B.C. A brane of this type is called a Dp -brane, where p is the (spatial) dimension of the brane and D here stands for Dirichlet, implying the brane cannot move in a direction perpendicular to its own p dimensional space.

Though the string ends are constrained to move only on the brane, the rest of the string can oscillate in all directions, the two of the brane shown plus the x^3 direction (in which direction its ends cannot move). For higher dimensions, the same idea applies. The string endpoints can't move off the Dp -brane (they can slide “tangentially” along it, but not normal to it). The rest of the string, however, can have oscillation modes in the brane (tangentially) or outside of the brane (normally).

For higher dimensions branes beyond $p = 2$, “tangentially” means motion within the brane itself; “normal” means motion outside the brane. Typically string endpoints are restricted to tangential motion, but string oscillations are both tangential and normal to the brane. The “(NN)” symbol in Fig. 1 next to a coordinate axis means the first and second ends of the string have Neumann B.C.'s in that dimension, i.e., the ends can slide in that direction. The “(DD)” symbol means the string endpoints both have Dirichlet B.C.'s, the ends cannot move in that direction.

In short, string endpoint motion is limited to the brane (in p dimensions), but the rest of the string can oscillate throughout all space (in all d dimensions).

Confusion can arise because sometimes strings are referred to as one dimensional branes. But, it is best to consider them as completely different animals, at least while one is first learning the theory.

2 Strings Between Branes

Strings don't have to begin and end on the same brane. They can begin on one brane and end on another, as for two of the strings in Fig. 2.

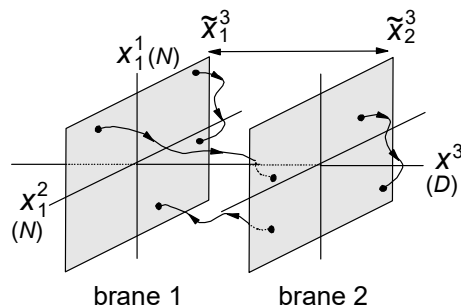


Fig. 2. Strings Can Also Begin on One Brane and End on Another

Further, unlike the branes shown in Fig. 2, different branes can have different dimensions. A string might start on a $D2$ -brane, for example, but end on a $D1$ or $D6$ -brane, as but two examples. Further, two different branes could be coincident. In Fig. 2, this would mean brane 2 occupies the same space as (overlaps) brane 1, i.e., $\tilde{x}_2^3 - \tilde{x}_1^3 = 0$. One end of a string would be on one brane, and the other end, on the other brane, even though the two ends are constrained to move (slide) only in the same x^1 - x^2 plane, and not normal to that plane.

When this situation arises (commonly in string theory), diagrams often show the coincident branes in different places, i.e., as not coincident. This is only because one cannot distinguish visually between two coincident branes if they are drawn in their actual locations. So, keep this in mind when you see strings extending from one brane to another, and be clear on which branes are coincident and which are not.

As one eventually learns, different branes can not only have a different number of spatial dimensions or extend over different subspaces, but have different characteristics, as well, i.e., they can be of different types. The kind of elementary particle a string represents is determined by the type of brane it begins on and the type of brane it ends on.

For the $6d$ compactified space of superstring theory, a number of different types of branes are used. Some of these are coincident with others; some are not. Strings reaching between different type branes represent different particles of the standard model.

3 A Simplified Look at String Theory's Connection to Elementary Particle Theory

Barton Zwiebach published a nice simplified overview of string theory's application to particle theory in the *MIT Physics Annual 2004*, which is included later in Sect. 5 herein. It is a good roadmap to the theory that can help one keep sight of the forest, in spite of the trees, as one progresses through learning the details of that theory.

I believe this can also be found on the website for Zwiebach's book *A First Course in String Theory* (Cambridge 2009),

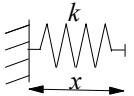

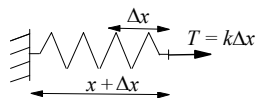
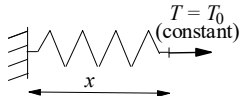


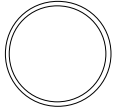
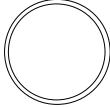
3. Energy in a Superstring

Bob Klauber Orig: 20 June 2020 2nd rev: April 27, 2023

1 Tension Energy: Classical Strings vs Superstrings

The tension force as modeled in a superstring is a bit weird by classical standards, as it does not depend (as a classical elastic string obeying Hooke's law would) on how much the string has been stretched. The tension in the superstring is the same constant value for any amount of stretching. It is even more weird in that the initial length of the superstring is considered to be zero. This means the potential energy in the string is directly proportional to the length of the string (unlike a classical elastic string). This is summarized in Wholeness Chart 1.

Wholeness Chart 1. Tension and Stretch in Classical vs “Modern” String Theory

	<u>Classical String</u>	<u>Superstring</u>	<u>Comment</u>
Open			
Unstretched	 Elastic string modeled as spring		Superstring considered to have zero length unstretched, unlike classical elastic string.
Stretched	 $T = k\Delta x$	 $T = T_0$ (constant)	Classical force proportional to stretch Δx . Superstring force constant for any stretch, and stretch is x (unlike classical string Δx).
Potential Energy	$E = V = \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}T\Delta x$	$E = V = T_0x$	Classical tension (potential) energy proportional to $(\Delta x)^2$; superstring proportional to x .
Tension Force	$T = \frac{dV}{dx} = k\Delta x$ (varies with Δx)	$T = \frac{dV}{dx} = T_0$ (constant)	Classical tension grows with stretch Δx ; superstring is constant with stretch x .
Closed			
Unstretched	 Radius R	 Zero radius	Superstring considered to have zero circumference unstretched, unlike classical rubber band.
Stretched	 $T = k\Delta C$ $\Delta C = 2\pi\Delta R$	 $T = T_0$ (constant)	Classical force proportional to stretch ΔC . Superstring force constant for any stretch, and stretch is C (unlike rubber band ΔC).
Potential Energy	$E = V = \frac{1}{2}k(\Delta C)^2 = \frac{1}{2}T\Delta C$	$E = V = T_0C$	Classical tension (potential) energy proportional to $(\Delta C)^2$; Superstring energy proportional to C .
Tension Force	$T = \frac{dV}{dC} = k\Delta C$ (varies with ΔC)	$T = \frac{dV}{dC} = T_0$ (constant)	Classical tension grows with stretch ΔC ; superstring is constant with stretch C .

2 Mass

Mass in relativistic string theory can arise from the internal energy the string holds, which manifests externally as mass via $m = E/c^2$, where E is the internal energy.

So, for a traditional relativistic classical string theory, we would have contributions to the mass from i) the string's inherent rest mass, which we designate by m_0 , and ii) mass from the internal energy of the string, which for the simple case of a non-vibrating string would equal the energy from string tension E_t divided by c^2 .

$$E_t = \int_0^l T dx \quad (\text{for constant } T = T_0, E_t = T_0 l) \quad m_t = \frac{E_t}{c^2}. \quad (1)$$

So, in traditional classical relativistic string theory, the string mass for a static string would be

$$m = m_0 + m_t. \quad (2)$$

However, “modern” relativistic string theory (both classical and quantum) take (Zwiebach, pg. 119)

$$m_0 = 0 \quad (\text{“modern” relativistic string theory}). \quad (3)$$

This makes analysis simpler, but one might question how valid the assumption is. The result of this and the prior sections are summarized in Wholeness Chart 2, where we also extrapolate mass to mass density (per unit length) along the string, a more relevant number for calculating things like natural frequency of the string.

Wholeness Chart 2. Aspects of Various String Theories

<u>Attribute of Theory</u>	<u>Classical Newtonian</u>	<u>True Classical Relativistic</u>	<u>“Modern” Classical Relativistic</u>	<u>“Modern” Quantum Relativistic</u>
Unstretched string	Finite length	Finite length	Zero length (point)	As at left
Tension vs stretch	Hooke’s Law $T = k\Delta x$ Δx = displacement from zero stretch position	Hooke’s Law with relativistic considerations	$T_0 = \text{constant}$ Tension independent of stretch	As at left
External mass, static string	$m = m_0 \neq 0$ m_0 = inherent mass No contribution from T	$m = m_0 + m_T$ m_0 = inherent mass $\neq 0$ Contribution from T : $m_T = \frac{\int T dx}{c^2}$	$m = m_T$ $m_0 = 0$ (no inherent mass) Contribution from T_0 : $m_T = \frac{T_0 l}{c^2}$	As at left
Mass density	$\rho = \rho_0 \neq 0$	$\rho = \rho_0 + \rho_T$ $\rho_0 \neq 0$	$\rho = \rho_T$ $\rho_0 = 0$	As at left
Label points on string?	Yes	Yes	No*	As at left

* See Zwiebach, pgs. 120 (Sect. 6.8 2nd paragraph), 123 (2nd line under (6.87)), 124 (2nd line up from bottom),

3 Superstring Loops and Tension Energy

We define the quantity α' and express string energy in terms of it and the radius of a superstring circular loop.

$$\text{Tension } E_{loop} = 2\pi RT \quad \frac{1}{\alpha'} \equiv 2\pi T \quad \Rightarrow \quad \text{Tension } E_{loop} = \frac{R}{\alpha'} \quad (4)$$

α' reflects the amount of tension in the string. It is proportional to the inverse of the tension. For multiple wrappings (loops) of the string

$$\text{Tension } E = \frac{mR}{\alpha'} \quad m = \text{number of wrappings (“wrapping, or winding, number”).} \quad (5)$$

Note the units.

$$T \text{ (tension force) units (from Newton's law) } \frac{ml}{s^2} \xrightarrow{\text{natural units}} \frac{1}{l^2} \text{ or } M^2 \quad (6)$$

So, from (4), α' has units of M^{-2} (or l^2 , as it is sometimes taken as cm^2).

4 Loops and Vibration Energy (= Kinetic Energy)

Quantized energy is

$$\text{Kinetic } E = \hbar\omega = \hbar \frac{2\pi c}{\lambda} \text{ (massless)} \xrightarrow{\text{natural units}} \frac{1}{l} \text{ or } M^1. \quad (7)$$

The lowest (first or fundamental) mode of vibration has wavelength $\lambda = \lambda_1$ equal to $C = 2\pi R$ in (7), giving, in natural units,

$$\text{Kinetic } E_{\substack{\text{lowest mode} \\ 1 \text{ wrapping}}} = \frac{1}{R}. \quad (8)$$

For more wrappings with the same wavelength (one circumference), kinetic energy would not change.

$$\text{Kinetic } E_{\substack{\text{lowest mode} \\ m \text{ wrappings.} \\ \text{same } \lambda_1}} = \frac{1}{R}. \quad (9)$$

But, we also have higher vibration modes to consider, where the shorter wavelength means higher kinetic energy.

$$\lambda = \lambda_n = \frac{\lambda_1}{n} \quad \text{Kinetic } E_{\substack{\text{nth mode} \\ m \text{ wrappings.} \\ \text{same } \lambda}} = \frac{n}{R} \quad n = \text{mode number} \quad (10)$$

So, greater wrapping number increases potential (tension) energy via (5), and higher modes increase kinetic (vibration) energy via (10). They both contribute to the total energy of the string, which is

$$E_{\substack{m \text{ wrappings.} \\ \text{nth mode}}} = \text{Tension } E + \text{Vibration } E = \frac{mR}{\alpha'} + \frac{n}{R} \quad \text{for given fundamental mode } \lambda_1. \quad (11)$$

If a massless superstring has energy E with its wrapping around dimensions in a higher dimensional compactified space, then to us in the non-compactified 4D spacetime, it looks to have (rest) mass

$$m = \frac{E}{c^2} \quad (12)$$

5 Ramifications

Note that one term in (11) has R in the numerator, and one has R in the denominator. Consider one case where (we will drop subscripting on E)

$$E = \frac{mR}{\alpha'} + \frac{n}{R} = \frac{10R}{1} + \frac{1}{R} \quad \frac{m}{\alpha'} = 10 \quad n = 1. \quad (13)$$

For small $R = .01$ this has large kinetic energy of 100 units, but small potential energy of 0.1 units with a total energy of 100.1 units.

Now, consider the case where $R = 10$. This has small kinetic energy 0.1 and large kinetic energy of 100, for the same total of 100.1 units.

One superstring has large radius; the other, small, but the string energy is the same, and via (12), they both manifest in our world as having the same mass, i.e., being the same particle, at least effectively.

Suppose one of these cases can be solved perturbatively, but the other not. We can get the same prediction for mass either way, so we take the easy way, the perturbative way, and the answer is good for both cases. A very large radius string wrapped in Calabi-Yau space has identical eigenvalues (like energy) as a very small radius string.

We can thus solve an intractable problem by solving a tractable equivalent problem. The answers to the easy problem are answers to the hard one. As one no doubt knows, this is a great boon.

This is called a duality. There are dual systems, which are different, yet effectively the same. In this case, it is a T-duality, where, depending on the author, T stands for i) “target space”, whose meaning is a topic for another day, ii) “topological”, as compactified space is characterized by its topology, which relates to how strings can be wrapped around openings within it, or iii) (probably most correct) “toroidal” (where a circle is a 1 dimensional toroid).

Caveat: This treatment is simplified. There is a bit more to calculating wound string energy beyond the scope of this discussion.

4. Simplified In-depth Intro to String Theory

Robert D. Klauber www.quantumfieldtheory.info May 2, 2026

Equation symbols such as (24.2) [569] refer to Zwiebach, *A First Course in String Theory*. For this example, it is equation (24.2) in Zwiebach on page 569.

1 String Theory Postulates

1.1 Classical Relativistic String Theory Postulates

- String tension T_0 is constant and not proportional to stretch length, i.e., not as in Hooke's law. That is $T_0 \neq k \Delta x$.
- The string is stretched from initial length zero (unlike rubber bands with finite length when not stretched). So, from a),

$$E_{tens} = T_0 a \quad a = \text{string length} \quad (1)$$

- Non-energy-related mass of string = 0. Rest (not vibrating) mass m_0 arises solely from stretch tension energy via $m_0 = E_{tens}/c^2$. There will be an additional mass contribution to the total mass M from string vibration (kinetic) energy.
- Assume the action is the area of the string worldsheet, and it is minimized for free strings. (As in other areas of physics, one obtains the Lagrangian density \mathcal{L} from the action, the Hamiltonian density from \mathcal{L} , and the equation of motion from the action minimization.)

1.2 Quantization Postulates

Parallel to the two postulates of QFT, we have the following postulates for quantum string theory.

- Take the classical relativistic string Hamiltonian density \mathcal{H} of Sect. 1.1(d) as the quantum string Hamiltonian density. (This is same as taking the classical Lagrangian density \mathcal{L} as the quantum \mathcal{L} , which is equivalent to taking the classical equation of motion as the quantum equation of motion, which is equivalent to taking the classical solution form for string motion X^μ , the solution to the equation of motion, as the form of the quantum string field).
- Impose certain commutation relations (parallel to those in QFT) on the string field and its conjugate momentum density. This, in turn, results in commutation relations for the coefficients of the Fourier expansion terms in the solution X^μ . And this turns the coefficients into creation and destruction operators.

There is one more assumption, not unlike other such assumptions we have seen in quantum mechanics. That is, we cannot detect (measure) the position of internal points of the string. They cannot be labeled, so we cannot follow their motion.

Note that the assumption of the prior paragraph plus postulates a, b, and c of Sect. 1.1 mean that quantization of classical theory, although often claimed, does not really occur in string theory. We do not simply quantize the classical theory of strings, but a conveniently modified version of it. And thus, the correspondence principle does not hold for strings. That is, in the classical limit, quantum string theory does not reduce to classical Newtonian theory, unlike all of our other theories of modern physics.

I have not seen the point of the prior paragraph made in the literature, though someone may have. It is not well known and, in fact, is often misrepresented.

2 Adopt a Class of Gauges

X^μ , the position in spacetime of points on the string, is expressed as a function of two parameters on the string worldsheet surface, τ and σ . $X^\mu = X^\mu(\tau, \sigma)$.

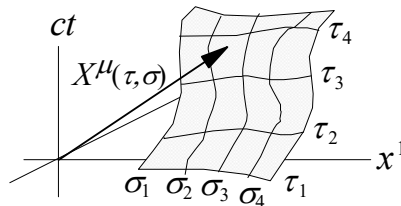


Figure 1. String Worldsheet Showing Lines of Constant τ and σ .

We might tend to think of τ as having the value of proper time, but it generally does not. It is more general in nature and just a parameter. This can be confusing, as in relativity theory, τ is typically used as the symbol for proper time, but in string theory, it is just a parameter.

In principle, these parameters can be virtually anything. In practice, string theory is like electromagnetism or many other physics theories, in that development and analysis can be greatly simplified by making a good (convenient) choice of gauge, here a good choice for τ and σ .

Early in this chapter, we will simply assume a good choice for a family (type or class) of gauges has been made, without getting into exact details. In the gauge family of choice, for any possible string trajectory in spacetime (wherein all points on the string must remain inside the light cone, i.e., cover timelike paths), τ increases as observer time t does (for any observer and for any choice of τ in this gauge family). That is, all timelike intervals have positive change in τ .

Also, in this gauge family, lines of constant τ and lines of constant σ are orthogonal to one another at every point (event) on the worldsheet. A spacelike interval moving from left to right on a spacetime diagram will always have increasing σ .

In practice, this means that for this class of gauge, τ is related to time, but generally is not equal to time. Similarly, σ is related to spatial distance, but generally not equal to it. τ increases monotonically with time t of any inertial observer, and σ varies monotonically in space from one end of the string to the other for any inertial observer. For open strings (not closed in a loop), by convention, we take σ values to extend from 0 at one end of the string to π at the other end. The center of the string is taken at $\sigma = \pi/2$.

In Appendix A and Sect. 6, we show the chosen gauge family mathematically. For now, we just accept it has the properties described.

The parameters τ and σ are dimensionless. Note that the surface area of the string worldsheet is invariant under variations in τ and σ (under transformations to different τ and σ). That is, changes in the spacing and arrangement of the parameter grid lines on that surface leave the area of the sheet unchanged. The gauge we will employ chooses a convenience spacing and arrangement of the grid lines among a literally infinite number of choices, all of which are consonant with minimum worldsheet surface area.

3 Field Equation and Its Solution in the Chosen Gauge Class

3.1 The Field Equation

When using this gauge family and the requirement of Sect. 1.1(d) of minimum worldsheet area (minimum action), after a fair amount of mathematical manipulation (Appendix A), the equation of motion can be deduced. It turns out to take the form we have seen many times, that of the wave equation,

$$\boxed{\ddot{X}^\mu - X^{\mu\prime\prime} = 0} \quad \text{where} \quad \dot{X}^\mu = \frac{\partial X^\mu}{\partial \tau} \quad X^{\mu\prime} = \frac{\partial X^\mu}{\partial \sigma} \quad \ddot{X}^\mu = \frac{\partial^2 X^\mu}{\partial \tau^2} \quad X^{\mu\prime\prime} = \frac{\partial^2 X^\mu}{\partial \sigma^2}. \quad (24.2) [569] \quad (2)$$

Take care, however, that the derivative symbols using dots and primes in string theory are with respect to the parameters τ and σ , which generally do not represent time (neither proper nor coordinate) or space (neither proper length nor coordinate length). This may take some getting used to, but it is the accepted practice.

3.2 Solution to Field Equation in Chosen Gauge Class

Even though the dots and primes in (2) are not generally with respect to time and space, we know the solution form for (2), if they were, from prior work. Here, we can just take the same solutions, but just substitute τ for t and σ for x .

Hence, the solution to (2) for an open string (not closed in a loop) is, where we discuss symbols and the meaning of each term below,

$$\boxed{X^\mu(\sigma, \tau) = x_0^\mu + \underbrace{\sqrt{2\alpha'}\alpha_0^\mu}_{2\alpha'p^\mu} \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma}. \quad (24.9)[570] \quad (3)$$

In (3), the coefficients of terms are defined in a way that will prove beneficial later on in the theory. One can check that (3) solves (2) by simple substitution. Note that the integer n ranges from $-\infty$ to $+\infty$, but does not include zero.

α' , for reasons we won't get into, is called the slope parameter, and where T_0 is the tension force in the string, equals

$$\boxed{\alpha' = \frac{1}{2\pi T_0}} \quad (\text{natural units}). \quad (8.76) [169] \quad (4)$$

First term

The constant term x_0^μ in (3) is the position of the middle of the string $\sigma = \pi/2$ (so the last term in (3) is zero) for $\tau = 0$ (so the penultimate term in (3) is zero). In a sense, one might think of this as initial position, but to be precise, for that we would need $t = \tau = 0$ initially. x_0^μ is the position of the center of the string along the line on the world sheet where $\tau = 0$.

Second term

In the second term after the equal sign in (3), α_0^μ is defined in terms of α' and the 4-momentum of the entire string p^μ , as shown in the under brackets. That term can be re-written with the slope parameter and with the tension T_0 of (4) as equal to the

string tension energy E_{tens} divided by the open string length a , as, where M is the mass of the string due to both tension and vibration energy,

$$2\alpha' p^\mu = \frac{p^\mu}{\pi T_0} \xrightarrow{\text{open string}} \frac{p^\mu}{\pi E_{tens} / a} = \frac{a}{\pi} \frac{p^\mu}{E_{tens}} = \frac{a}{\pi} \frac{p^\mu}{m_0} = \frac{a}{\pi} \frac{M u^\mu}{m_0} \quad \left(\begin{array}{l} m_0 = \text{mass due only to string tension} \\ u^\mu = 4\text{-velocity} \end{array} \right). \quad (5)$$

Thus, the second term in (3) becomes

$$\frac{a}{\pi} \frac{M}{m_0} u^\mu \tau, \quad (6)$$

which effectively represents the (non-oscillating) motion of the center of the string and makes some sense, as it entails velocity times a time parameter. Heuristically, imagine a special case where τ represents proper time (it virtually never does in string theory, as it is a more generalized parameter), and σ represents actual distance (it virtually never does) along the string from $\sigma = 0$ to $\sigma = \pi$, so $a = \pi$. If we have no vibration of the string, so $M = m_0$, then (6) for the spatial coordinates becomes

$$u^i \tau = \frac{v^i}{\sqrt{1-v^2}} t \sqrt{1-v^2} = v^i t, \quad (7)$$

the distance a non-accelerating object has moved after time t . Note this treatment is simply a mnemonic helpful in thinking about the second term after the equal sign in (3), as τ is a more generalized parameter and generally $M \neq m_0$, so quantities in (3) will not be exactly what we might otherwise think of them as being. The math in (3) still works, though the precise physical visualization of it may be challenging. Regardless, the second term gives us the part of X^μ due to free (non-accelerated) motion of the string as a function of the parameter τ , for whatever gauge we use to specify τ .

Third term

The third term represents the oscillating motion modes of the string (since it isn't just a solid object), which to get the LH side of (3), is superimposed upon the first and second term contributions to get string motion at every value of σ and τ . Though σ and τ don't represent spatial position and time, the math of (3) still holds, but the physical intuition on what is happening in spacetime is more problematic.

Note the cosine dependence on σ , a measure of distance along the string. The center of $\cos \sigma$ is at $\pi/2$, and there it equals zero. So, the last term in (3) describes oscillating motion about the string centerpoint, whose coordinates in spacetime are prescribed by the first two terms after the equal sign.

The α_n^μ ($n \neq 0$) are Fourier expansion coefficients, where the mode number of the string vibration is n . It can be confusing at first, but in string theory, the α_n^μ (alpha, not "a") coefficients actually are sort of an intermediary step to get to the type of coefficients one typically uses in QFT, represented by a_n^μ ("a", not alpha). Specifically,

$$\alpha_n^\mu = \sqrt{n} a_n^\mu \quad n \neq 0. \quad (8)$$

As we will discuss, the a_n^μ ("a") coefficients are siblings to the QFT operator coefficients, in that they have essentially the same commutation relations.

The Role of Mode Number n

Note also in (3) the form of the time factor oscillation $e^{-in\tau}$ in the last term,

$$e^{-in\tau} \quad (\text{string theory}). \quad (9)$$

Normally, for waves and oscillations, we see a factor like

$$e^{-i\omega_n t} \quad (\text{QFT}). \quad (10)$$

The parameter τ is not generally, as noted, time t for the observer, and this accounts for $\omega_n \neq n$. But, for string theory as formulated here, one can, in a sense, think of n as the frequency of the mode with respect to the τ parameter (not generally with respect to time). n effectively has units of oscillations per unit τ parameter. But also, n is a non-zero integer, and both τ and n are unitless.

$$\text{The role of } \omega_n \xrightarrow{\text{string theory}} n. \quad (11)$$

It can, at times, help in learning string theory to think in terms of (11).

4 The Hamiltonian in the Chosen Gauge Class

4.1 The String Lagrangian

From the string equation of motion (2), one can deduce the Lagrangian (density) for strings via trial and error substitution into the Euler-Lagrange equation. When one does this, the Lagrangian is found (which we confirm below) to be

$$\mathcal{L} = K(\dot{X} \cdot \dot{X} - X' \cdot X') \quad \text{notation } X \cdot X = X^\mu X_\mu \quad K \text{ taken} = \frac{1}{4\pi\alpha'} \text{ by convention.} \quad (12)$$

To check, with the usual 4D independent variables in that equation x^α taken as the 2D τ and σ ,

$$\frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \xrightarrow{x^\alpha = \tau, \sigma} \quad \frac{\partial}{\partial \tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{X}} \right) + \frac{\partial}{\partial \sigma} \left(\frac{\partial \mathcal{L}}{\partial X'} \right) - 0 = 0 \quad \rightarrow \quad \ddot{X} - X'' = 0, \quad (13)$$

which is our field equation (2).

Note that, by starting from (2) and deducing (12), we have gone a bit in reverse from the standard derivation of these results, because it may be easier to grasp the essence of string oscillation that way. In more formal presentations, one starts from the minimum world sheet area postulate, where that area A is the action, and minimizes that. The Lagrangian is then found, after significant effort, from the integrand of $A = \int dA = \int \mathcal{L} d\tau d\sigma$, where $\underline{\delta A} = 0$, and shown, in the chosen gauge, to be equivalent to (12). And then, one finds (2) from (12) via the Euler-Lagrange equation, as in (13). For details, see Appendix A of this chapter.

4.2 Hamiltonian in Terms of String Displacement

4.2.1 Finding the Hamiltonian

With (12), one finds the Hamiltonian (density) with the Legendre transformation.

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \dot{X}^\mu - \mathcal{L} = \frac{1}{4\pi\alpha'} 2\dot{X}^\mu - \frac{1}{4\pi\alpha'} (\dot{X} \cdot \dot{X} - X' \cdot X') = \frac{1}{4\pi\alpha'} (\dot{X} \cdot \dot{X} + X' \cdot X') \quad (14)$$

So, in the chosen gauge, the total Hamiltonian is

$$H = \int_0^\pi \frac{1}{4\pi\alpha'} (\dot{X} \cdot \dot{X} + X' \cdot X') d\sigma = \int_0^\pi \mathcal{H} d\sigma. \quad (24.6) \text{ with } (24.5) [570] \quad (15)$$

4.2.2 Re-written Hamiltonian in More Familiar Terms

For heuristic purposes, again imagine τ as time and σ as spatial distance (which they aren't, but it can help intuitively). Then, in (4), we have, where the physical length of an open string is π ,

$$\alpha' = \frac{1}{2\pi T_0} \quad \xrightarrow{\text{open strings}} \quad = \frac{1}{2\pi E_{\text{tens}} / \pi} = \frac{1}{2m_0} \quad (\sigma \text{ taken as spatial distance here}), \quad (16)$$

so (15) becomes

$$H = \int_0^\pi \left(\frac{m_0}{2\pi} \dot{X} \cdot \dot{X} + \frac{T_0}{2} X' \cdot X' \right) d\sigma = \int_0^\pi \underbrace{\left(\frac{\rho}{2} \dot{X} \cdot \dot{X} + \frac{T_0}{2} X' \cdot X' \right)}_{\mathcal{H}} d\sigma \quad \begin{array}{l} (\rho = \text{mass density, mass per unit } \sigma \text{ length}) \\ (\mathcal{H} = \text{Hamiltonian density, } H \text{ per unit } \sigma \text{ length}) \end{array} \quad (17)$$

Note in (17) that the Hamiltonian density has a kinetic energy density term (in mass density $m_0/\pi = \rho$) and timelike derivatives of position) and a potential energy term (in tension and spacelike derivatives of position). This, as we noted above, would only be strictly true for σ as position and τ as time in the observer's reference frame, but it is a good mnemonic and can help intuitively.

4.2.3 Comparison with QFT Hamiltonian

Note further how \mathcal{H} of (15) parallels the classical scalar Hamiltonian density, as shown in Klauber, Vol. 1, (3-31), pg. 49, where μ , rest mass not counting tension there equals zero here, superscript 0 means scalar, subscript 0 means free field, and K is a constant,

$$\mathcal{H}_0^0 = \pi_0^0 \dot{\phi} - \mathcal{L}_0^0 = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} \dot{\phi} - \mathcal{L}_0^0 = K(\dot{\phi} \dot{\phi} + \nabla \phi \cdot \nabla \phi). \quad (\text{classical free scalar field}) \text{ Klauber (3-31) [49]} \quad (18)$$

Time and space derivatives in (18) are with respect to actual space and time coordinates, whereas in (15) they are with respect to parameters σ and τ , which are more general in nature. H in (15) is actually not energy, since τ and σ are used there instead of spacetime coordinates. H in string theory works to give us the action, the Lagrangian, and the field equations for strings, so all the variational calculus math works out. But H there is generally not energy. Nevertheless, one can see the striking parallels between the string Hamiltonian density (15) in the special case of (17) and that of classical (as well as quantum) field theory.

4.3 Hamiltonian in Terms of Quantum Operator Coefficients

We now employ the quantization postulate of Sect. 1.2(b), i.e., we take the coefficients in (3) to be operators obeying certain commutation relations, similar to those of QFT. But, first a review of QFT Hamiltonians.

4.3.1 For QFT

For Scalars

As shown in Klauber, Vol. 1, Chap. 3, pgs. 49 to 54, we can substitute the field equation solution (ϕ there) into the Hamiltonian (the integral of (18) above and taking $K=1$), employ the second quantization requirement of the commutation relations for the coefficients in the solution, and get the Hamiltonian in terms of those coefficients (which are creation and destruction operators). The result of this is Klauber's (3-55) [54], where symbols should be familiar to the reader,

$$H_0^0 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + \frac{1}{2} + b^\dagger(\mathbf{k}) b(\mathbf{k}) + \frac{1}{2} \right) = \sum_{\mathbf{k}} \omega_{\mathbf{k}} (N_a(\mathbf{k}) + N_b(\mathbf{k})) + \sum_{\mathbf{k}} \omega_{\mathbf{k}}. \quad (\text{quantum scalar field}). \quad (19)$$

For a discrete system each $\omega_{\mathbf{k}}$ is an integer multiple of the fundamental frequency ω_1 , so we can re-express (19) as

$$H_0^0 = \sum_{m=1}^{\infty} \omega_1 m (N_a(m) + N_b(m)) + \sum_{m=1}^{\infty} \omega_1 m \quad m = \text{mode number, an integer, not mass and not winding number}. \quad (20)$$

For a real field, the antiparticle is the same as the particle, so we only have the "a" type particle. Eliminating the "b" type terms in (19), we have

$$H_0^0 = \omega_1 \sum_{m=1}^{\infty} m N_a(m) + \frac{\omega_1}{2} \sum_{m=1}^{\infty} m = \omega_1 \sum_{m=1}^{\infty} m a_m^\dagger a_m + \frac{\omega_1}{2} \sum_{m=1}^{\infty} m. \quad (21)$$

Note that $a_m^\dagger a_m$ is a number operator, which, when operating on states, gives us the number of particles in the m th oscillation mode. We expect energy to be proportional to the number of particles. The $\omega_1 m$ is the frequency of the m th mode, and we expect energy to also be proportional to frequency, as it is here. Three times ($m=3$) the frequency for the same number of particles, three times the energy. Twice the number of particles (eigenvalue of $N_a(m)=2$) at given frequency m , twice the energy.

For Vectors

The above is for a scalar. For a massless real vector field, such as the photon, one finds that in 4D (see Klauber, Vol. 1, (5-57), pg. 149) the free Hamiltonian is (22).

$$H_0^1 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{r=0}^3 \zeta_r a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}) + \frac{1}{2} \sum_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{r=0}^3 \zeta_r \quad \begin{array}{l} 4 \text{ polarizations assumed, } \zeta_0 = -1 \quad \zeta_1 = \zeta_2 = \zeta_3 = 1 \\ \text{number operator } N_r(\mathbf{k}) = \zeta_r a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}). \end{array} \quad (22)$$

Carrying out the summation for the last term in (22), we have

$$H_0^1 = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \sum_{r=0}^3 \zeta_r a_r^\dagger(\mathbf{k}) a_r(\mathbf{k}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} = \omega_1 \sum_{m=1}^{\infty} m \zeta_r a_r^\dagger(m) a_r(m) + \omega_1 \sum_{m=1}^{\infty} m \quad \left(\begin{array}{l} \text{assume 4 polarizations} \\ \text{in } D=4 \text{ spacetime} \end{array} \right). \quad (23)$$

For a coordinate system where the axes are aligned with the polarizations, $r = \mu$, and for massless 4-vectors like photons, where the longitudinal and timelike polarizations cancel, are undetectable, and can be ignored,

$$\begin{aligned} H_0^1 &= \omega_1 \sum_{m=1}^{\infty} m \zeta_\mu a_\mu^\dagger(m) a_\mu(m) + \frac{D'}{2} \omega_1 \sum_{m=1}^{\infty} m & D' = 2 \text{ here = number} \\ & & \text{of detectable polarization modes;} \\ &= \omega_1 \sum_{m=1}^{\infty} m \sum_{\mu} N_\mu(m) + \frac{D'}{2} \omega_1 \sum_{m=1}^{\infty} m & \text{polarizations } r=1,2 \\ & & \text{aligned with } \mu=1,2 \text{ axes.} \end{aligned} \quad (24)$$

For 4D, where $D=4$, $D'=D-2$, and (24) becomes

$$\begin{aligned} H_0^1 &= \omega_1 \sum_{m=1}^{\infty} m \zeta_\mu a_\mu^\dagger(m) a_\mu(m) + \frac{D-2}{2} \omega_1 \sum_{m=1}^{\infty} m & \text{No timelike or longitudinal modes;} \\ & & D=4 \text{ here = spacetime dimensions;} \\ &= \omega_1 \sum_{m=1}^{\infty} m a^{\mu\dagger}(m) a_\mu(m) + \frac{D-2}{2} \omega_1 \sum_{m=1}^{\infty} m & \text{polarizations } r \text{ aligned with } \mu \text{ axes.} \end{aligned} \quad (25)$$

For higher spacetime dimensions than 4, the same relation can be expected to hold, just with different D .

Note the number operator in the first row of (25) has all lowered indices. If the destruction operator had a raised index and the construction operator had a lowered index, they would be destroying and creating different (contravariant vs covariant) states. We need both operators to have either a covariant index or both have a contravariant index if together they are to represent a number operator. But then we need to insert a minus sign for the $a_0^\dagger(m)a_0(m)$ term. The $\zeta\mu$ in the first row of (25) accomplishes that. We can have effectively the same thing, by simply raising one index, as in the second row, since raising the index only changes the sign of the $\mu = 0$ term.

4.3.2 For String Theory

We do a similar thing in string theory as we did in QFT. We take the Hamiltonian of (15), plug (3) into it, and postulate commutation relations between the coefficients α_n^μ and their complex conjugates similar in form to those employed in QFT.

When we do this (see Appendix B, relation (79)), we get, with the underbrace in (3) to convert the $\alpha_{0,\mu}$ factors, (don't confuse 4-momentum p^μ with mode number p)

$$H = \frac{1}{2}\alpha_0^\mu\alpha_{0,\mu} + \sum_{p=1}^{\infty} p a_p^{\dagger\mu} a_{p,\mu} + \frac{D'}{2} \sum_{p=1}^{\infty} p \quad (26)$$

p = mode number (we use n in (3) and (79), but that is used for something else later)

D' = number of modes string is assumed to oscillate in (all but longitudinal and time dimension)

The last term in (26) arises from the non-zero commutation relations, as one can see in Appendix B.

As noted in Appendix B, if we assume the string lives in D dimensions, where there is no oscillation in the longitudinal nor timelike directions, then (26) becomes (see Appendix B, which has the final result (82))

$$H = \frac{1}{2}\alpha_{0,\mu}\alpha_0^\mu + \sum_{p=1}^{\infty} p a_{-p,\mu} a_p^\mu + \frac{D-2}{2} \sum_{p=1}^{\infty} p = \frac{1}{2}\alpha_{0,\mu}\alpha_0^\mu + \sum_{p=1}^{\infty} p a_{p,\mu}^\dagger a_p^\mu + \frac{D-2}{2} \sum_{p=1}^{\infty} p \quad (27)$$

$D' = D - 2$ = number of modes string is assumed to oscillate in.

From (11), we know that p (n in (11)) acts like frequency in (27), like $\omega_m = m\omega_1$ in (23) (22). And $a_{p,\mu}^\dagger a_p^\mu$ is a number operator (like $a_r^\dagger(\mathbf{k})a_r(\mathbf{k})$ in (22)) that gives us the number of strings in the p th mode, oscillating in the μ direction (and we sum over these different directions as each is considered a separate string). So, as in QFT, the Hamiltonian is proportional to the frequency times the number of particles oscillating at that frequency. (But remember that p is the frequency per unit τ parameter, which isn't usually the actual physically measured frequency.)

As noted in Appendix B, a complicated analysis determines that for bosonic strings, Lorentz invariance only holds for $D = 26$ in (27).

5 The Infinite Sums

Note the similarity of the infinite sums in the last terms of (25) and (27). In QFT, the issue of the infinite sum is interpreted as an infinite number, or as an enormous number if one sums only up to the Planck energy, and in theory constitutes an (infinite or enormous) energy in the vacuum. This, of course, is not observed, and the issue is widely considered to be unresolved. When making scattering or decay calculations, the infinite sum is simply ignored.

In string theory, however, the seemingly infinite sum of the last term in (27) is treated in, what euphemistically can be called, an "odd" way. Through identification of that sum with a particular entity known as the zeta function and extensive, sophisticated mathematical gyrations, string theorists accept the following.

$$\sum_{p=1}^{\infty} p = 1 + 2 + 3 + \dots = -\frac{1}{12}. \quad (28)$$

If you have trouble accepting (28), you are not alone, but string theory depends on it critically. It is the only value of the sum for which one can have Lorentz invariance in string theory (and then, only in the bosonic string spacetime of 26 dimensions, i.e., $D = 26$). In other words, all of string theory hinges on the validity of (28).

Thus, for string theory, (27) becomes

$$H = \frac{1}{2} \alpha_{0\mu} \alpha_0^\mu + \sum_{p=1}^{\infty} p a_{p\mu}^\dagger a_p^\mu - 1 \quad \text{string theory Hamiltonian.} \quad (29)$$

One could ask that if (28) is indeed valid, then we have no issue with the infinite sum of the last term in (25). If (28) is true, then that sum equals $-\omega_1/12$, and we don't have a large vacuum energy (ZPE) problem. Yet physicists simultaneously consider there to be an unresolved issue with (25), but not with (27).

Time to move on.

6 Key Result for the Hamiltonian

In Appendix A, we show the gauge of choice for τ and σ is defined, in part, by (55), repeated below.

$$X'^2 + \dot{X}^2 = 0. \quad (30)$$

Note that in this gauge, our Hamiltonian density of (14) is zero, so our total Hamiltonian H of (15) is zero. This may seem strange but recall what we said below (18). The Hamiltonian in our case is not physical energy, but merely a mathematical entity that helps us in analysis. So, it is OK that it equals zero. Our system can still have non-zero energy.

However, we are working in quantum field theory (the string = the field) where the Hamiltonian is an operator, not a number, and it operates on string states. So, what we are saying is actually that H acting on a string state yields zero. That is, where $|\phi\rangle$ is any state containing one or more strings,

$$H|\phi\rangle = 0. \quad (31)$$

This may seem strange, as we are used to the Hamiltonian acting on a state to yield the energy eigenvalue of the state times the original state. But that is when the Hamiltonian is the total energy operator. Here, by our choice, it is not.

7 Determining the Masses of Strings

From (27) and (31), we have

$$H|\phi\rangle = \left(\frac{1}{2} \alpha_{0\mu} \alpha_0^\mu + \sum_{p=1}^{\infty} p a_p^{\dagger\mu} a_{p\mu} + \frac{D-2}{2} \sum_{p=1}^{\infty} p \right) |\phi\rangle = 0. \quad (32)$$

Employing the underbrace relation of (3), followed by (4) and the relativistic mass squared relation, gives us

$$\begin{aligned} H|\phi\rangle &= \left(\alpha' p_\mu p^\mu + \sum_{p=1}^{\infty} p a_p^{\dagger\mu} a_{p\mu} + \frac{D-2}{2} \sum_{p=1}^{\infty} p \right) |\phi\rangle = \left(\frac{p_\mu p^\mu}{2\pi T_0} + \sum_{p=1}^{\infty} p a_p^{\dagger\mu} a_{p\mu} + \frac{D-2}{2} \sum_{p=1}^{\infty} p \right) |\phi\rangle \\ &= \left(\alpha' M^2 + \sum_{p=1}^{\infty} p a_p^{\dagger\mu} a_{p\mu} + \frac{D-2}{2} \sum_{p=1}^{\infty} p \right) |\phi\rangle. \end{aligned} \quad (33)$$

where M is the mass of the string viewed by an external observer and includes contributions from tension and vibration energy (m_0 is only from tension energy). Where N is the string number operator (and we switch the dummy index p to n), we have

$$M^2 |\phi\rangle = \frac{1}{\alpha'} \left(\sum_{n=1}^{\infty} n a_{n\mu}^\dagger a_n^\mu + \frac{D-2}{2} \sum_{p=1}^{\infty} p \right) |\phi\rangle = \frac{1}{\alpha'} \left(N + \frac{D-2}{2} \sum_{p=1}^{\infty} p \right) |\phi\rangle \quad N = \sum_{n=1}^{\infty} n a_{n\mu}^\dagger a_n^\mu \quad (34)$$

If we accept (28), this becomes

$$\boxed{M^2 |\phi\rangle = \frac{1}{\alpha'} \left(\sum_{n=1}^{\infty} n a_{n\mu}^\dagger a_n^\mu - \frac{D-2}{24} \right) |\phi\rangle = \frac{1}{\alpha'} \left(N - \frac{D-2}{24} \right) |\phi\rangle.} \quad (35)$$

Relation (35) lets us determine the mass M , the eigenvalues for mass of a quantum string state $|\phi\rangle$ (LHS), by evaluating the operation on the RHS. Note that the N operator has a factor of n , the mode number, which we saw in (11) is related to string vibration energy for the n th mode (one contribution for each μ direction). The factor $a_{n\mu}^\dagger a_n^\mu$ acts like a similar part of the number operator in QFT. It yields the number of strings oscillating in the n th mode (summed for each μ direction). So, one of these strings

has an energy contribution proportional to n , its mode number. If $a_{n\mu}^\dagger a_n^\mu$ gives us the number 2 as the number of strings oscillating in that mode, we would have an energy contribution related to $2n$, and etcetera for greater numbers of strings.

Since the first term inside the parentheses of (35) is an integer, one would expect the second term to be so, as well. That means the minimum number of dimensions D (assuming $4 \leq D$) is 26. Extensive analysis of Lorentz invariance for strings yields the same number 26. For that, we find (35) becomes

$$M^2 |\phi\rangle = \frac{1}{\alpha'} \left(\sum_{n=1}^{\infty} n a_{n\mu}^\dagger a_n^\mu - 1 \right) |\phi\rangle = \frac{1}{\alpha'} (N - 1) |\phi\rangle. \quad (36)$$

Note that in our special case of (16), where σ equals physical distance along a string of length π , (35) becomes

$$M^2 |\phi\rangle = 2m_0 \left(\sum_{n=1}^{\infty} n a_{n\mu}^\dagger a_n^\mu - 1 \right) |\phi\rangle = 2m_0 (N - 1) |\phi\rangle \quad (\text{special case for } \sigma = \text{physical distance along string}). \quad (37)$$

This lets us see that when N (which has units of energy) is multiplied by rest mass solely from tension m_0 , we get units of energy (or mass) squared on the RHS, which is what we have on the LHS. But, again (37) does not represent the general gauge family, so it should only be used in a heuristic sense, to get a feeling for (36).

Note that for the zero mode state, where $n = 0$, M is imaginary, i.e., a tachyon. For states where $N = 1$, we have massless string states. At the origin of the universe, before the Higgs field bestowed mass on particles, all particles had zero mass. Thus, we need the negative number -1 in (35) and (36) (from (28)) if string theory is to have any chance of being the underlying theory of the universe. Indeed, string descriptions of standard model particles invariably employ the $N = 1$ (massless) string states as SM particle states.

8 An Overview of String Theory

8.1 Different Formulations to String Theory

What we have focused on in this article is known as the covariant formulation to string theory, because we used Minkowski coordinates and Lorentz covariance is something we have considerable familiarity with, in those coordinates.

There is an alternative formulation which uses something called light-cone coordinates, effectively replacing the x^0 and x^1 coordinate axes with the edges of the light cone in the first and second quadrants. The light-cone formulation also employs a gauge, and that is similar to the gauge we have been working with..

The light-cone formulation is, in my opinion, a considerably more difficult way to be introduced to string theory, but doing so is quite common, and this is the way Zwiebach does it. One needs to learn both formulations, eventually, of course, but the covariant way may be the easier gateway to understanding the theory.

8.2 Open vs Closed Strings

We have looked herein at open strings, i.e., strings that are not loops. Closed strings are strings that form loops and are treated in a generally similar manner as we have done here, but with some modifications to account for the difference in the two types of string.

8.3 Free vs Interacting Strings

Everything we have done here is solely for free strings, those that are not interacting with other strings. As with QFT, we need to walk (with free fields/strings) before we can run (with interacting fields/strings).

8.4 Bosonic vs Supersymmetric Strings

In this article we have only dealt with bosonic strings. We have quantized (more or less) classical strings, which like bosons in QFT, are something we consider manifest in our macroscopic world. But, just as Dirac was needed to make the breakthrough of including fermions in QFT, supersymmetry is needed in string theory, in order to include fermions into the theory.

It turns out that while bosonic strings require 26 dimensions, supersymmetric string (superstring) theory requires only 10. One needs to learn bosonic string theory before making the jump to superstrings.

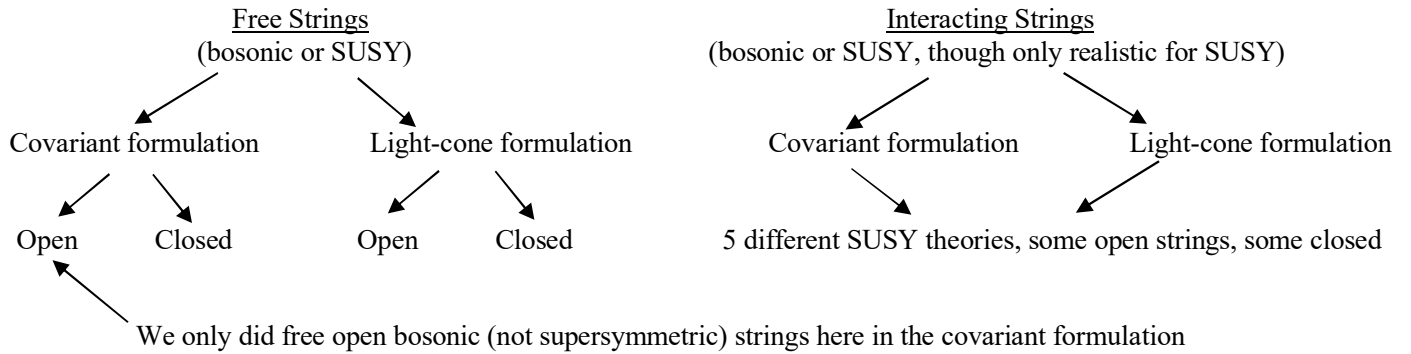
8.5 Different String Theories

There are five known string theories, all of which start with the basics we have presented herein. They differ with regard to whether they contain open or closed strings, whether they include bosonic strings along with superstrings, whether the strings have an inherent direction (called “orientation”) along their length or not, and certain other factors. It is suspected that they are unified, along with supergravity, into one grand theory called M theory, but no one has actually been able to prove that.

8.6 Summary Overview

Wholeness Chart 1 displays the different aspects of string theory and how they relate to one another.

Wholeness Chart 1. Overview of the Structure of String Theory



9 Appendix A. Minimum Worldsheet Area and the Lagrangian

One needs a background in general relativity, or at least differential geometry, to understand this appendix.

9.1 Background

The worldsheet area, as noted earlier, is taken as the action to be minimized and goes by the formal name Nambu-Goto action. Recall from differential geometry, the metric for any set of coordinates in a given space, where X^α represents the position vector in (an oftentimes higher dimensional) Cartesian (Minkowski in spacetime) coordinates, and x^μ represent the generalized coordinates in the given space of dimension D , is

$$g_{\mu\nu} = \begin{bmatrix} \frac{\partial X^\alpha}{\partial x^0} \frac{\partial X_\alpha}{\partial x^0} & \frac{\partial X^\alpha}{\partial x^0} \frac{\partial X_\alpha}{\partial x^1} & \cdots & \frac{\partial X^\alpha}{\partial x^0} \frac{\partial X_\alpha}{\partial x^D} \\ \frac{\partial X^\alpha}{\partial x^1} \frac{\partial X_\alpha}{\partial x^0} & \frac{\partial X^\alpha}{\partial x^1} \frac{\partial X_\alpha}{\partial x^1} & \cdots & \frac{\partial X^\alpha}{\partial x^1} \frac{\partial X_\alpha}{\partial x^D} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial X^\alpha}{\partial x^D} \frac{\partial X_\alpha}{\partial x^0} & \frac{\partial X^\alpha}{\partial x^D} \frac{\partial X_\alpha}{\partial x^1} & \cdots & \frac{\partial X^\alpha}{\partial x^D} \frac{\partial X_\alpha}{\partial x^D} \end{bmatrix} = \frac{\partial X^\alpha}{\partial x^\mu} \frac{\partial X_\alpha}{\partial x^\nu} \quad \mu, \nu = 0, 1, 2, \dots, D. \quad (38)$$

Recall further, that the differential volume element (volume in a 2D space = area; volume in a 3D space = what we think of as volume; volume in an a more general space is D dimensional), which is the volume we would measure with instruments (i.e., the “physical” volume V , not the coordinate volume), is

$$dV = \sqrt{|g|} dx^0 dx^1 \dots dx^D \quad \text{All spatial space, } g = \text{determinant of } g_{\mu\nu}. \quad (39)$$

In spacetime, due to the minus sign on g_{00} , this becomes

$$dV = \sqrt{-g} dx^0 dx^1 \dots dx^D \quad \text{Spacetime of dimension } D. \quad (40)$$

9.2 Apply GR Principles to 2D Worldsheet

Let’s apply all of this to our 2D worldsheet in a higher dimensional spacetime of D dimensions, and our generalized coordinates on the worldsheet are the parameters τ, σ . Thus, where we use the symbol Zwiebach does for the $2D$ spacetime metric on the worldsheet of γ ,

$$g_{\mu\nu} = \gamma_{\mu\nu} = \begin{bmatrix} \frac{\partial X^\alpha}{\partial \tau} & \frac{\partial X_\alpha}{\partial \tau} & \frac{\partial X^\alpha}{\partial \sigma} & \frac{\partial X_\alpha}{\partial \sigma} \\ \frac{\partial X^\alpha}{\partial \sigma} & \frac{\partial X_\alpha}{\partial \sigma} & \frac{\partial X^\alpha}{\partial \sigma} & \frac{\partial X_\alpha}{\partial \sigma} \end{bmatrix} = \begin{bmatrix} \dot{X} \cdot \dot{X} & \dot{X} \cdot X' \\ X' \cdot \dot{X} & X' \cdot X' \end{bmatrix} = \begin{bmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ X' \cdot \dot{X} & (X')^2 \end{bmatrix} \quad \alpha = 0, 1, 2, \dots, D. \quad (41)$$

The volume element in 2D is an area element, so from (40) and (40),

$$dA = \sqrt{-\gamma} d\tau d\sigma = \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} d\tau d\sigma \quad 2D \text{ surface of worldsheet.} \quad (42)$$

For our minimization of worldsheet area postulate, we have the

$$\text{Nambu-Goto action} = \text{worldsheet area} = A = \int dA \quad \xrightarrow{\text{for } \delta A = 0} \quad = \int \underbrace{\mathcal{L} d\tau d\sigma}_{dA} \text{ for } \delta A = 0. \quad (43)$$

That is, for minimization of worldsheet area dA must be equal to $\mathcal{L} d\tau d\sigma$.

Equating dA of (42) and (43), we have

$$\mathcal{L} = \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} \quad (\text{true of any gauge}), \quad (44)$$

which is true for any choice of grid lines for τ and σ , i.e., for any gauge, as long as the worldsheet area is minimized.

9.3 Picking a Convenient Gauge

Note that putting (44) into the Euler-Lagrange equation produces a bit of an unsightly mess, which is not easy to use in analysis. We won't do it here, but you might want to try it and see.

The good news is that we have flexibility in choosing the gauge (grid lines for τ, σ). If we take

$$\dot{X} \cdot X' = \dot{X}^\mu X'_\mu = g_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} = 0 \quad (\text{gauge choice}), \quad (45)$$

then (44) simplifies to

$$\mathcal{L} = \sqrt{-(\dot{X})^2 (X')^2}, \quad (46)$$

which we want to put into the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\alpha} \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \xrightarrow{\substack{\phi = X \\ x^\alpha = \tau, \sigma}} \quad \frac{\partial}{\partial \tau} \left(\frac{\partial \mathcal{L}}{\partial \dot{X}} \right) + \frac{\partial}{\partial \sigma} \left(\frac{\partial \mathcal{L}}{\partial X'} \right) - 0 = 0 \quad (47)$$

to yield the equation of motion. Note, that still gets a little complicated, as

$$\frac{\partial \mathcal{L}}{\partial \dot{X}} = -\frac{1}{2\pi\alpha'} \frac{\dot{X} X'^2}{\sqrt{-(\dot{X} \cdot X')^2}} \quad \frac{\partial \mathcal{L}}{\partial X'} = -\frac{1}{2\pi\alpha'} \frac{X' \dot{X}^2}{\sqrt{-(\dot{X} \cdot X')^2}}. \quad (48)$$

The good news is that we have not used up all of our gauge freedom with (45). The constraint limited our widest choice of τ, σ parameters to a smaller family. But it is still a family, not a single gauge. There are still further constraints we can make. So, to simplify the left-hand relation in (48), we choose a gauge in which

$$\frac{X'^2}{\sqrt{-(\dot{X} \cdot X')^2}} = 1, \quad (\text{gauge choice}) \quad (49)$$

$$\text{so,} \quad \frac{\partial \mathcal{L}}{\partial \dot{X}} = -\frac{1}{2\pi\alpha'} \dot{X}. \quad (50)$$

That is great, but how about the right-hand relation in (48)? We can simplify that too by noting what we get by re-arranging (49), where in the penultimate step we use the first step over,

$$X'^2 = \sqrt{-(\dot{X} \cdot X')^2} \quad \rightarrow \quad X'^4 = -(\dot{X} \cdot X')^2 \quad \rightarrow \quad X'^2 = -\dot{X}^2 \quad \rightarrow \quad \sqrt{-(\dot{X} \cdot X')^2} = -\dot{X}^2. \quad (51)$$

And thus, from the last step of (51),

$$\frac{\dot{X}^2}{\sqrt{-(\dot{X} \cdot X')^2}} = -1, \tag{52}$$

which, when used in the right-hand side of (48), gives us

$$\frac{\partial \mathcal{L}}{\partial X'} = \frac{1}{2\pi\alpha'} X' . \tag{53}$$

Putting (50) and (53) into (47), yields

$$-\frac{\partial}{\partial \tau} \left(\frac{1}{2\pi\alpha'} \dot{X} \right) + \frac{\partial}{\partial \sigma} \left(\frac{1}{2\pi\alpha'} X' \right) = 0 \quad \rightarrow \quad \boxed{\ddot{X} - X'' = 0}, \tag{54}$$

which has the (simple and useful) form of the wave equation.

Note that we got the wave equation (54) by incorporating the gauge restraints (45) and (49). Taking (49) in the form of the third relation in (51), we can summarize our gauge choice via

$$\boxed{\dot{X} \cdot X' = 0 \quad X'^2 + \dot{X}^2 = 0} \quad \text{(re-expressing our two gauge conditions).} \tag{55}$$

As an aside, recall that Maxwell’s equation takes the form of the wave equation only for the Lorenz gauge ($\partial_\mu A^\mu = 0$). Otherwise, it is far more complicated. (See Klauber, Vol. 1, pg. 141, Wholeness Chart 5-1.) So, this trick of getting the simple form of the wave equation is one we have used before (by fixing the gauge).

9.4 Switching to an Equivalent Lagrangian

Note that if we use the Lagrangian of (12), we get the same field equation, the same equation of motion (the wave equation). Compare (13) to (54) Thus, the two different Lagrangians (12) and (46) describe exactly the same physical phenomenon, so they are equivalent. We can use either and make the same real world predictions. String theorists have found (12) to have some advantages and use that, for the most part.

And that is what we have employed in the rest of this chapter. Take care that the solutions to (13) (= (54)) we work with throughout are only valid for the particular gauge fixing of (55). Otherwise, the equation of motion is no longer in the form of the classic wave equation, and its solutions are far harder to find and far harder to work with.

9.5 Things to Note about Our Gauge Constraint Choice

Recall we wanted two things for our gauge (in addition to giving us the wave equation): 1) that the lines of τ and σ are orthogonal, and 2) that τ is timelike and σ is spacelike.

9.5.1 Orthogonality

Those well versed in differential geometry will know right away that (45) implies lines of τ and σ are everywhere perpendicular. If you are not so well versed, in Fig. 2, you can see that the derivative of X^α with respect to τ is tangent to a line of constant σ . Similarly, the derivative of X^α with respect to σ is tangent to a line of constant τ . That is, in our symbolism, \dot{X} is tangent to σ grid lines, and X' is tangent to τ grid lines. So, if (45) is true, the tangents are orthogonal, and thus the grid lines are orthogonal. Our gauge choice ensures this. (Note, in Fig. 2 we show lines that are not always orthogonal to make a point. In our applications, they are generally orthogonal at every point.)

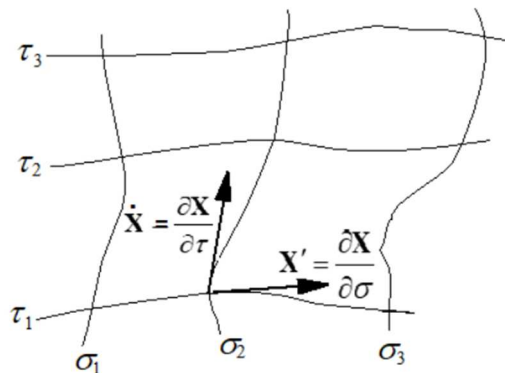


Figure 2. Derivatives and Tangent Vectors

9.5.2 Timelike τ and Spacelike σ

One might think that our wave equation (54) for this gauge fixing could actually have any τ and σ , as long as the grid lines are orthogonal. That would mean τ could be timelike, spacelike, or lightlike. Ditto for σ . It will help analysis and its interpretation a lot if we restrict τ to timelike choices and σ to spacelike. That is, where τ increases with increase in time t of any observer, and σ increases with purely spatial intervals of any observer.

The particular subset we want is all those grid line (gauge) choices for which τ is timelike and σ is spacelike.

$$\dot{X}^2 + X'^2 = 0 \quad (\text{one gauge condition}). \quad (56)$$

We can re-write this more explicitly as

$$\frac{\partial X^\alpha}{\partial \tau} \frac{\partial X_\alpha}{\partial \tau} + \frac{\partial X^\alpha}{\partial \sigma} \frac{\partial X_\alpha}{\partial \sigma} = 0 \quad \rightarrow \quad \frac{\partial X^\alpha}{\partial \tau} \frac{\partial X_\alpha}{\partial \tau} = - \frac{\partial X^\alpha}{\partial \sigma} \frac{\partial X_\alpha}{\partial \sigma} \quad (57)$$

The spacetime intervals, as one moves on the world sheet via parameter increases $\Delta\tau$ and $\Delta\sigma$, for each parameter are found from

$$(\Delta s_\tau)^2 = \frac{\partial X^\alpha}{\partial \tau} \Delta\tau \frac{\partial X_\alpha}{\partial \tau} \Delta\tau \quad (\Delta s_\sigma)^2 = \frac{\partial X^\alpha}{\partial \sigma} \Delta\sigma \frac{\partial X_\alpha}{\partial \sigma} \Delta\sigma. \quad (58)$$

But, from (57) and (58),

$$(\Delta s_\tau)^2 = -(\Delta s_\sigma)^2. \quad (59)$$

Moving purely in the τ direction yields a spacetime interval squared of opposite sign from that of moving purely in the σ direction. Therefore, one direction must be timelike (negative sign for $(\Delta s)^2$) and the other, spacelike (positive sign for $(\Delta s)^2$). By convention, we choose the τ direction to be timelike, so the σ direction must be spacelike.

Thus, (56) gives us one of the desired criteria we had for our parameters τ and σ , that the former is always timelike, and the latter, spacelike.

9.5.3 One Relation Including Both Gauge Conditions

As we show after (60), we can re-write the two gauge conditions (55) as

$$\boxed{(\dot{X} \pm X')^2 = 0} \quad (\text{alternate form for gauge conditions (55)}). \quad (60)$$

Proof:

Note that (60) is actually two relations,

$$\begin{aligned} (\dot{X} + X')^2 = 0 & \quad \rightarrow \quad \dot{X}^2 + 2\dot{X} \cdot X' + X'^2 = 0 & (a) \\ (\dot{X} - X')^2 = 0 & \quad \rightarrow \quad \dot{X}^2 - 2\dot{X} \cdot X' + X'^2 = 0 & (b) \end{aligned} \quad (61)$$

Add (61) (a) to (b) and we get the right-hand side of (55). Subtract the two, and we get left-hand side of (55). QED.

9.6 Summary of Effects of Postulates and Chosen Gauge

9.6.1 Classical Strings

Requiring minimum world sheet area for a string in spacetime,

$$1. \quad \text{yields the Lagrangian (44), } \mathcal{L} = \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}$$

The gauge fixing of (60) (= (55)) results in

1. orthogonal parameter grid lines of τ and σ
2. timelike τ and spacelike σ
3. the wave equation as the equation of motion

Because the same wave equation of motion can be found with a different Lagrangian,

1. we choose to use that equivalent, more convenient Lagrangian (12), $\mathcal{L} = K(\dot{X} \cdot \dot{X} - X' \cdot X')$,

which is the same form as the Lagrangian one finds from relativistic dynamic analysis of a string with no forces acting on it (but with $t, x \rightarrow \tau, \sigma$).

All of the above (this Sect. 9.6.1) comprises Postulate (d) of Sect. 1.1 (pg. 1).

9.6.2 Quantum Strings

Postulate (a) of Sect. 1.2, the first postulate (see Klauber Vol. 1, pg. 4) in forming a quantum field theory (of which strings is one type), takes the form of the classical string Lagrangian (12) as the starting point for quantum string theory.

10 Appendix B: Deriving H in Terms of Operator Coefficients

Take (24.6) [570] re-expressed as (15) and repeated below for convenience,

$$H = \int_0^\pi \frac{1}{4\pi\alpha'} (\dot{X} \cdot \dot{X} + X' \cdot X') d\sigma = \int_0^\pi \mathcal{H} d\sigma. \quad (62)$$

Use (3) to find the derivatives with respect to τ and σ ,

$$\dot{X}^\mu(\sigma, \tau) = \frac{\sqrt{2\alpha'}\alpha_0^\mu}{2\alpha'p^\mu} + i\sqrt{2\alpha'} \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \cos n\sigma \quad (63)$$

$$X'^\mu(\sigma, \tau) = i\sqrt{2\alpha'} \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \sin n\sigma \quad (64)$$

The First Term in (62)

The first term in (62), then becomes

$$\int_0^\pi \frac{1}{4\pi\alpha'} \dot{X}^\mu \dot{X}_\mu d\sigma = \int_0^\pi \frac{1}{4\pi\alpha'} \left(\sqrt{2\alpha'}\alpha_0^\mu + \sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \alpha_n^\mu e^{-in\tau} \cos n\sigma \right) \left(\sqrt{2\alpha'}\alpha_{0,\mu} + \sqrt{2\alpha'} \sum_{\substack{p=-\infty \\ p \neq 0}}^{\infty} \alpha_{p,\mu} e^{-ip\tau} \cos p\sigma \right) d\sigma \quad (65)$$

$$= \int_0^\pi \frac{1}{4\pi\alpha'} \left(\begin{aligned} &2\alpha' \alpha_0^\mu \alpha_{0,\mu} + \sqrt{2\alpha'} \alpha_0^\mu \sqrt{2\alpha'} \sum_{\substack{p=-\infty \\ p \neq 0}}^{\infty} \alpha_{p,\mu} e^{-ip\tau} \cos p\sigma + \sqrt{2\alpha'} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \alpha_n^\mu e^{-in\tau} \cos n\sigma \sqrt{2\alpha'} \alpha_{0,\mu} \\ &+ 2\alpha' \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \sum_{\substack{p=-\infty \\ p \neq 0}}^{\infty} \alpha_n^\mu \alpha_{p,\mu} e^{-i(n+p)\tau} \cos p\sigma \cos n\sigma \end{aligned} \right) d\sigma \quad (66)$$

In integrating over σ from 0 to π , the first term in (66) is constant, so it is just multiplied by π . The second and third terms (each with a factor of $\cos n\sigma$) go to zero. Under that integration, the last term is only non-zero for $n = p$ or $n = -p$. Specifically,

$$\int_0^\pi \cos n\sigma \cos p\sigma d\sigma = \frac{\pi}{2} \text{ if } n = p \text{ or } n = -p \quad n, p \text{ integers.} \quad (67)$$

We thus have

$$\begin{aligned} \int_0^\pi \frac{1}{4\pi\alpha'} \dot{X} \cdot \dot{X} d\sigma &= \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \frac{1}{4} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\alpha_n^\mu \alpha_{n,\mu} e^{-i(n+n)\tau} + \alpha_n^\mu \alpha_{-n,\mu} e^{-i(n-n)\tau} \right) \\ &= \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \frac{1}{4} \sum_{n=1}^{\infty} \left(\alpha_n^\mu \alpha_{n,\mu} e^{-i2n\tau} + \alpha_{-n}^\mu \alpha_{-n,\mu} e^{i2n\tau} + \alpha_n^\mu \alpha_{-n,\mu} e^0 + \alpha_{-n}^\mu \alpha_{n,\mu} e^0 \right). \end{aligned} \quad (68)$$

With (12.58)[245]) extended to the covariant formulation,

$$\alpha_{-n} = \alpha_n^\dagger \quad n \text{ a positive integer,} \quad (69)$$

and we have

$$\int_0^\pi \frac{1}{4\pi\alpha'} \dot{X} \cdot \dot{X} d\sigma = \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \frac{1}{4} \sum_{n=1}^{\infty} \left(\alpha_n^\mu \alpha_{n,\mu} e^{-i2n\tau} + \alpha_n^\dagger \alpha_{n,\mu}^\dagger e^{i2n\tau} + \alpha_n^\mu \alpha_{n,\mu}^\dagger + \alpha_n^\dagger \alpha_{n,\mu} \right). \quad (70)$$

The Second Term in (62)

The second term in (62), with (64), becomes

$$\begin{aligned} \int_0^\pi \frac{1}{4\pi\alpha'} X' \cdot X' d\sigma &= \int_0^\pi \frac{1}{4\pi\alpha'} \left(i\sqrt{2\alpha'} \sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \sin n\sigma \right)^2 d\sigma \\ &= -\int_0^\pi \frac{1}{2\pi} \left(\sum_{n \neq 0} \alpha_n^\mu e^{-in\tau} \sin n\sigma \right) \left(\sum_{p \neq 0} \alpha_{p,\mu} e^{-ip\tau} \sin p\sigma \right) d\sigma. \end{aligned} \quad (71)$$

Then, use the following integral formula to evaluate (71).

$$\int_0^\pi \sin n\sigma \sin p\sigma d\sigma = 0 \text{ for } p \neq n \text{ or } -n; \quad = \frac{\pi}{2} \text{ for } p = n; \quad = -\frac{\pi}{2} \text{ for } p = -n. \quad (72)$$

Thus,

$$\begin{aligned} \int_0^\pi \frac{1}{4\pi\alpha'} X' \cdot X' d\sigma &= -\frac{1}{4} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\alpha_n^\mu \alpha_{n,\mu} e^{-i(n+n)\tau} - \alpha_n^\mu \alpha_{-n,\mu} e^{-i(n-n)\tau} \right) \\ &= -\frac{1}{4} \sum_{n=1}^{\infty} \left(\alpha_n^\mu \alpha_{n,\mu} e^{-i2n\tau} + \alpha_{-n}^\mu \alpha_{-n,\mu} e^{i2n\tau} - \alpha_n^\mu \alpha_{-n,\mu} - \alpha_{-n}^\mu \alpha_{n,\mu} \right) \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left(-\alpha_n^\mu \alpha_{n,\mu} e^{-i2n\tau} - \alpha_n^{\dagger\mu} \alpha_{n,\mu}^{\dagger} e^{i2n\tau} + \alpha_n^\mu \alpha_{n,\mu}^{\dagger} + \alpha_n^{\dagger\mu} \alpha_{n,\mu} \right). \end{aligned} \quad (73)$$

The Hamiltonian: Add the First and Second Terms in (62)

Adding (70) and (73) gives us the total string Hamiltonian for free strings,

$$\begin{aligned} H &= \int_0^\pi \frac{1}{4\pi\alpha'} \dot{X} \cdot \dot{X} d\sigma + \int_0^\pi \frac{1}{4\pi\alpha'} X' \cdot X' d\sigma \\ &= \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \frac{1}{4} \sum_{n=1}^{\infty} \left(\alpha_n^\mu \alpha_{n,\mu} e^{-i2n\tau} + \alpha_n^{\dagger\mu} \alpha_{n,\mu}^{\dagger} e^{i2n\tau} + \alpha_n^\mu \alpha_{n,\mu}^{\dagger} + \alpha_n^{\dagger\mu} \alpha_{n,\mu} \right) + \frac{1}{4} \sum_{n=1}^{\infty} \left(-\alpha_n^\mu \alpha_{n,\mu} e^{-i2n\tau} - \alpha_n^{\dagger\mu} \alpha_{n,\mu}^{\dagger} e^{i2n\tau} + \alpha_n^\mu \alpha_{n,\mu}^{\dagger} + \alpha_n^{\dagger\mu} \alpha_{n,\mu} \right) \\ &= \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \frac{1}{2} \sum_{n=1}^{\infty} \left(\alpha_n^\mu \alpha_{n,\mu}^{\dagger} + \alpha_n^{\dagger\mu} \alpha_{n,\mu} \right). \end{aligned} \quad (74)$$

$$\text{Using} \quad \alpha_n = \sqrt{n} a_n, \quad n \text{ a positive integer}, \quad (75)$$

$$H = \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \frac{1}{2} \sum_{n=1}^{\infty} \left(n a_n^{\dagger\mu} a_{n,\mu} + n a_{n,\mu} a_n^{\dagger\mu} \right). \quad (76)$$

As noted in the paragraph after (25), (76) can be re-expressed with all lower spacetime indices, for which $a_n^{\dagger\mu} a_{n,\mu} = N_a(n)$ is the number operator, as

$$H = \frac{1}{2} \alpha_{0\mu} \alpha_{0\mu} + \frac{1}{2} \sum_{n=1}^{\infty} \left(n \zeta_\mu a_n^{\dagger\mu} a_{n\mu} + n \zeta_\mu a_{n\mu} a_n^{\dagger\mu} \right) \quad \zeta_0 = -1, \quad \zeta_1 = \zeta_2 = \zeta_3 = 1 \quad (77)$$

We then employ the commutation relations, similar to those of QFT,

$$\begin{aligned} \left[a_{n\mu}, a_{n\mu}^{\dagger} \right] &= \eta_{\mu\mu} \\ a_{n\mu} a_{n\mu}^{\dagger} &= a_{n\mu}^{\dagger} a_{n\mu} + \eta_{\mu\mu} \end{aligned} \quad \text{no sum on } \underline{\mu}, \quad \eta_{\mu\mu} = g_{\mu\mu} \text{ in Minkowski spacetime.} \quad (78)$$

With (78) for the last summation term in (77), we get

$$\begin{aligned} H &= \frac{1}{2} \alpha_{0\mu} \alpha_{0\mu} + \frac{1}{2} \sum_{n=1}^{\infty} \left(n \zeta_\mu \left(a_{n\mu}^{\dagger} a_{n\mu} + a_{n\mu} a_{n\mu}^{\dagger} \right) + \eta_{\mu\mu} \right) = \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \sum_{n=1}^{\infty} n a_n^{\dagger\mu} a_{n,\mu} + \frac{1}{2} \sum_{n=1}^{\infty} \underbrace{\zeta_\mu \eta_{\mu\mu}}_{=D'} n \\ &= \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \sum_{n=1}^{\infty} n N_a(n) + \frac{D'}{2} \sum_{n=1}^{\infty} n. \end{aligned} \quad (79)$$

At this point, as it turns out, one needs to postulate that for the last term in (79) strings have no measurable longitudinal nor timelike modes of oscillation, or string theory will not work out. This is, in fact, true for massless fields in our world (like photons or any field before Higgs symmetry breaking), and may be (in fact, must be) considered a property of quantum relativistic strings.

Since we assume (see Sect. 1.2) that we cannot detect position of a point on the string, we can only measure motion transverse to the string and not along it (not longitudinally). Additionally, oscillation is motion in space over time. So, considering there are no longitudinal nor temporal oscillation of quantum strings is quite reasonable.

New postulate: Since two oscillation modes do not exist, the number of independent transverse modes is two less than the number of dimensions D (which from (79) equals D). See Zwiebach, pg. 580, sentence after (24.55). Then, (79) becomes

$$H = \frac{1}{2} \alpha_0^\mu \alpha_{0,\mu} + \sum_{n=1}^{\infty} n a_n^{\dagger\mu} a_{n,\mu} + \frac{D-2}{2} \sum_{n=1}^{\infty} n. \quad (80)$$

Problem with the postulate: Of course, if there were no longitudinal or timelike modes of oscillation, then we would not have the $\mu=0$ oscillation terms in the summations of (80) and the other (spatial) dimensions would be constrained such that no longitudinal oscillation could occur either for the string. Yet, theorists keep $\mu=0$ terms in (80). (See Zwiebach, pg. 579, discussion after (24.19).)

This may give one justifiable pause with regard to string theory's internal consistency.

Re-writing (80) using (16), and the underbrace relation of (24.9) [also in (63) herein],

$$\alpha_0^\mu = \sqrt{2\alpha'} p^\mu \quad (81)$$

we have

$$H = \alpha' p^\mu p_\mu + \sum_{n=1}^{\infty} n a_n^{\dagger\mu} a_{n,\mu} + \frac{D-2}{2} \sum_{n=1}^{\infty} n. \quad (82)$$

A complicated analysis determines that for bosonic strings Lorentz invariance only holds for $D = 26$.

11 Appendix C. Deducing H the Way Most Texts Do It

11.1 Constraint for Our Choice of Gauge Family

From Appendix A (60), we know the expression

$$(\dot{X}^\mu \pm X'^\mu)(\dot{X}_\mu \pm X'_\mu) = (\dot{X} \pm X')^2 = 0 \quad (24.13)[571] \quad (83)$$

represents constraints on τ and σ (through the derivatives) that make τ and σ behave in ways we have previously described in Sect. 2. That is, (83) will limit these parameters to a gauge family where τ always increases with increasing t , and lines of constant σ are orthogonal to lines of constant τ . (83) essentially defines that gauge family (though you can't see why without study of Appendix A). And thus, (83) yields the \mathcal{H} of (14), which was found using this same gauge constraint.

By plugging the τ and σ derivatives of X^μ (see (63) and (64) in Appendix B), into (83), we find, after some algebra,

$$(\dot{X} \pm X')^2 = \frac{1}{2\pi T_0} \sum_{n \in \mathbb{Z}} L_n e^{-in(\tau \pm \sigma)} = 0 \quad \text{where} \quad L_n = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^\mu \alpha_{p,\mu}. \quad (84)$$

In classical theory, this would mean $L_n = 0$ for each n . But in quantum theory, where L_n is an operator, it means, where $|\phi\rangle$ is any (quantum string) state, we expect, where we note L_n is not normal ordered,

$$L_n |\phi\rangle = 0. \quad (85)$$

(85) is the equivalent of our gauge condition (83). The L_n are called Viasoro operators. We herein will only be interested in L_0 .

$$L_0 = \frac{1}{2} \alpha_{0,\mu} \alpha_0^\mu + \frac{1}{2} \sum_{\substack{p \in \mathbb{Z} \\ p \neq 0}} \alpha_{-p}^\mu \alpha_{p,\mu}. \quad (86)$$

There are additional subtleties with (85) we won't delve into here. Suffice for present purposes that (85) is true as stated for $n = 0$.

11.2 Re-expressing L_0 to Find It Equals the String Hamiltonian

With Appendix B relation (69), from (12.58)[245] of Zwiebach,

$$\alpha_{-p} = \alpha_p^\dagger \quad p \text{ a positive integer}, \quad (87)$$

and Appendix B relation (75), from (12.56)[244],

$$\alpha_p = \sqrt{p} a_p, \quad p \text{ a non-zero integer}, \quad (88)$$

(86) becomes

$$\begin{aligned} L_0 &= \frac{1}{2} \alpha_{0\mu} \alpha_0^\mu + \frac{1}{2} \sum_{p=1}^{\infty} (\alpha_p^\mu)^\dagger \alpha_{p\mu} + \frac{1}{2} \sum_{p=1}^{\infty} \alpha_p^\mu (\alpha_{p\mu})^\dagger = \frac{1}{2} \alpha_{0\mu} \alpha_0^\mu + \frac{1}{2} \sum_{p=1}^{\infty} p (a_p^\mu)^\dagger a_{p\mu} + \frac{1}{2} \sum_{p=1}^{\infty} p a_p^\mu (a_{p\mu})^\dagger \\ &= \frac{1}{2} \alpha_{0\mu} \alpha_0^\mu + \frac{1}{2} \sum_{p=1}^{\infty} p \left((a_p^\mu)^\dagger a_{p\mu} + a_p^\mu (a_{p\mu})^\dagger \right). \end{aligned} \quad (89)$$

Employing commutation relations, such as postulated in all quantum field theories (of which string theory is one),

$$\left[(a_p^\mu)^\dagger a_{q\nu}, a_{q\nu} (a_p^\mu)^\dagger \right] = \delta_\nu^\mu \delta_{pq}, \quad (90)$$

to substitute for the last term in (89), we have

$$L_0 = \frac{1}{2} \alpha_{0\mu} \alpha_0^\mu + \sum_{p=1}^{\infty} p \left(a_{p\mu}^\dagger a_p^\mu + \frac{1}{2} \delta_\mu^\mu \right) = \frac{1}{2} \alpha_{0\mu} \alpha_0^\mu + \sum_{p=1}^{\infty} p a_p^\dagger{}^\mu a_{p\mu} + \frac{D'}{2} \sum_{p=1}^{\infty} p, \quad (91)$$

where D' is the number of dimensions the string oscillates in. Since, as we've said, quantum strings do not oscillate in the time or transverse directions (see (27),

$$D' = D - 2, \quad (92)$$

where D is the total number of spacetime dimensions. Thus, (91) becomes

$$L_0 = \frac{1}{2} \alpha_{0\mu} \alpha_0^\mu + \sum_{p=1}^{\infty} p a_p^\dagger{}^\mu a_{p\mu} + \frac{D-2}{2} \sum_{p=1}^{\infty} p. \quad (93)$$

Lo and behold, (93) equals our string Hamiltonian (27). That is (with no normal ordering),

$$H = L_0. \quad (94)$$

From our gauge condition (85) defining the family of gauges we are working in, where $|\phi\rangle$ is a string state,

$$L_0 |\phi\rangle = H |\phi\rangle = 0 \quad \text{for strings in this gauge family.} \quad (95)$$

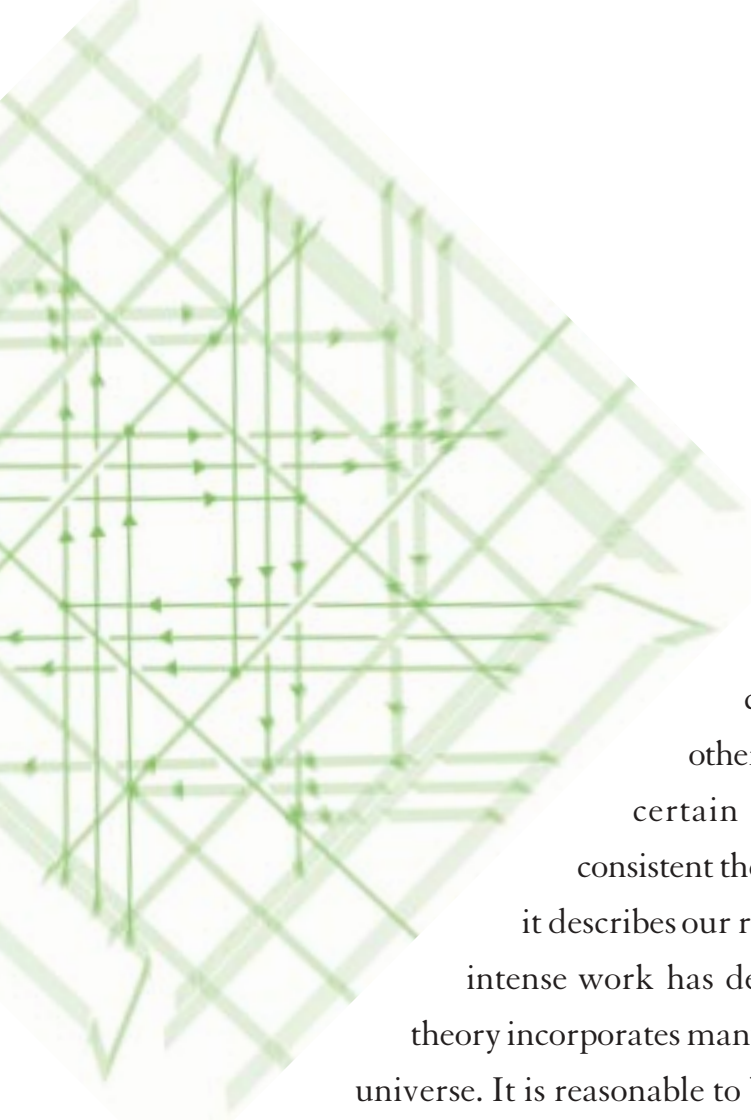
As noted in the main text, this may seem strange, as we are used to the Hamiltonian acting on a state to yield the energy eigenvalue of the state times the original state. But recall what we said earlier, below (18). The Hamiltonian here is *not* energy. It is an operator in string theory that helps us in analysis, and from which we can obtain things like the Lagrangian (via the Legendre transformation), but it is not energy.

Finding L_0 from our gauge condition ((83), equivalent to (85)) allowed us to show that it equals the string Hamiltonian, as in (95). The bottom line is that particular finding tells us that the Hamiltonian acting on any string state yields zero.

Barton Zwiebach

5 From Vibrating Strings to a Unified Theory of All Interactions

For the last twenty years, physicists have investigated String Theory rather vigorously. The theory has revealed an unusual depth. As a result, despite much progress in our understanding of its remarkable properties, basic features of the theory remain a mystery. This extended period of activity is, in fact, the second period of activity in string theory. When it was first discovered in the late 1960s, string theory attempted to describe strongly interacting particles. Along came Quantum Chromodynamics—a theory of quarks and gluons—and despite their early promise, strings faded away. This time string theory is a credible candidate for a theory of all interactions—a unified theory of all forces and matter. The greatest complication that frustrated the search for such a unified theory was the incompatibility between two pillars of twentieth century physics: Einstein’s General Theory of Relativity and the principles of Quantum Mechanics. String theory appears to be



the long-sought quantum mechanical theory of gravity and other interactions. It is almost certain that string theory is a consistent theory. It is less certain that it describes our real world. Nevertheless, intense work has demonstrated that string theory incorporates many features of the physical universe. It is reasonable to be very optimistic about the prospects of string theory.

Perhaps one of the most impressive features of string theory is the appearance of gravity as one of the fluctuation modes of a closed string. Although it was not discovered exactly in this way, we can describe a logical path that leads to the discovery of gravity in string theory. One considers a string, similar in many ways to the vibrating strings with tension and mass that are studied in freshman physics. This time, however, the string is relativistic. This means that the *classical* mechanics of this string is consistent with Einstein's special theory of relativity. A relativistic string is, in fact, a very interesting and subtle object with a rich spectrum of vibration modes. These classical vibrations, however, cannot be identified with physical particles. But quantum theory comes to the rescue: the quantum mechanics of the relativistic string gives vibration modes that *can* be identified with physical particles! A particular quantum vibration mode of the closed string describes a *graviton*, the quantum of the gravitational field. A particular quantum vibration of an open string describes a *photon*, the quantum of the electromagnetic field. It is the magic of quantization that makes these results possible. In string theory all particles—matter particles and force carriers—arise as quantum fluctuations of the relativistic string. Physicists struggled to invent a quantum theory of gravity during much of the twentieth century, and the answer came from the quantization of classical

relativistic strings. We are reminded of an opinion expressed by Dirac in 1966 [1]:

“The only value of the classical theory is to provide us with hints for getting a quantum theory; the quantum theory is then something that has to stand in its own right. If we were sufficiently clever to be able to think of a good quantum theory straight away, we could manage without classical theory at all. But we’re not that clever, and we have to get all the hints that we can to help us in setting up a good quantum theory.”

The purpose of this article is to explain some of the unusual features of relativistic strings and to show one way in which string theory may describe the Standard Model of particle physics.

What are relativistic strings?

To gain some understanding of relativistic strings, we can compare them with the more familiar nonrelativistic strings. Nonrelativistic strings are typically characterized by two independent parameters: a string tension T_0 and a mass per unit length m_0 . Each of the four strings on a violin, for example, has a different tension and mass density. When a string with fixed endpoints is also static, the direction along the string is called the *longitudinal direction*. Such a string can exhibit small

transverse oscillations (*Figure 1a*). In this case, the velocity of any point on the string is orthogonal to the longitudinal direction. The velocity v of a transverse wave moving along the string is a simple function of the tension and the mass per unit length:

$$v = \sqrt{T_0/m_0}. \quad (1)$$

A nonrelativistic string may support a different type of oscillation. When we have a longitudinal oscillation, the velocity of any point on the string remains along the string (*Figure 1b*). In a longitudinal oscillation the wave velocity does not involve the tension, but rather a tension coefficient that describes

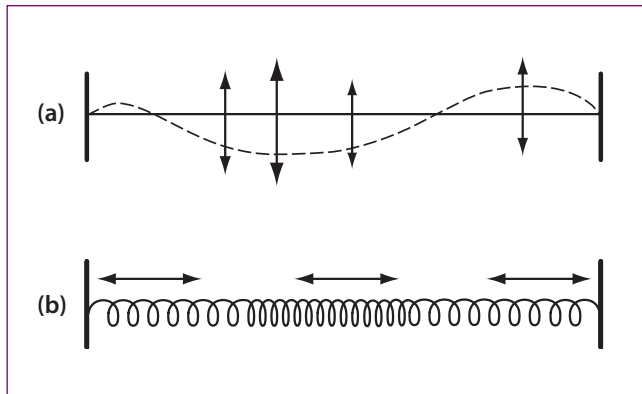


FIGURE 1

(a) In a transverse oscillation the motion of any point on the string is perpendicular to the longitudinal direction. (b) In a longitudinal oscillation the motion of any point on the string (represented by a thin slinky) is along the direction of the string. In order to detect longitudinal motion we must be able to tag the points along the string.

how the tension changes upon small stretching of the string. More important, a longitudinal wave requires the existence of structure along the string. In order to tell that the various points of the string are really oscillating we must be able to tag them. If this is not possible, a longitudinal oscillation is undetectable because, as a whole, the string does not move. Transverse motion is less subtle; we can always tell when the string moves away from the equilibrium longitudinal direction.

It takes a significant amount of imagination to construct the classical mechanics of relativistic strings. In fact, the mechanics is simplest for the so-called “massless relativistic string.” This is the string that one quantizes to obtain string theory. To gain intuition, let’s discuss four surprising properties of these strings.

- (1) The relativistic string is characterized by its tension T_0 alone—there is no independent mass density parameter. The velocity of transverse waves on

continues on page 46

String Theory for Undergraduates? The Story Behind 8.251

When, in the fall of 2001, distinguished string theorist Professor Barton Zwiebach first proposed to the Physics Education Committee a new elective for the Department's undergraduate curriculum based upon his upcoming textbook, "A First Course in String Theory," the response was a moment of startled silence. How, the Committee members wondered, could an area of physics be taught to undergraduates that was built upon intellectual concepts viewed as challengingly opaque by not a few of the faculty?

Nevertheless, Zwiebach's talent and reputation as one of the most gifted instructors at MIT was well-established, so the funds were granted to develop the new elective, "8.251: String Theory for Undergraduates." Launched in the Spring 2002 semester and repeated annually, the class continues to attract an equal number of undergraduate and graduate students. It has been so successful that Zwiebach received the 2002 Everett Moore Baker Memorial Award for Excellence in Undergraduate Teaching, the only MIT prize whose winner is chosen solely by undergraduates.

Comments from his students show a keen appreciation of both the topic and the teacher.

"The class itself was often fantastic. Midway through the term, when we'd reached Chapter 10 in his book, Prof. Zwiebach announced that we had done three semesters of quantum field theory in one lecture. It was a heady feeling..."

"I'll always be grateful for 8.251. Unlike most classes around here, it left a warm and fuzzy spot in my heart. It has had a practical payoff, too: learning to handle commutator relations early gave me a jump over my 8.05 [Quantum Physics II] compadres, and seeing Lagrangian dynamics early let me delve into journal articles with less trepidation. I had a great time with my 8.06 [Quantum Physics III] term paper, mostly because Prof. Zwiebach's class introduced me to fruitful new concepts I could then turn around and apply elsewhere, giving me that spine-tingling shiver of knowledge fitting together."

—Blake C. Stacey (SB '04)

"Barton Zwiebach's course finally bridges the gap between theoretical physics as taught on the undergraduate level and its current frontier, string theory. Before taking this course, I was convinced one would need to learn very sophisticated mathematical tools

before one could try to understand, even on a basic level, what string theory is about. [Thus] it was very impressive, and intellectually very satisfying, to see from Zwiebach's class that basic knowledge about classical and quantum mechanics is sufficient to get a head start in this subject....To make this theory accessible to students at the undergraduate level can hardly be overestimated in its importance."

—Martin Zwierlein, Graduate Student, Atomic Physics

"Originally I decided to take the class because string theory is...on the frontier of physics, and this class proposed to teach me the subject (at least some parts of it) with a minimum of previously required knowledge..."

"The class itself was a novel way of teaching the topic and...quite different from the way string theory is taught in other texts. Instead of beginning with abstract field theoretic concepts, 8.251 started rooted in the physics that we were all familiar with: the mechanics of a simple string. It all started there and quickly went through many iterations until arriving at the quantum mechanics of relativistic strings. Though at times the math was difficult, as is unavoidable in this subject, the concepts were very clear throughout the journey, which also included a discussion of the theory of branes, T-duality and a few exotic topics like string thermodynamics and black holes."

"One thing that is for certain is that the class would simply not be the same without Prof. Zwiebach; his clear lecturing and willingness (and ability) to answer questions was great. I enjoyed the class to the point of volunteering to help look over the chapters of the textbook that were yet to be written, because I really wanted to see more material on this subject."

"All in all, the class was very exciting; it was unlike most other physics classes at MIT and remains among my favorites."

—Alan M. Dunn (SB '04)

For a more detailed look at 8.251: STRING THEORY FOR UNDERGRADUATES, visit the class home page at <http://mit.edu/8.251/www/>. The class textbook, *A First Course in String Theory* [2], is available from Cambridge University Press (<http://publishing.cambridge.org/stm/physics/strings/>).

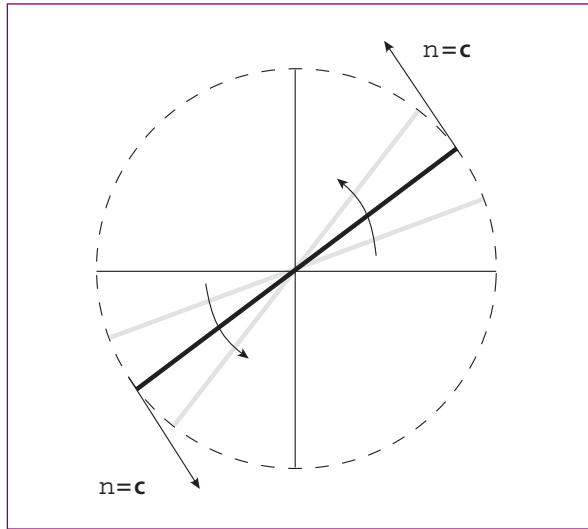


FIGURE 2

A relativistic open string can rotate rigidly about its midpoint. The angular velocity must be such that the endpoints move with the speed of light.

continued from page 32

this string is the velocity of light c , so using (1) the mass density m_0 is fixed once T_0 is fixed:

$$c = \sqrt{T_0/m_0} \quad m_0 = T_0/c^2. \quad (2)$$

Special relativity tells us that mass and energy are interchangeable, but familiar examples involve quantum processes, such as massive particles that annihilate into energetic (zero-mass) photons. In the relativistic string, energy/mass conversion occurs classically. Imagine beginning with an infinitesimally short relativistic string and stretching it out to some length L . Since the string tension is constant, the work done on the string is equal to the product T_0L of the tension times the length. This energy makes up the rest mass of the string. Energy is converted into rest mass by stretching the string! The mass is equal to the energy divided by c^2 , so it equals T_0L/c^2 . Consequently,

the mass per unit length is T_0/c^2 , as anticipated in (2). The relativistic string has no intrinsic mass; the mass arises from work done against the tension.

- (2) The relativistic string does not support longitudinal oscillations. This is a revealing fact: it tells us that the string has no substructure. The points along the relativistic string cannot be tagged in an unambiguous way. When a string moves a little, we cannot really tell which point went where. There is a minor exception: if we have an open string, we can keep track of the motion of the endpoints, which, after all, are points. Many times people ask, What is the string made of? The lack of longitudinal oscillations tells us that no meaningful answer can be provided: the classical relativistic string has no constituent parts that can be identified.
- (3) The endpoints of a free relativistic open string move with the speed of light. For familiar strings, oscillations require that the motion of the endpoints be constrained. The simplest constraint is to fix the endpoints; the string can then have a nonzero tension and oscillations are possible. Nontrivial motion is possible for relativistic open strings even if the endpoints are not fixed. Elementary mechanics suggests that the effective tension of the string must vanish at the endpoints. This is actually achieved when the endpoints move at the speed of light. One of the simplest open string motions is that of an open string that rotates rigidly about its midpoint (Figure 2). This motion has an unusual property: the angular momentum J of this string is linearly proportional to the square of the energy E of the string:

$$J = \alpha E^2 \quad (3)$$

The constant of proportionality α is called the slope parameter. The above property was the reason why physicists attempted (and still attempt!) to use some kind of string theory to describe strongly interacting particles. Indeed, hadronic resonances fit rather accurately a linear relation between angular momentum

and the square of the mass (or the square of the energy). This relation is completely unusual: for a rigid bar rotating nonrelativistically about its midpoint, one finds the rather different $J \propto E$. Equation (3) can be understood roughly by assuming that the mass of the string is concentrated at the endpoints. Since the speed of the endpoints is constant and equal to the speed of light, the angular momentum is proportional to the length of the string times the mass of the string. Given that both the length and the mass of the string are proportional to its energy, the angular momentum is proportional to the energy squared.

- (4) A relativistic string has an orientation which determines the sign of the string charge. Consider an electron and its antiparticle, the positron. They are oppositely charged point particles. Being zero-size points, there is no intrinsic geometrical property that distinguishes their charges. This is different in string theory. Relativistic strings come with an orientation. For a closed string, the orientation is an arrow that defines a preferred direction along the string. One can travel along a closed string in two directions; the orientation picks one out of these two (see Figure 3). It turns out that oppositely oriented strings have opposite *string* charges. In contrast to the case of point particles, in string theory the sign of charge has a geometrical basis. While string charge is a novel concept, the implications for open strings are readily understood. To specify an open string we must choose a direction or draw an arrow along the string. This arrow creates a clear-cut distinction between the two, previously similar, endpoints: the arrow points away from one endpoint, called the beginning endpoint, and towards the other endpoint, called the final endpoint. A surprising effect then takes place: the string charge forces the open string endpoints to acquire opposite *electric* charges! String

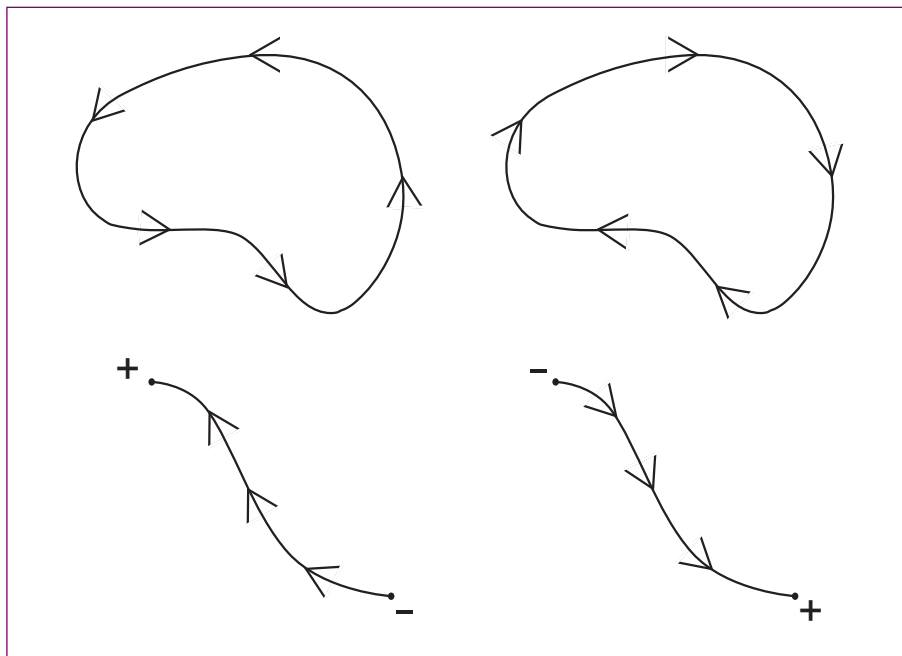


FIGURE 3

Relativistic strings carry orientation, a direction of travel along the string indicated by arrows. **Top line:** two oppositely oriented closed strings are states with opposite string charge. **Bottom line:** two oppositely oriented open strings. The endpoints of open strings carry ordinary electric charge. The charges at the open string endpoints are opposite: (+) at the final endpoint and (-) at the beginning endpoint.

charge transmutes into electric charge. The orientation points from the negatively charged to the positively charged endpoint. Since open strings carry electric charges, we may attempt to identify known charged particles with excitations of open strings.

The above properties, derived in the classical theory of strings, remain true in the quantum theory of strings. Further surprises emerge, however, when relativistic strings are quantized. One finds that quantum mechanical strings cannot propagate consistently in spacetimes of arbitrary dimensionality. For the simplest—bosonic strings—the spacetime must be twenty-six dimensional. For superstrings, strings whose excitations include bosons and fermions, the dimensionality of spacetime is ten, one of time and nine of space. Quantization also implies that strings have quantum states of oscillation. This allows us to identify the oscillations of strings with particles, which are themselves quanta of familiar fields. The masses of the particles associated with string oscillations are computed using the quantum theory. While closed string oscillations that could be identified with gravitons have positive mass in the classical theory, their mass turns out to be exactly zero in the quantum theory! This is precisely what is needed, since gravitons are exactly massless particles. There was no reason to expect gravity to arise from fluctuating strings, but it does. Quantum relativistic strings provide a theory of quantum gravity. A related effect occurs for open strings: massless oscillations of open strings represent photons.

Building blocks of the Standard Model

There are four known forces in nature. The Standard Model of particle physics summarizes the present-day understanding of three of them. It describes the electromagnetic force, the weak force and the strong force, but leaves out the gravitational force. The Standard Model also describes the elementary particles that have been discovered so far.

The electromagnetic force is transmitted by photons, the quanta of the electromagnetic field. The weak force is responsible for the process of nuclear beta decay, in which a neutron decays into a proton, an electron and an anti-neutrino. The strong force or color force holds together the constituents of the neutron, the proton, the pions and many other subnuclear particles. These constituents, called quarks, are held so tightly by the color force that they cannot be seen in isolation.

In the late 1960s the Weinberg-Salam model of *electroweak* interactions put together electromagnetism and the weak force into a consistent, unified framework. The theory is initially formulated with four massless particles that carry the forces. A process of symmetry breaking gives mass to three of these particles: the W^+ , the W^- , and the Z^0 . These particles are the carriers of the weak force. The particle that remains massless is the photon. The theory of the color force is called quantum chromodynamics (QCD). The carriers of the color force are eight massless particles, colored gluons that, just as the quarks, cannot be observed in isolation. The quarks respond to the gluons because they carry color; in fact, quarks come in three colors. The electroweak theory together with QCD form the Standard Model of particle physics.

Since we aim to show how the familiar particles and interactions may arise in string theory, we now summarize the particle content of the Standard Model. We have already said that gravity appears automatically in string theory as a fluctuation of closed strings. Therefore, we will not focus on gravity, but rather on the other force carriers and the matter particles, both of which arise from vibrations of open strings.

The Standard Model includes twelve force carriers: eight massless gluons, the W^+ , W^- , Z^0 , and the photon. All of them are bosons. There are also many matter particles, all of which are fermions. The matter particles are of two types: leptons and quarks. The leptons include the electron e^- , the muon m^- , the tau τ^- , and the associated neutrinos ν_e , ν_m , and ν_τ . We can list them as

$$\text{Leptons : } (\nu_e, e^-), (\nu_m, m^-), \text{ and } (\nu_\tau, \tau^-). \quad (4)$$

Since we must include their antiparticles, this adds up to a total of twelve leptons. The quarks carry color charge electric charge and respond to the weak force, as well. There are six different types or “flavors” of quarks: up (u), down (d), charm (c), strange (s), top (t), and bottom (b). We can list them as

$$\text{Quarks : } (u, d), (c, s), \text{ and } (t, b). \quad (5)$$

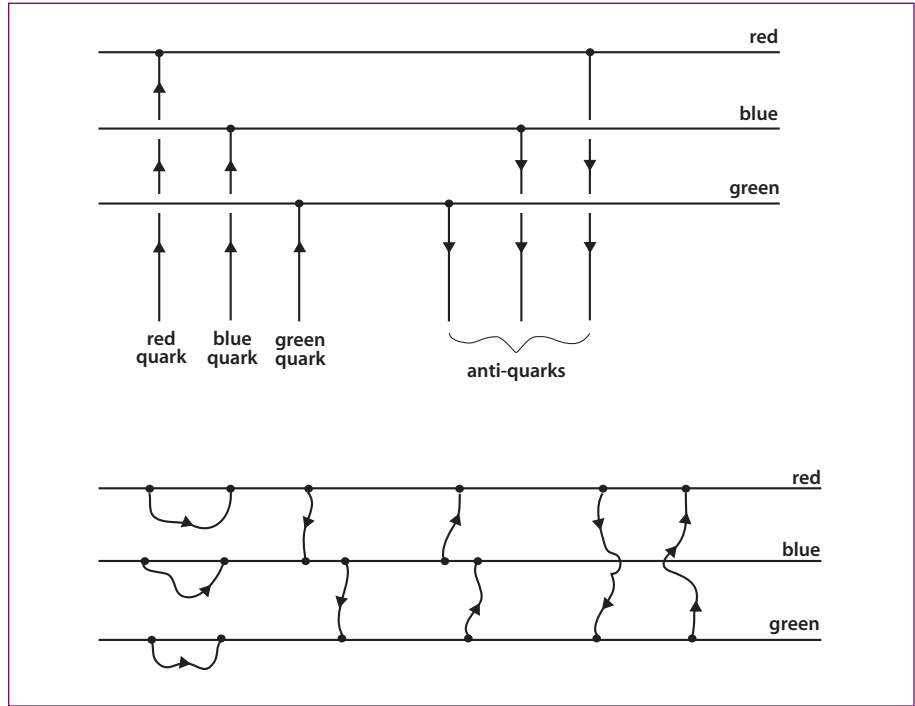
The u and d quarks, for example, carry different electric charges and respond differently to the weak force. Each of the six quark flavors listed above comes in three colors, so this gives $6 \times 3 = 18$ particles. Including the antiparticles, we get a total of 36 quarks. Adding leptons and quarks together we have a grand total of 48 matter particles.

Although the matter particles displayed above and some of the gauge bosons have masses, these masses are in some sense remarkably small. Consider the fundamental constants of nature: Newton’s gravitational constant, the speed of light and Planck’s constant. Since there are three basic units—those of mass, length and time—there is a unique way to construct a quantity with the units of mass using only the three fundamental constants. The resulting mass is called the Planck mass and its numerical value is about 2.23×10^{-5} grams. While ordinary by the standards of macroscopic objects, this prototype mass is extraordinarily large when compared with the masses of elementary particles: it is twenty-two orders of magnitude larger than the mass of the electron, for example. It is in this sense that elementary particles are essentially massless.

The *chirality* of the electroweak interactions guarantees that the matter particles cannot acquire masses until electroweak symmetry breaking takes place. If one adjusts the scale of electroweak symmetry breaking to be small, the matter particles will be light. To understand the meaning of chirality, we recall that particles with spin are described in terms of left-handed and right-handed states. If the spin angular momentum points along the direction of the motion, the particle is said to be right-handed; if the spin angular momentum points opposite to the direction of motion, the particle is said to be left-handed. A left-handed electron, for example, is denoted as e_L^- and a right-handed electron is denoted as e_R^- . The

FIGURE 4

Top: Three parallel D-branes (shown as horizontal lines) are needed to produce the color interactions. The branes can be labeled by colors: red, blue and green. The left-handed quarks are open strings that end on the colored branes. A red quark, for example, is a string that ends on the red brane. The left-handed antiquarks are open strings that begin on the colored branes. **Bottom:** The open strings that begin and end on the brane configuration are gauge bosons. This brane configuration supports nine gauge bosons, eight of which are the gluons of QCD.



electroweak interactions are chiral because the left-handed states and the right-handed states of the Standard Model particles respond differently to the weak forces; there is a fundamental left-right asymmetry. If we focus on the electron and the neutrino, for example, we have:

$$\begin{pmatrix} \nu_{eL} \\ e_{eR} \end{pmatrix}, e_{eR}^-, \nu_{eR}. \tag{6}$$

The left-handed states in the doublet feel the weak interactions, while the right-handed electron and the right-handed neutrino states do not. A similar situation holds for the quarks. The left-handed states of the u and d quarks feel the weak interactions while the right-handed states do not:

$$\begin{pmatrix} u_L \\ d_L \end{pmatrix}, u_R, d_R. \tag{7}$$

The existence of mass requires couplings between left- and right-handed states that are not allowed as long as chirality holds.

D-branes and the Standard Model

D-branes are extended objects in string theory. Whenever we have open strings we also have D-branes, since the endpoints of open strings must lie on them. D-branes come in various dimensionalities. A Dp -brane is a D-brane that has p spatial dimensions. A D2-brane, for example, may look like a sheet of paper, and a D1-brane may look like a string. In four-dimensional spacetime, a D3-brane may fill the full extent of the three spatial dimensions, in which case we have a space-filling brane. Since D-branes extend in various dimensions we can imagine observers that live on D-branes.

String endpoints carry electric charge so, in order to represent a charged particle, we arrange to have a string with one endpoint lying on the D-brane. The other string endpoint must lie on another, possibly separate D-brane, otherwise the string would represent two oppositely charged particles, for a net of zero charge. A positively charged particle is represented by a string that ends on the D-brane, while a negatively charged particle is represented by a string that begins on the D-brane. In fact, the photons that couple to these charges arise from open strings with *both* endpoints on the D-brane. As required, these states have zero charge.

How can we get quarks using D-branes and strings? Since color is just a whimsical label for a kind of charge, to obtain three types of color we simply use three different D-branes: a red brane, a blue brane, and a green brane (*Figure 4*). To represent quarks we use strings that have one endpoint on a colored brane while the other endpoint lies on a different collection of branes, to be specified later. A string that ends on a green brane is a green quark, a string that ends on a blue brane is a blue quark, and a string that ends on a red brane is a red quark. Strings that *begin* on the colored branes are antiquarks. On the other hand, the gluons—the carriers of the color force—arise from strings that begin *and* end on the colored branes. With three D-branes, there are a total of nine such strings. Out of these, eight of them are the gluons we are looking for.

For any quark, one endpoint of the corresponding string lies on a color brane. Where does the other endpoint lie? The answer becomes clear once we consider the weak interactions. In order to produce the four gauge bosons of the electroweak interactions we need two new D-branes, two “weak branes.” To draw the brane configuration, it is convenient to use a plane and represent the D-branes as lines. We take the weak branes to intersect the color branes, as shown in *Figure 5*. Let’s now consider the two flavors of quarks indicated in (7). Since the left-handed

u and d quarks feel the weak interactions (in addition to the color force) the strings that represent them must have their other endpoint on a weak brane. Given that we have two weak branes, we have a perfect fit: the left-handed u quarks are strings stretched from one of the weak branes to the color branes, while the left-handed d quarks are strings stretched from the other weak brane to the color branes.

The strings that represent the left-handed quarks stretch from one kind of brane to the other. The string tension forces the strings to have the minimum possible length, which in this case is zero, so they represent massless states that live at the points

FIGURE 5

The color branes intersect the two weak branes (shown vertically). Open strings localized near the intersection that stretch from the weak branes to the color branes are left-handed quarks. The two flavors of quarks (u and d) arise because there are two weak branes.

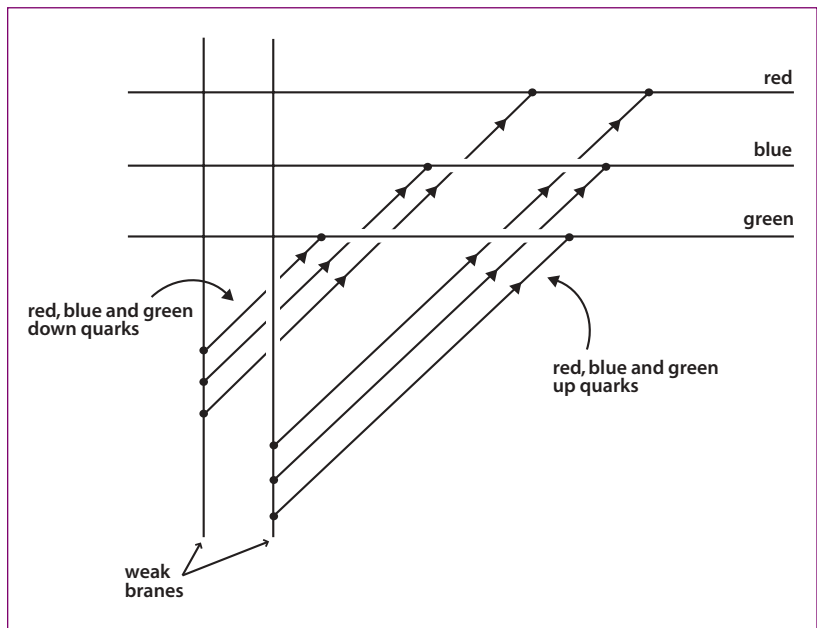
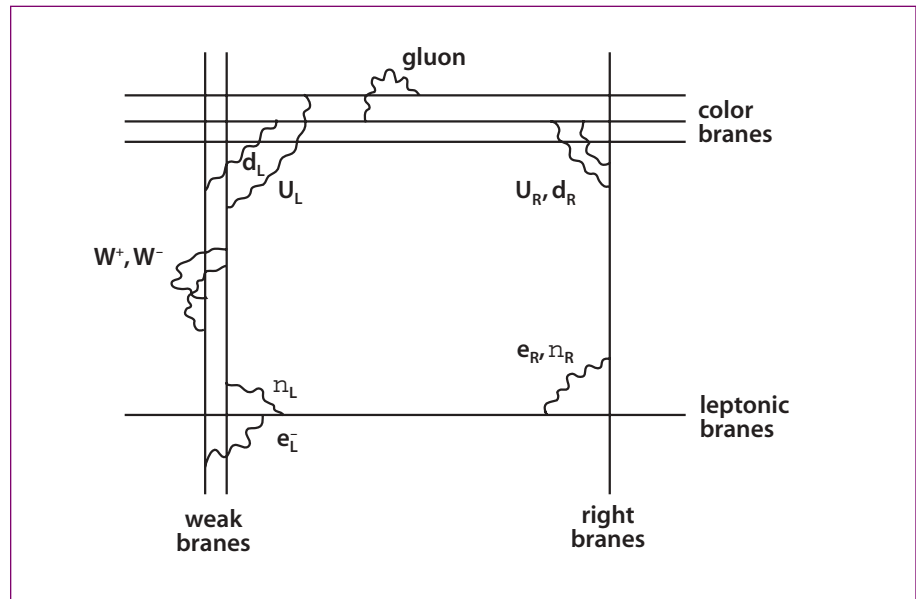


FIGURE 6

The full D-brane configuration in which open strings represent the familiar particles of the Standard Model. The vertical branes to the right are called “right branes” because they support the right-handed quarks (which do not feel the weak interactions). The horizontal branes at the bottom are called “leptonic branes” since they support the left-handed leptons (to the left) and the right-handed leptons (to the right).



where the branes intersect. Chirality is a property that guarantees that mass cannot be readily acquired. When D-branes intersect, there is no small displacement of the branes that eliminates the intersection, so the strings that stretch from one brane to the other cannot acquire mass. Chirality is a property of states that arise at brane intersections.

How do we get the right-handed quarks listed in (7)? Since these states feel the color force, the strings have one end on the color branes. Since they do not feel the weak interactions, they cannot have their other endpoint on the weak branes and consequently we need new branes. These new “right branes” must intersect the color branes for the states to be massless. As shown in *Figure 6*, the right-handed quarks stretch from the right branes to the color branes.

Let’s now consider the lepton doublet that includes the left-handed neutrino and electron [see (6)]. These particles feel the weak interactions, so they are represented by strings that have one endpoint on the weak branes. Since they do not feel the strong interactions, the other endpoint in those strings must end on new “leptonic branes.” The left-handed leptons are shown as strings localized at the intersection of the weak branes and the leptonic branes. Finally, let’s consider the right-handed electron e_R^- and the right-handed neutrino ν_R . These particles feel neither the color force nor the weak force. They are represented by strings that stretch from right branes to leptonic branes, as shown at the bottom right corner of *Figure 6*.

We have exhibited the states that comprise one family of the Standard Model. The Standard Model has two additional families, with states completely analogous to those described in (6) and (7). These are obtained with additional intersections. The first models to give the precise spectrum of the Standard Model were constructed in 2001 by Ibanez, Cremades and Marchesano [3].

We have so far imagined the D-branes as D1-branes that are stretched on a two-dimensional plane. Let us finally show how the brane configuration fits into a ten-dimensional superstring theory. A physical setup requires an effective four-dimensional spacetime, so six of the spatial dimensions must curl up into a compact space of small volume. To visualize the brane configuration we assume that two out of the six extra dimensions are curled up into a two-dimensional torus (*Figure 7*). The D-branes are all chosen to be D4-branes, and three out of their four spatial directions fill our space. The last direction is chosen to appear as a line on the two-dimensional compact torus. So, in fact, *Figure 6* was a picture of the D-branes as seen on the torus, a close-up that does not quite show how the D-branes are fully wrapped around the torus. The strings that represent the Standard Model particles are localized at the brane intersections and are perceived as particles.

While there are string constructions that give precisely the matter content of the Standard Model, no one claims to have a derivation of particle physics from strings. For this, one must also show that symmetry breaking works out correctly and particles acquire their familiar masses. This has not yet been done. I hope, however, to have demonstrated that familiar features of our observed universe can emerge from string theory.

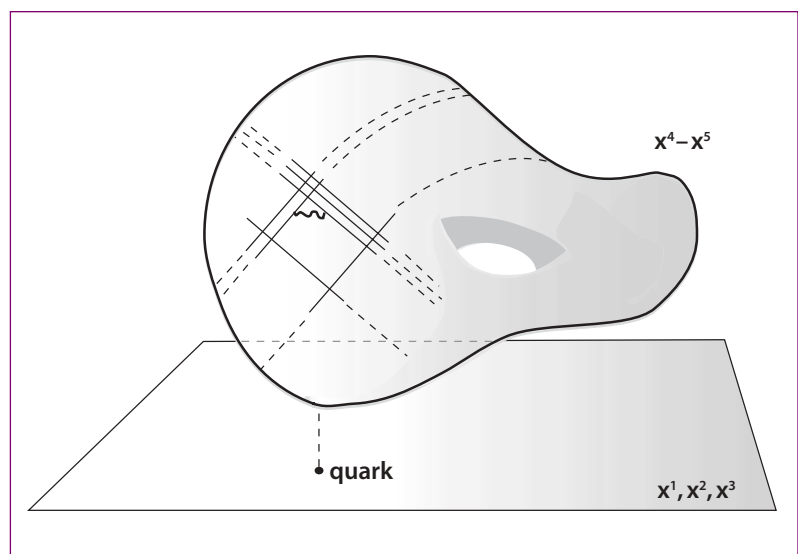
Outlook

We may wonder what are the possible outcomes of an exhaustive search for a realistic string model. One possible outcome (the worst one) is that no string model reproduces the Standard Model. This would rule out string theory. Another possible outcome (the best one) is that one string model reproduces the Standard Model. Moreover, the model represents a well-isolated point in the space of all string solutions. The parameters of the Standard Model are thus predicted. The number of string models may be so large that a strange possibility emerges: there may exist many string models with almost identical properties, all of which are consistent with the Standard Model to the accuracy that it is presently known. In this possibility there may be a significant loss of predictive power. Other outcomes may be possible.

New experimental input will also help us determine if string theory describes our universe. The recent discovery of a nonzero positive cosmological constant has suggested new directions of investigation based on cosmological properties of strings. A discovery of supersymmetry would be a strong indication that string theory is

FIGURE 7

The intersecting D-brane configuration. To visualize a compactification we must imagine that a compact torus, such as the one shown in the figure, exists on top of each point of ordinary three space (represented as a plane with coordinates $X^1, X^2,$ and X^3). The D4-branes fill three-space and have one direction along the torus. The D-branes appear as lines on the torus. A left-handed quark is an open string that stretches from weak to color branes on the torus. It is perceived as a particle in three-space.



correct because supersymmetry is generic in string theory—it is almost a prediction. The discovery of extra dimensions, perhaps surprisingly large ones, would also have dramatic implications. Most likely, finding out if string theory describes our universe will require a greater mastery of the theory. String theory is in fact an unfinished theory. Much has been learned, but there is no complete formulation of the theory and its conceptual foundation remains largely mysterious. String theory is an exciting research area because the central ideas remain to be found.

REFERENCES

- [1] *Lectures in Quantum Field Theory*, P.A.M. Dirac. Monograph Series, Belfer Graduate School of Science, Yeshiva University. New York, 1966. See Section 8, entitled “Value of the Classical Theory.”
- [2] *A First Course in String Theory*, B. Zwiebach. Cambridge University Press, UK, 2004.
- [3] Cremades, D., Ibanez, L.E. and Marchesano, F. “More about the Standard Model at Intersecting Branes.” arXiv:hep-ph/0212048.

BARTON ZWIEBACH is a Professor of Physics at the Massachusetts Institute of Technology. He was born in Lima, Peru, where he earned an undergraduate degree in Electrical Engineering from the Universidad Nacional de Ingenieria in 1977.

Zwiebach's graduate work was in Physics, at the California Institute of Technology. He received his Ph.D. in 1983, working under the supervision of Murray Gell-Mann. Zwiebach has held postdoctoral positions at the University of California, Berkeley, and at MIT, where he became an Assistant Professor of Physics in 1987, and a tenured member of the faculty in 1994.

6. Gravity Units, E/M Units, and Planck Length in Higher Dimensions

Robert D. Klauber, corrected version Feb 28, 2023

Gravity

D = 4			D ≥ 4		
d = D - 1 = 3			d = D - 1 ≥ 3		
Quantity	Units	Point Source	Quantity	Units	Point Source
$\mathbf{F} = m\mathbf{g}$	F (dynes)	$F = -\frac{GmM}{r^2}$	$\mathbf{F}^{(D)} = m\mathbf{g}^{(D)}$	Same as 4D	$F^{(D)} = -\frac{G^{(D)}mM}{r^{d-1}}$
$G = \frac{Fr^2}{mM}$	$\frac{F \cdot l^2}{m^2} \left(\frac{\text{dynes} \cdot l^2}{\text{g}^2} = \frac{\text{ergs} \cdot \text{cm}}{\text{g}^2} \right)$	N/A	$G^{(D)} = \frac{F^{(D)}r^{d-1}}{mM} = Gr^{d-3}$ See above far right box.	$\frac{F \cdot l^{d-1}}{m^2} \left(\frac{\text{dynes} \cdot l^{d-1}}{\text{g}^2} = \frac{\text{ergs} \cdot \text{cm}^{d-2}}{\text{g}^2} \right)$	N/A
M	m (g)	“	M	Same as 4D	“
test mass m	m (g)	“	test mass m	Same as 4D	“
$\mathbf{g} = \frac{\mathbf{F}}{m} = -\nabla V_g$	$\frac{F}{m} = \frac{\text{Energy}}{m \cdot l} \left(\frac{\text{dynes}}{\text{g}} = \frac{\text{ergs}}{\text{g} \cdot \text{cm}} \right)$	$\mathbf{g} = -\frac{GM}{r^2} \frac{\mathbf{r}}{ \mathbf{r} }$	$\mathbf{g}^{(D)} = \frac{\mathbf{F}^{(D)}}{m} = -\nabla V_g^{(D)}$	Same as 4D	$\mathbf{g}^{(D)} = -\frac{G^{(D)}M}{r^{d-1}} \frac{\mathbf{r}}{ \mathbf{r} }$
V_g	$\frac{E}{m} \left(\frac{\text{ergs}}{\text{g}} \right)$	$V_g = -\frac{GM}{r}$	$V_g^{(D)}$	Same as 4D	$V_g^{(D)} = -\frac{1}{d-2} \frac{G^{(D)}M}{r^{d-2}}$
Newton grav law	$\nabla \cdot \mathbf{g} = -4\pi G \rho_m$	$\frac{F}{m \cdot l} \left(\frac{\text{dynes}}{\text{g} \cdot l} = \frac{\text{ergs}}{\text{g} \cdot \text{cm}^2} \right)$	$\nabla \cdot \mathbf{g} = -4\pi GM \delta^{(3)}(\mathbf{r})$	Same as 4D	$\nabla \cdot \mathbf{g}^{(D)} = -4\pi G^{(D)} \rho_m^{(D)}$
	$\nabla^2 V_g = 4\pi G \rho_m$	“	$\nabla^2 V_g = 4\pi GM \delta^{(3)}(\mathbf{r})$	Same as 4D	$\nabla^2 V_g^{(D)} = 4\pi G^{(D)} \rho_m^{(D)}$
ρ_m	$\frac{m}{l^3} \left(\frac{\text{g}}{\text{cm}^3} \right)$	$M \delta(r)$	$\rho_m^{(D)}$	$\frac{m}{l^d} \left(\frac{\text{g}}{\text{cm}^d} \right)$	$M \delta^{(3)}(\mathbf{r})$

For each spatial dimension greater than 3 (D greater than 4), the units of $G^{(D)}$ increase by a factor of length (cm) and the units of mass density are reduced by a factor of length (cm) in the denominator. All other quantities keep the same units as in $3d (= 4D)$. In particular, total mass (M) units are the same for any D. Mass plays the role of “charge” in gravitation theory, but as seen by comparison with electromagnetism on the next page, charge in e/m and mass in gravity are not treated the same way in higher dimensional theory.

After reading the following page, be aware that the higher dimensional geometric factor c_d employed in e/m theory is absorbed into the higher dimensional gravitation constant $G^{(D)}$. So, it doesn't show up in the gravity chart above, whereas it does in the e/m chart on the next page.

Electromagnetism

D = 4			D ≥ 4			
$d = D - 1 = 3$ $c_3 = \frac{1}{4\pi}$			$d = D - 1 \geq 3$ $c_d = \frac{\Gamma(d/2)}{2\pi^{d/2}} = \frac{1}{Vol(S^{d-1})}$ (unitless)			
Quantity	Units	Point Source	Quantity	Units	Point Source	
$\mathbf{F} = q\mathbf{E}$	F (dynes)	$\mathbf{F} = \frac{qQ}{4\pi r^2} \frac{\mathbf{r}}{ \mathbf{r} }$	$\mathbf{F}^{(D)} = q\mathbf{E}^{(D)}$	$\mathbf{F}^{(D)}$ same as 4D	$\mathbf{F}^{(D)} = c_d \frac{q^{(D)}Q^{(D)}}{r^{d-1}} \frac{\mathbf{r}}{ \mathbf{r} }$	
No constant in e/m comparable to G in gravity			No constant in e/m like G in gravity. So, units of q (and Q) must change. Plus, need c_d factor.			
Q	q (esu)	N/A	$Q^{(D)}$	To cancel r factors, $q^{(D)}Q^{(D)} = (\text{esu})^2 r^{d-3}$ $q^{(D)}$ units $\rightarrow q\sqrt{l^{d-3}} = \text{esu} \cdot \text{cm}^{(d-3)/2}$	N/A	
test charge q	q (esu)	“	test charge $q^{(D)}$	“	“	
$\mathbf{E} = \frac{\mathbf{F}}{q} = -\nabla\Phi$	$\frac{F}{q} = \frac{\text{Energ}}{q \cdot l} \left(\frac{\text{dynes}}{\text{esu}} = \frac{\text{ergs}}{(\text{esu}) \cdot \text{cm}} \right)$	$\mathbf{E} = \frac{Q}{4\pi r^2} \frac{\mathbf{r}}{ \mathbf{r} }$	$\mathbf{E}^{(D)} = \frac{\mathbf{F}}{q^{(D)}} = -\nabla\Phi^{(D)}$	$\frac{F}{q^{(D)}} = \frac{\text{Energ}}{q^{(D)} \cdot l} \left(\frac{\text{dynes}}{\text{esu} \cdot \text{cm}^{(d-3)/2}} \right)$	$\mathbf{E}^{(D)} = -c_d \frac{Q^{(D)}}{r^{d-1}} \frac{\mathbf{r}}{ \mathbf{r} }$	
Φ	$\frac{\text{Energ}}{q} \left(\frac{\text{ergs}}{\text{esu}} \right)$	$\Phi = \frac{Q}{4\pi r}$	$\Phi^{(D)}$	$\frac{\text{Energ}}{q^{(D)}} = \left(\frac{\text{ergs}}{\text{esu} \cdot \text{cm}^{(d-3)/2}} \right)$	$\Phi^{(D)} = \frac{c_d}{d-2} \frac{Q^{(D)}}{r^{d-2}}$	
Max eq	$\nabla \cdot \mathbf{E} = \rho$	$\frac{F}{q \cdot l} \left(\frac{\text{dynes}}{\text{esu} \cdot l} = \frac{\text{ergs}}{\text{esu} \cdot \text{cm}^2} \right)$	$\nabla \cdot \mathbf{E} = Q\delta^{(3)}(\mathbf{r})$	$\nabla \cdot \mathbf{E}^{(D)} = \rho^{(D)}$	$\frac{\text{Energ}}{q^{(D)} \cdot l^2} = \left(\frac{\text{ergs}}{\text{esu} \cdot \text{cm}^{(d-3)/2} \cdot \text{cm}^2} = \frac{\text{ergs}}{\text{esu} \cdot \text{cm}^{(d+1)/2}} \right)$	$\nabla \cdot \mathbf{E}^{(D)} = Q^{(D)} \delta^{(d)}(\mathbf{r})$
	$\nabla^2 \Phi = -\rho$	“	$\nabla^2 \Phi = -Q\delta^{(3)}(\mathbf{r})$	$\nabla^2 \Phi^{(D)} = \rho^{(D)}$	This is same as above box and (in different, but equivalent, units) below box.	$\nabla^2 \Phi^{(D)} = -Q^{(D)} \delta^{(d)}(\mathbf{r})$
	ρ	$\frac{Q}{l^3} \left(\frac{\text{esu}}{\text{cm}^3} \right)$	$-Q\delta(r)$	$\rho^{(D)}$	$\frac{q^{(D)}}{l^d} \left(\text{esu} \cdot \text{cm}^{(d-3)/2} \frac{1}{\text{cm}^d} = \frac{\text{esu}}{\text{cm}^{(d+3)/2}} \right)$	$Q^{(D)} \delta^{(d)}(\mathbf{r})$

In the force expressions (see first row for both type forces) in $D > 4$, we get extra factors of length units (from the r exponent) in the denominator on the RHS. In gravity theory, we absorb those extra units into the definition of $G^{(D)}$, such that units for other gravitational quantities, like mass, potential, and force per unit mass remain unchanged.

In e/m, there is no constant in the force expressions like G in gravity. If we assume the units for force do not change in higher dimensions in e/m, as it is in gravity, we have to absorb these extra units into our higher dimension definition of charge. We also need the dimensional geometric factor c_d , which was absorbed into $G^{(D)}$ in gravity.

Since there are two factors of charge in the last block of the first row in the e/m chart above, the charge units must be $(\text{esu}) \cdot \sqrt{\text{cm}^{(d-3)}} = (\text{esu}) \cdot \text{cm}^{(d-3)/2}$. Since $Q^{(D)}$ appears in all other relations as a single factor, all of those other relations must change units from what they had in 4D. Charge density, for example, is no longer esu divided by (higher D) volume, so \mathbf{E} and Φ units are weird, as well. (Mass density in gravity however scales directly with the inverse of higher D volume alone).

For each extra dimension above $d = 3$, charge units increase by a factor of the square root of length, in cgs, $\text{cm}^{1/2}$.

On the other hand, if we had wanted to keep Φ (and thus \mathbf{E}) as having the same units, then we would need to define charge units differently than we did above, and then \mathbf{F} units are weird.

Planck Length and Gravitational Constant in Higher Dimensions

Robert D. Klauber www.quantumfieldtheory.info Feb 29, 2024

<u>Step</u>	<u>Description</u>	<u>4D</u>	<u>5D</u>	<u>Any D</u>	<u>Comment</u>
1	Gravity point source force	$F = G \frac{M_1 M_2}{r^2}$	$F^{(5)} = G^{(5)} \frac{M_1 M_2}{r^3}$	$F^{(D)} = G^{(D)} \frac{M_1 M_2}{r^{D-2}}$	Force spread in extra dimension(s). Same units for $F^{(D)}$ in any D .
	Units of $G^{(D)}$	[G]	[G X m]	[G X m^{D-4}]	m = meters
2	Dimensional analysis for Planck length	$l_p^2 = \frac{G\hbar}{c^3}$	$(l_p^{(5)})^3 = \frac{G^{(5)}\hbar}{c^3}$	$(l_p^{(D)})^{D-2} = \frac{G^{(D)}\hbar}{c^3}$	l_p = Planck length for $D=4$ $l_p^{(D)}$ = Planck length, any D
	A ratio of interest	$\frac{l_p^2}{G} = \frac{\hbar}{c^3}$	$\frac{(l_p^{(5)})^3}{G^{(5)}} = \frac{\hbar}{c^3} = \frac{l_p^2}{G}$	$\frac{(l_p^{(D)})^{D-2}}{G^{(D)}} = \frac{\hbar}{c^3} = \frac{l_p^2}{G}$	$(l_p^{(D)})^{D-2} = \frac{l_p^2 G^{(D)}}{G}$
3	Mass spread in circle in extra dimension. Derived in Zwiebach pgs 65-66	N/A	$G^{(5)} = G l_c$	$G^{(D)} = G l_c^{D-4}$	l_c = length of each circumference of circular compact dimension(s)
				$G^{(D)} = G V_c$	V_c = volume of extra dimensions for tori
4	Combine step 2 (ratio) and step 3		$(l_p^{(5)})^3 = l_p^2 \frac{G^{(5)}}{G} = l_p^2 l_c$	$(l_p^{(D)})^{D-2} = l_p^2 l_c^{D-4} = l_p^2 V_c$	If $l_c = l_p$, then $l_p^{(D)} = l_p$.

This page intentionally left blank for best arrangement of subsequent pages.

7. Summary of String Gauges Employed in String Theory and Coordinate Systems

Augments Zwiebach Chaps 9-11. (See last paragraph of Sect 11.5, pg. 229)

By Robert D. Klauber March 10, 2024

	GAUGES: Fixing τ and/or σ					Comment
	Static	Special Static	A Particular Gauge Family		Light-Cone	
Fix (define) τ	$\tau = t = X^0$ $-n \cdot X = \tau$	as at left	$n \cdot X = \beta \alpha' (n \cdot p) \tau$ p constant, any n . This fixes τ for chosen n		At left, with n aligned with light cone edge	4D here, also valid for higher D. n and all vector quantities can be expressed in any coord system
Fix (define) σ		$d\sigma = \frac{ds}{\sqrt{1 - v_T^2/c^2}}$ From $d\sigma = \frac{dE}{T_0}$	$n \cdot \mathcal{P}^\tau = \frac{\beta}{2\pi} n \cdot p$ Fixes σ for chosen n , but not simple to see how		At left with n above	
Other defined		$\dot{X} \cdot X' = 0$		All of below left column true for any n	“	Relations left & below good in any coord system
Results			$n \cdot \mathcal{P}^\sigma = 0$	← eq motion, $n \cdot \mathcal{P}^\tau$ above, BCs	“	
Constraints			$\dot{X} \cdot X' = 0 \quad \dot{X}^2 + X'^2 = 0$ equivalent $(\dot{X} \pm X')^2 = 0$ Same fixing of gauge family as top two blocks above	← 1 st from $n \cdot \mathcal{P}^\sigma = 0$, $\mathcal{P}^\sigma = \frac{\partial \mathcal{L}}{\partial X'}$ and \mathcal{L} ; 2 nd from 1 st , $\mathcal{P}^\tau = \frac{\partial \mathcal{L}}{\partial \dot{X}}$ & 2 blocks, top left column	“	Using these two relations is the same as using the two in the top two rows. Both define the same gauge family.
Conj mom densities	Complicated	$\mathcal{P}_\mu^\tau = \frac{T_0}{c^2} \dot{X}_\mu \quad \mathcal{P}_\mu^\sigma = -T_0 X'_\mu$	$\mathcal{P}_\mu^\tau = \frac{1}{2\pi\alpha'} \dot{X}_\mu \quad \mathcal{P}_\mu^\sigma = \frac{1}{2\pi\alpha'} X'_\mu$	← \mathcal{L} in $\mathcal{P}^\tau, \mathcal{P}^\sigma$ above	“	
Eq motion	$\frac{\partial \mathcal{P}^\tau}{\partial \tau} + \frac{\partial \mathcal{P}^\sigma}{\partial \sigma} = 0$ Details complicated	$\frac{1}{c^2} \ddot{X}_\mu - X''_\mu = 0$	$\ddot{X}_\mu - X''_\mu = 0 \quad c = 1$ here	← eq motion with $\mathcal{P}^\tau, \mathcal{P}^\sigma$ above left	“	
Solution	Complicated		X^μ soltn to above (Neumann BC), (9.56) pg 186,		“	This soln good in any coord system

NOTE: Each of the above possible gauges can be expressed in any coordinate system. The two coordinate systems we work with most commonly are the observer spacetime (Minkowski) system with $x^\mu = (ct, x^1, x^2, x^3) = (x^0, x^1, x^2, x^3)$ and the light-cone coordinate system $x^\mu = (x^+, x^-, x^2, x^3)$.

Quantities like \mathcal{P}^τ and \mathcal{P}^σ are different things in each gauge, because they are defined in terms of derivatives of the Lagrangian with respect to τ and σ derivatives of X^μ . τ and σ are defined differently in different gauges, so \mathcal{P}^τ and \mathcal{P}^σ are different things in different gauges.

For a given gauge, the four vectors \mathcal{P}^τ and \mathcal{P}^σ (μ index suppressed) can be expressed in different coordinate systems. They are the same thing physically in a given gauge, regardless of coordinate system chosen. But their components in different coordinate systems will be different. This is like a 3D velocity vector that will have one set of components in one coordinate system and another set in a different coordinate system (say, one rotated with respect to the first system). It is the same thing expressed in two different coordinate systems.

Similarly, our choice of the four-vector n determines our gauge. But n has different components in different coordinate systems, even though (for the same gauge choice for n), it is the same physical entity regardless of the coordinate system chosen.

If we go with a particular gauge (like the light-cone gauge), we can express everything we deal with in any coordinate system. But, it turns out the light-cone coordinate system is the easiest to use with the light-cone gauge.

Note: $E_{tension} = T_0 l_{string}$ From definition of $d\sigma = dE/T_0 \rightarrow E_{tension} \& \text{oscillation} = T_0 \pi$.

COORDINATE SYSTEMS WITH DIFFERENT GAUGES			
	Static Gauge	A Particular Gauge Family	Light-Cone Gauge
Minkowski Coords	$n = n^\mu = (n^0, n^1, n^2, n^3) = (1, 0, 0, 0)$	arbitrary $n = n^\mu = (a, b, c, d)$	specific $n = n^\mu = (n^+, n^-, n^2, n^3) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$
	$n \cdot X = \beta \alpha' (n \cdot p) \tau$ $n \cdot X = -ct = \beta \alpha' (-E/c) \tau$ Natural units, open spring with $\beta = 2$ $t = \frac{2}{2\pi T_0} E \tau \xrightarrow{E=T_0 \pi} \tau = t$	$n \cdot X = \beta \alpha' (n \cdot p) \tau$	$n \cdot X = \beta \alpha' (n \cdot p) \tau \rightarrow \frac{-X^0 + X^1}{\sqrt{2}} = \beta \alpha' (-p^0 + p^1) \tau$
Light-cone coords	$n = n^\mu = (n^+, n^-, n^2, n^3) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$	$n = n^\mu = (\frac{1}{\sqrt{2}}(a+b), \frac{1}{\sqrt{2}}(a-b), c, d)$	$n = n^\mu = (n^+, n^-, n^2, n^3) = (1, 0, 0, 0)$
	$n \cdot X = \beta \alpha' (n \cdot p) \tau$ $n \cdot X = \frac{-X^+ - X^-}{\sqrt{2}} = \beta \alpha' \frac{(-p^+ - p^-)}{\sqrt{2}} \tau$	$n \cdot X = (\frac{1}{\sqrt{2}}(a+b), \frac{1}{\sqrt{2}}(a-b), c, d) \begin{bmatrix} -X^1 \\ -X^0 \\ X^2 \\ X^3 \end{bmatrix}$	$n \cdot X = \beta \alpha' (n \cdot p) \tau \rightarrow X^+ = \beta \alpha' p^+ \tau$
			$n \cdot \mathcal{P}^\tau = \frac{\beta}{2\pi} n \cdot p \rightarrow \mathcal{P}^{\tau+} = \frac{\beta}{2\pi} p^+$
			Similarly easier than Minkowski coordinates for other quantities.

8. String Gauge Relations Visualized in Spacetime Diagrams

Robert D. Klauber www.quantumfieldtheory.info correction March 10, 2024 to Jan 29 & 21, 2024 (had n^μ not n_μ unit vec)

1 The τ Parameter

A family of gauges used in string theory of the string world sheet parameters τ and σ is, from Zwiebach (9.27) [181], where n_μ is a timelike or lightlike unit vector, X^μ is spacetime coordinates on the string world sheet, p^μ is four-momentum of the entire string, and \mathcal{P}^μ is the conjugate 4-momentum density for X^μ ,

$$n_\mu X^\mu = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad \beta = 1 \text{ for closed strings; } = 2 \text{ for open} \quad (1)$$

$$n_\mu \mathcal{P}^\mu = \frac{\beta}{2\pi} n_\mu p^\mu \quad (2)$$

An analysis of what the planes of constant τ look like for various choices of n_μ is shown on subsequent pages and, for generality, includes spacelike n_μ . A summary is shown in Wholeness Chart 1 below.

Note we define a unit vector in a Cartesian, rather than Minkowskian, sense. For example, the lightlike vector

$$n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \quad n_\mu n^\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right) \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right)^T = -\frac{1}{2} + \frac{1}{2} = 0, \quad (3)$$

has zero length (like any lightlike vector) in a Minkowski sense, but in a Cartesian sense, we can consider it has unit length. Otherwise, for the light-cone gauge, where n_μ is aligned with the surface of the light cone, n_μ would not be a unit vector, but a zero length vector.

Wholeness Chart 1 is summary of the analysis of (1) above, which can be found below in Sect. 5. Fig. 1 is a visualization of these results.

Key takeaway: For n_μ timelike or lightlike, any object (which must travel inside the light cone) will have increasing τ parameter as time t passes in an observer's frame, and as proper time τ_{proper} passes on the object. For particles/strings, the parameter τ never decreases with time.

Wholeness Chart 1. Overview of τ for Various n_μ and Massless String

<u>Unit Vector</u>		<u>Nature of n_μ</u>	<u>Constant τ Plane Orientation</u>	<u>τ</u>
$n_\mu = (1, 0, 0, 0)$	↑	timelike	Visually \perp to n_μ (time axis)	timelike $\tau = t$
$n_\mu = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0 \right)$	↗	timelike	Visually \perp to n_μ	timelike
$n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0 \right)$	↗	lightlike	Visually \perp to n_μ (light-cone edge)	lightlike
$n_\mu = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0, 0 \right)$	→	spacelike	Visually \perp to n_μ	spacelike
$n_\mu = (0, 1, 0, 0)$	→	spacelike	Visually \perp to n_μ (space axis)	spacelike

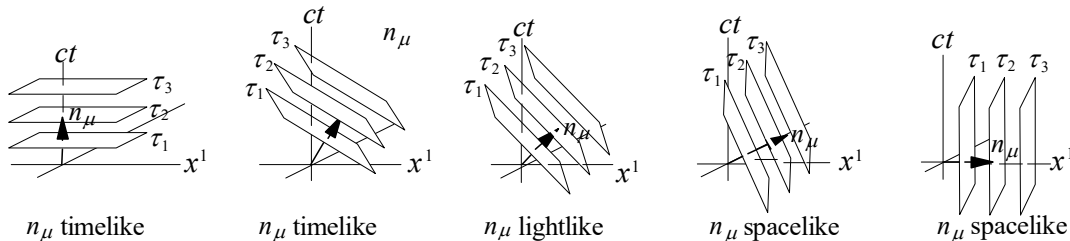


Figure 1. Planes of Constant τ for Different n_μ and Massless String ($\tau_1 < \tau_2 < \tau_3$)

As noted, for an object traveling (which is always inside the light cone), τ must always increase along its path for n_μ timelike or lightlike. Not so for n_μ spacelike. The gauge family of (1) and (2) always takes n_μ as timelike or lightlike.

2 The σ Parameter

Gauge constraint relations (1) and (2) can be expressed in different ways, as Zwiebach shows in Chap. 9. One of these is

$$\dot{X} \cdot X' = 0 \quad \text{and} \quad \dot{X}^2 + X'^2 = 0 \quad (\text{same gauge family as (1) and (2)}, \quad (9.33) \text{ [182]} \quad (4)$$

which can be re-expressed as

$$(\dot{X} \pm X') = 0. \quad (9.34) \text{ [182]} \quad (5)$$

Each of (1) and (2), (4), and (5) is equivalent to any of the others. Each is a set of two equations that constrain τ and σ , which makes sense – two equations to constrain two parameters.

\dot{X} is a vector obtained by a partial derivative in which we fix (keep constant) σ , so it points in the direction of increasing τ and is tangent to a constant σ line. Similarly, X' is a vector tangent to a line of constant τ . So, the first relation in (4) tells us the lines of constant σ are orthogonal (in spacetime) to the lines of constant τ at every point on the world sheet.

3 Intersection of World Sheet with Plane of Constant τ

The intersection of a string world sheet and a plane of constant τ looks like Fig. 2. Note, to help in illustration we have used a massive (not massless) string, so all points on the string travel inside the light cone.

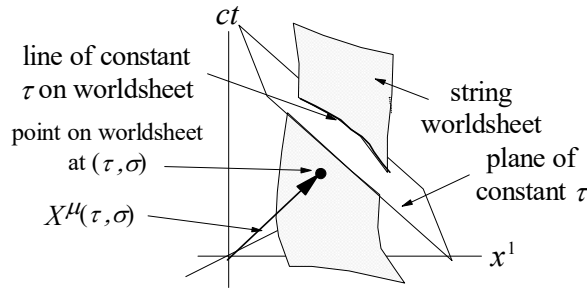


Figure 2. Intersection of a String Worldsheet with a Plane of Constant τ

4 n_μ in the Second Quadrant

For n_μ pointing up and to the left in a spacetime diagram (instead of up and to the right), we would find the planes of constant τ as a mirror image of those in Fig. 1.

5 Supporting Calculations for Massless String

In (1), $\beta \frac{1}{2\pi T_0}$ is constant. For simplicity, we will assume this is unity, and we will use the same simple light-like 4-momentum for each example. That is, for our examples, in this section,

$$\text{we take } \beta \frac{1}{2\pi T_0} = 1 \quad \text{and} \quad p^\mu = (1 \ 1 \ 0 \ 0 \dots). \quad (6)$$

We will then consider cases where $\tau = 1, 2, 3$ to see where the planes of constant τ lie in a spacetime diagram.

Note we are choosing a massless string, so the string 4-momentum is lightlike. In the next section, we look at a massive string, where the 4-momentum is timelike.

5.1 n_μ Aligned with Time Axis

Take

$$n_\mu = (1, 0, 0, \dots) \quad \rightarrow \quad n^\mu = (-1, 0, 0, \dots), \quad (7)$$

so (1) becomes

$$n_\mu X^\mu = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad \rightarrow \quad n_\mu X^\mu = \tau \quad \text{1st constant } \tau \text{ plane } \tau = 1; \text{ 2nd constant } \tau \text{ plane } \tau = 2; \text{ 3rd, } \tau = 3. \quad (8)$$

$$n_\mu X^\mu = X^0 = \tau \quad \rightarrow \quad X^0 = \tau \quad \rightarrow \quad X^0 = t = 1 \quad \text{on 1st constant } \tau \text{ plane; } = 2 \text{ on 2nd; } = 3 \text{ on 3rd.} \quad (9)$$

This is what the left-hand diagram in Fig. 1 shows. Planes of constant τ appear horizontal, perpendicular to the X^0 axis, and perpendicular to n_μ in this example. Also, τ increases with increasing $X^0 = t$.

5.2 n_μ Timelike, but Not Aligned with Time Axis

Take

$$n_\mu = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0, \dots \right) \quad \rightarrow \quad n^\mu = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0, \dots \right) \quad (10)$$

so (1) becomes

$$n_\mu X^\mu = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad \rightarrow \quad n_\mu X^\mu = \left(\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} \right) \tau \quad \text{1st plane } \tau = 1; \text{ 2nd, } \tau = 2; \text{ 3rd, } \tau = 3 \quad (11)$$

$$n_\mu X^\mu = \frac{2}{\sqrt{5}} X^0 + \frac{1}{\sqrt{5}} X^1 = \frac{3}{\sqrt{5}} \tau \quad \rightarrow \quad 2X^0 + X^1 = 3\tau. \quad (12)$$

$$\begin{aligned} \tau_1 = 1 &\rightarrow 2X^0 + X^1 = 3 &\rightarrow X^1 = 0, X^0 = 1.50 & X^0 = 0, X^1 = 3 \\ \tau_2 = 2 &\rightarrow 2X^0 + X^1 = 6 &\rightarrow X^1 = 0, X^0 = 3.00 & X^0 = 0, X^1 = 6 \\ \tau_3 = 3 &\rightarrow 2X^0 + X^1 = 9 &\rightarrow X^1 = 0, X^0 = 4.50 & X^0 = 0, X^1 = 9 \end{aligned} \quad (13)$$

(13) represent lines sloping from the upper left to lower right in the $X^0 X^1$ plane, but planes when we include the X^2 axis. They are represented in the second diagram of Fig. 1. These plane of constant τ appear visually in a spacetime diagram like they are perpendicular to n_μ , though in 4D spacetime they would not be considered orthogonal². Note that τ increases with increasing $X^0 = t$. Hence, τ increases along the world line of any object.

5.3 n_μ Lightlike

Take

$$n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \dots \right) \quad (14)$$

so (1) becomes

$$n_\mu X^\mu = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad \rightarrow \quad n_\mu X^\mu = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \tau = \frac{2}{\sqrt{2}} \tau \quad \text{1st plane } \tau = 1; \text{ 2nd, } \tau = 2; \text{ 3rd, } \tau = 3 \quad (15)$$

$$n_\mu X^\mu = \left(\frac{1}{\sqrt{2}} X^0 + \frac{1}{\sqrt{2}} X^1 \right) \tau = \frac{2}{\sqrt{2}} \tau \quad \rightarrow \quad X^0 + X^1 = 2\tau \quad (16)$$

$$\begin{aligned} \tau_1 = 1 &\rightarrow X^0 + X^1 = 2 &\rightarrow X^1 = 0, X^0 = 2 & X^0 = 0, X^1 = 2 \\ \tau_2 = 2 &\rightarrow X^0 + X^1 = 4 &\rightarrow X^1 = 0, X^0 = 4 & X^0 = 0, X^1 = 4 \\ \tau_3 = 3 &\rightarrow X^0 + X^1 = 6 &\rightarrow X^1 = 0, X^0 = 6 & X^0 = 0, X^1 = 6 \end{aligned} \quad (17)$$

¹ Note that we have taken $\beta/2\pi T_0 = 1$ and $p^\mu = (1, 1, 0, 0)$ to make things simple, but actually, for open strings, $\beta/2\pi T_0 = E$, where E is energy from tension and oscillation of the string, and $p^\mu = (E, |p|, 0, 0)$. So, $n_\mu p^\mu = E$, and $n_\mu X^\mu = t$. Thus, from the left side of (8), $\tau = t$.

² The vector orthogonal to $n_\mu = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0 \right)$ in spacetime is $v_\mu = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0, 0 \right)$, since $n_\mu v^\mu = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0 \right) \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0, 0 \right)^T = 0$.

But in a spacetime diagram, v_μ does not look perpendicular to n_μ . The vector $\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0, 0 \right)$, however, looks perpendicular to n_μ in this example visually, but in spacetime is not orthogonal to it. Similarly, the vector $(2, -1, 0, 0)$, which aligns with a constant τ plane in this example, looks visually perpendicular to n_μ (but is not orthogonal to it in spacetime).

These are planes as represented in the middle diagram of Fig. 1. These planes of constant τ do look visually in a spacetime diagram like they are perpendicular to the edge of the light cone, i.e., perpendicular to n_μ in this example. Note that τ increases with increasing $X^0 = t$. Hence, τ increases along the world line of any object.

5.4 Spacelike n_μ

Showing the results of the last two diagrams in Fig. 1 is left to the reader. Note that the gauge family we are considering always has timelike or lightlike n_μ , so deducing and showing the constant τ planes is more of an academic than practical exercise. You can skip it if you like with little impact on the learning process.

5.5 Conclusions for Massless Strings

For n_μ timelike or lightlike, τ increases monotonically with increasing t , and thus, always increases along the spacetime path of any object. One exception: for lightlike n_μ along the right edge of the light cone, τ is constant for a massless object traveling along the left edge of the light cone.

For n_μ spacelike, we could have objects traveling spacetime paths where τ decreases with increasing t , as you can see by considering what such paths would look like in the last two diagrams of Fig. 1.

Hence, we can conclude that in this gauge family (that only includes timelike or lightlike n_μ), τ is a useful parameter because, for any possible particle/string motion, it never decreases with passage of time t . The same conclusion holds for massive particles/strings, as one could see, if one wished, by plotting the results of Sect. 6.

6 Supporting Calculations for Massive String

For the sake of completeness, in this section, we show similar calculations for a massive string. The strings of concern in almost all of string theory are the massless ones, so the following is of academic, but not great practical, value, and you may wish to just skip it.

We again note that $\beta \frac{1}{2\pi T_0}$ is constant and assume, for simplicity, that it equals unity, as before. However, we now consider a massive string, and the same simple 4-momentum for each example. That is, for our examples,

$$\text{we take } \beta \frac{1}{2\pi T_0} = 1 \quad \text{and} \quad p^\mu = (2 \ 1 \ 0 \ 0 \dots). \quad (18)$$

We will then consider cases where $\tau = 1, 2, 3$ to see where the planes of constant τ lie in a spacetime diagram. Note that the string 4-momentum is timelike, in contrast with the lightlike 4-momentum of the massless string in Sect. 5.

6.1 n_μ Aligned with Time Axis

Take

$$n_\mu = (1, 0, 0, \dots) \quad \rightarrow \quad n^\mu = (-1, 0, 0, \dots), \quad (19)$$

so (1) becomes

$$n_\mu X^\mu = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad \rightarrow \quad n_\mu X^\mu = 2\tau \quad \text{1st constant } \tau \text{ plane } \tau = 1; \text{ 2nd constant } \tau \text{ plane } \tau = 2; \text{ 3rd, } \tau = 3. \quad (20)$$

$$n_\mu X^\mu = X^0 = 2\tau \quad \rightarrow \quad X^0 = 2\tau \quad \rightarrow \quad X^0 = ct = 2 \text{ on 1st constant } \tau \text{ plane; } = 4 \text{ on 2nd; } = 6 \text{ on 3rd.} \quad (21)$$

Constant τ planes in this case are horizontal in spacetime diagrams and look visually to be perpendicular to n_μ . Again, from the logic of footnote 1 on pg. 3, in the realistic case, $\tau = t$, here.

6.2 n_μ Timelike, but Not Aligned with Time Axis

Take

$$n_\mu = \left(\frac{3/2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0, \dots \right) \quad \rightarrow \quad n^\mu = \left(-\frac{3/2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0, \dots \right) \quad (22)$$

so (1) becomes

$$n_\mu X^\mu = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad \rightarrow \quad n_\mu X^\mu = \left(\frac{3/2}{\sqrt{5}} \cdot 2 + \frac{1}{\sqrt{5}} \right) \tau \quad \text{1st plane } \tau = 1; \text{ 2nd, } \tau = 2; \text{ 3rd, } \tau = 3 \quad (23)$$

$$n_\mu X^\mu = \frac{3}{\sqrt{5}} X^0 + \frac{1}{\sqrt{5}} X^1 = \frac{4}{\sqrt{5}} \tau \quad \rightarrow \quad 3X^0 + X^1 = 4\tau. \quad (24)$$

$$\begin{aligned} \tau_1 = 1 &\rightarrow 3X^0 + X^1 = 4 &\rightarrow X^1 = 0, X^0 = \frac{4}{3} & X^0 = 0, X^1 = 4 \\ \tau_2 = 2 &\rightarrow 3X^0 + X^1 = 8 &\rightarrow X^1 = 0, X^0 = \frac{8}{3} & X^0 = 0, X^1 = 8 \\ \tau_3 = 3 &\rightarrow 3X^0 + X^1 = 12 &\rightarrow X^1 = 0, X^0 = 4 & X^0 = 0, X^1 = 12 \end{aligned} \quad (25)$$

(25) represents lines sloping from the upper left to lower right in the X^0X^1 plane, but planes when we include the X^2 axis. These do not look visually to be perpendicular to n_μ in this example. Nor do they look perpendicular to p^μ .

6.3 n_μ Lightlike

Take

$$n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0, \dots \right) \quad (26)$$

so (1) becomes

$$n_\mu X^\mu = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad \rightarrow \quad n_\mu X^\mu = \left(\frac{2}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \tau = \frac{3}{\sqrt{2}} \tau \quad \text{1st plane } \tau = 1; \text{ 2nd, } \tau = 2; \text{ 3rd, } \tau = 3 \quad (27)$$

$$n_\mu X^\mu = \left(\frac{2}{\sqrt{2}} X^0 + \frac{1}{\sqrt{2}} X^1 \right) \tau = \frac{3}{\sqrt{2}} \tau \quad \rightarrow \quad 2X^0 + X^1 = 3\tau \quad (28)$$

$$\begin{aligned} \tau_1 = 1 &\rightarrow 2X^0 + X^1 = 3 &\rightarrow X^1 = 0, X^0 = 1.50 & X^0 = 0, X^1 = 3 \\ \tau_2 = 2 &\rightarrow 2X^0 + X^1 = 6 &\rightarrow X^1 = 0, X^0 = 3.00 & X^0 = 0, X^1 = 6 \\ \tau_3 = 3 &\rightarrow 2X^0 + X^1 = 9 &\rightarrow X^1 = 0, X^0 = 4.50 & X^0 = 0, X^1 = 9 \end{aligned} \quad (29)$$

These plane of constant τ do not look visually in a spacetime diagram like they are perpendicular to the edge of the light cone. But, they do look visually to be perpendicular to p^μ .

6.4 Spacelike n_μ but not Aligned with Spatial Axis

Take

$$n_\mu = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0, 0, \dots \right) \quad \rightarrow \quad n^\mu = \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0, 0, \dots \right) \quad (30)$$

$$n_\mu X^\mu = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad \rightarrow \quad n_\mu X^\mu = \left(\frac{1}{\sqrt{5}} \cdot 2 + \frac{2}{\sqrt{5}} \cdot 1 \right) \tau \quad \text{1st plane } \tau = 1; \text{ 2nd, } \tau = 2; \text{ 3rd, } \tau = 3 \quad (31)$$

$$n_\mu X^\mu = \frac{1}{\sqrt{5}} X^0 + \frac{2}{\sqrt{5}} X^1 = \frac{4}{\sqrt{5}} \tau \quad \rightarrow \quad X^0 + 2X^1 = 4\tau \quad (32)$$

$$\begin{aligned} \tau_1 = 1 &\rightarrow X^0 + 2X^1 = 4 &\rightarrow X^1 = 0, X^0 = 4 & X^0 = 0, X^1 = 2 \\ \tau_2 = 2 &\rightarrow X^0 + 2X^1 = 8 &\rightarrow X^1 = 0, X^0 = 8 & X^0 = 0, X^1 = 4 \\ \tau_3 = 3 &\rightarrow X^0 + 2X^1 = 12 &\rightarrow X^1 = 0, X^0 = 12 & X^0 = 0, X^1 = 6 \end{aligned} \quad (33)$$

The constant τ planes in this case are sloped at less than a 45° angle from the time axis, and thus, there are spacetime paths an object could travel where τ would decrease over time.

6.5 Conclusions for Massive Strings

Massive strings are similar to massless ones in that for n_μ timelike or lightlike, τ increases monotonically with increasing t , and thus, always increases along the spacetime path of any object. Unlike massless strings, however, there is no exception, since the planes of constant τ can never form angles less than 45° with the time axis.

For n_μ spacelike, we could have objects traveling spacetime paths where τ decreases with increasing t , as you can see by considering what such paths would look like in the last two diagrams of Fig. 2.

Hence, as with massless strings, for massive strings, we can conclude that in this gauge family (that only includes timelike or lightlike n_μ), τ is a useful parameter because, for any possible particle/string motion, it never decreases with passage of time t .

Wholeness Chart 2 and Fig. 3 summarize the results for massive strings.-

Wholeness Chart 2. Overview of τ for Various n_μ and Massive String

<u>Unit Vector</u>		<u>Nature of n_μ</u>	<u>Constant τ Plane Orientation</u>	<u>τ</u>
$n_\mu = (1,0,0,0)$	↑	timelike	Visually \perp to n_μ (time axis)	timelike
$n_\mu = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0, 0\right)$	↗	timelike	Not visually \perp to n_μ , nor to p^μ	timelike
$n_\mu = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0\right)$	↗	lightlike	Not visually \perp to n_μ , but do look \perp to p^μ	lightlike
$n_\mu = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0, 0\right)$	↘	spacelike	Not visually \perp to n_μ , nor to p^μ	spacelike
$n_\mu = (0,1,0,0)$	→	spacelike	Visually \perp to n_μ (space axis)	spacelike

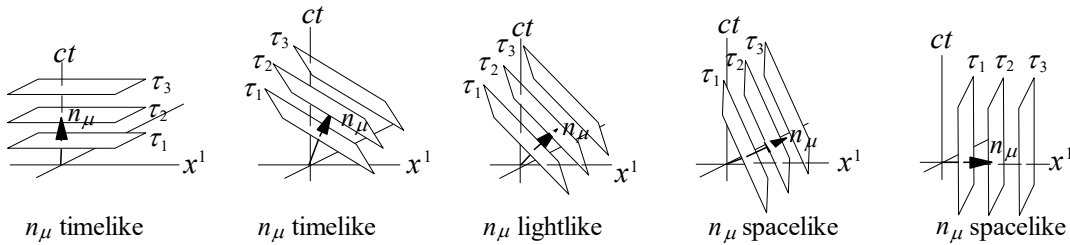


Figure 3. Planes of Constant τ for Different n_μ and Massive- String ($\tau_1 < \tau_2 < \tau_3$)

9. Symmetries on the World Sheet Summary

Bob Klauber March 2, 2022, revised & corrected Jan 13, 2023

Field	Symmetry in \mathcal{L}	4 Current	Conserved Charge	Conjugate Momentum	Physical Quantities Conserved
$\phi(x^\eta)$	$\phi \rightarrow \phi + \varepsilon$	$j^\alpha = \frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} \frac{\partial \phi}{\partial \varepsilon}$ (external)	$j_{,\alpha}^\alpha = 0 \rightarrow \int j^0 dV$ conserved 1 symmetry parameter ε \rightarrow 1 conserved quantity	$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_{,0}} = j^0$ ($\sim \dot{\phi}_{,0} = \dot{\phi}$ typically) Note: π is only time derivative, none spatial	$p_\mu = \pi \frac{\partial \phi}{\partial x^\mu} = \text{4-momentum density (Vol. 1, pg. 23, (B2-2.3))}$ $p_0 = \int p_0 dV = \int \pi \dot{\phi} dV = \int \dot{\phi} \dot{\phi} dV = \int \frac{2\omega_{\mathbf{k}} \omega_{\mathbf{k}}}{(\sqrt{2\omega_{\mathbf{k}} V})^2} dV = \omega_{\mathbf{k}} N_{\mathbf{k}}$ $p_i = \int p_i dV = \int \pi \dot{\phi}_{,i} dV = \int \dot{\phi}_{,i} \dot{\phi} dV = \int \frac{2k_i \omega_{\mathbf{k}}}{(\sqrt{2\omega_{\mathbf{k}} V})^2} dV = k_i N_{\mathbf{k}}$ Since we have only proven $\int \pi dV$ is conserved, it may not be obvious that p_μ is. Also, physically, there are four conserved quantities p_μ , though Noether's theorem only predicts one.
	$\phi \rightarrow \phi e^{i\beta}$	$j^\alpha = \frac{\partial \mathcal{L}}{\partial \phi_{,\alpha}} \frac{\partial \phi}{i\phi}$ (internal)	$j_{,\alpha}^\alpha = 0 \rightarrow \int j^0 dV$ conserved 1 symmetry parameter β \rightarrow 1 conserved quantity	N/A	$Q = q \int j^0 dV = q \int \frac{\partial \mathcal{L}}{\partial \phi} i\phi dV = q \int \frac{2\omega_{\mathbf{k}}}{2\omega_{\mathbf{k}} V} (N_a - N_b) dV = q(N_a - N_b)$
$A^\rho(x^\eta)$	$A^\mu \rightarrow A^\mu + \varepsilon^\mu$	$j_{\mu}^\alpha = \frac{\partial \mathcal{L}}{\partial A^{\rho}_{,\alpha}} \frac{\partial A^\rho}{\delta \varepsilon^\mu}$ $= \frac{\partial \mathcal{L}}{\partial A^{\rho}_{,\alpha}}$ (external)	$j_{\mu}^\alpha{}_{,\alpha} = 0 \rightarrow \int j_{\mu}^0 dV$ conserved for each μ 4 symmetry parameters ε^μ \rightarrow 4 conserved quantities	$\pi_\mu = \frac{\partial \mathcal{L}}{\partial A^{\rho}_{,0}} = \frac{\partial \mathcal{L}}{\partial \dot{A}^\rho} = j_{\mu}^0$ ($\pi_\mu \sim A_{\mu,0} = \dot{A}_\mu$ typically) Note: π_0 is conjug to time comp of A_μ ; π_i is conjug to space comp. Derivative always wrt time, never space.	$p_\mu = \pi_\rho \frac{\partial A^\rho}{\partial x^\mu}$ Each component of the field makes a contribution to the physical 4- momentum density $p_0 = \int p_0 dV = \int \pi_\rho A^{\rho}_{,0} dV = \int \dot{A}_\rho \dot{A}^\rho dV = \frac{2\omega_{\mathbf{k}}^2}{2\omega_{\mathbf{k}} V} V N_{\mathbf{k}} = \omega_{\mathbf{k}} N_{\mathbf{k}}$ $p_i = \int p_i dV = \int \pi_\rho A^{\rho}_{,i} dV = \int \dot{A}_\rho A^{\rho}_{,i} dV = \frac{2\omega_{\mathbf{k}} k_i}{2\omega_{\mathbf{k}} V} V N_{\mathbf{k}} = k_i N_{\mathbf{k}}$ Note: transformation not on x^η (independent variables), but on A^μ (dependent variables = fields).
	$A^\mu \rightarrow A^\mu e^{i\beta}$	$j^\alpha = \frac{\partial \mathcal{L}}{\partial A^{\rho}_{,\alpha}} \frac{\partial A^\rho}{iA^\rho}$ (internal)	$j_{,\alpha}^\alpha = 0 \rightarrow \int j^0 dV$ conserved 1 symmetry parameter β \rightarrow 1 conserved quantity	N/A	$Q = q \int j^0 dV = q \int \frac{\partial \mathcal{L}}{\partial A^\rho} iA^\rho dV = q \int \dot{A}_\rho iA^\rho dV$ $= q \int \frac{2\omega_{\mathbf{k}}}{2\omega_{\mathbf{k}} V} N_{\mathbf{k}} dV = q N_{\mathbf{k}}$

String Theory

Indices parallel Zwiebach pg. 158, except using ρ instead of a to label fields in \mathcal{L} , so not to confuse a with α (which labels independent variables $\varepsilon^\alpha = \tau, \sigma$). Note also, that Zwiebach uses μ for i , after introducing the meaning of the i index. Also see Notes on next page.

Field	Symmetry in \mathcal{L}	4 Current	Conserved Charge	Conjugate Momentum	Physical Quantities Conserved
$X^\rho(\xi^\alpha)$ $= X^\rho(\xi^1, \xi^2)$ $(\xi \text{ spatial } \uparrow)$ $(\alpha = \tau, \sigma \downarrow)$ $= X^\rho(\tau, \sigma)$ $= X^\rho(\xi^0, \xi^1)$ $\rho = 0, 1, 2, 3$ $\alpha = 0, 1$	4D translation $X^\mu \rightarrow X^\mu + \varepsilon^\mu$ $\mu = 0, 1, 2, 3$ in Zwiebach, μ sometimes as i (external)	$j_\mu^\alpha = (j_\mu^\tau, j_\mu^\sigma)$ $= (j_\mu^0, j_\mu^1)$ $= \frac{\partial \mathcal{L}}{\partial X^{\rho, \alpha}} \frac{\partial X^\rho}{\delta \varepsilon^\mu}$ $= \frac{\partial \mathcal{L}}{\partial X^{\mu, \alpha}} = (\mathcal{P}_\mu^\tau, \mathcal{P}_\mu^\sigma)$ <p style="text-align: center;">Zwiebach notation</p> $= \mathcal{P}_\mu^\alpha$	$j_{\mu, \alpha}^\alpha = 0 \rightarrow \int j_\mu^0 d\sigma$ conserved for each μ 4 symmetry parameters ε^μ \rightarrow 4 conserved quantities Aside 1 st eq above \rightarrow eq of motion $\frac{\partial \mathcal{P}_\mu^\tau}{\partial \tau} + \frac{\partial \mathcal{P}_\mu^\sigma}{\partial \sigma} = 0$ <p style="text-align: center;">Zwiebach notation</p>	$\pi_\mu = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial X^\mu}{\partial \xi^0} \right)} = \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial X^\mu}{\partial \tau} \right)}$ $= \frac{\partial \mathcal{L}}{\partial X^{\mu, 0}} = \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = j_\mu^\tau = j_\mu^0 = \mathcal{P}_\mu^\tau$ $(\pi_\mu \sim X_{\mu, 0} = \dot{X}_\mu \text{ typically?})$ Note: π_0 is conjugate to time comp X^0 ; π_1 or 2 or 3 is conjugate to space comp X^1 or 2 or 3. Derivative always wrt τ , not σ .	Note that x^η are the 4D coordinates in the frame of the observer and are independent (whereas X^ρ are 4D coordinates dependent on τ, σ). 4-momentum density (per unit σ) $\mathcal{P}_\mu = \pi_\rho \frac{\partial X^\rho}{\delta x^\mu} = \pi_\mu = j_\mu^\tau = j_\mu^0 = \mathcal{P}_\mu^\tau(\tau, \sigma)$ total string 4-momentum $p_\mu = \int j_\mu^\tau d\sigma = \int j_\mu^0 d\sigma = \int \mathcal{P}_\mu^\tau(\tau, \sigma) d\sigma$ p_μ conserved (for each μ) via 1 st eq, 4 th column
As above	$X^i \rightarrow X^i e^{i\beta}$	Not in Zwiebach as of pg 177 (internal)			
As above	4D Lorentz $X'^\mu = \Lambda_\nu^\mu X^\nu$ $= \Lambda^{\mu\nu} X_\nu$ Infinitesimal $\varepsilon^{\mu\nu} \ll 1$ $X'^\mu = X^\mu + \varepsilon^{\mu\nu} X_\nu$ $\delta X'^\mu = \varepsilon^{\mu\nu} X_\nu$ $(\varepsilon^{\mu\nu} \text{ antisym with } 3 \text{ indep boosts } + 3 \text{ indep rotations } \rightarrow 6 \text{ indep variables})$ (external)	$j_{\mu\nu}^\alpha = (j_{\mu\nu}^\tau, j_{\mu\nu}^\sigma)$ $= (j_{\mu\nu}^0, j_{\mu\nu}^1)$ $= \frac{\partial \mathcal{L}}{\partial X^{\rho, \alpha}} \frac{\partial X^\rho}{\delta \varepsilon^{\mu\nu}}$ $= j_\mu^\alpha \left(\frac{\partial \varepsilon^{\rho\xi}}{\partial \varepsilon^{\mu\nu}} \right) X_\xi$ $= j_\mu^\alpha \begin{pmatrix} \delta_\mu^\rho \delta_\nu^\xi \\ -\delta_\mu^\xi \delta_\nu^\rho \end{pmatrix} X_\xi$ $= j_\mu^\alpha X_\nu - j_\nu^\alpha X_\mu$ $= \underbrace{\mathcal{P}_\mu^\alpha X_\nu - \mathcal{P}_\nu^\alpha X_\mu}_{\text{Zwiebach notation}}$ $= \text{Lorentz current}$	$j_{\mu\nu, \alpha}^\alpha = 0 \rightarrow \int j_{\mu\nu}^0 d\sigma = M_{\mu\nu}$ = Lorentz charge Conserved for each indep set of μ and ν 6 symmetry parameters = six indep comps of $\varepsilon^{\mu\nu}$ \rightarrow 6 conserved quantities $\mu = 0, \nu = 1, 2, 3,$ 3 boost charges Other off diagonal terms, 3 rotation charges. Note the term "charge" is usually reserved for internal symmetries, but here used for Lorentzian.	N/A	For $j, k \neq 0$, $L_i = \frac{1}{2} \varepsilon_{ijk} M_{jk}$ = angular momentum, which is conserved because M_{jk} are. Boost M_{0j} is related to initial position, i.e., it is conserved during string motion.

Notes

$j_\mu^\alpha(\tau, \sigma) = (j_\mu^\tau(\tau, \sigma), j_\mu^1(\tau, \sigma)) = (j_\mu^0(\tau, \sigma), j_\mu^1(\tau, \sigma)) = (\mathcal{P}_\mu^\tau(\tau, \sigma), \mathcal{P}_\mu^\sigma(\tau, \sigma))$ is a vector dependent on τ and σ , so, visually, we can imagine a field of vectors whose base points lie on the world sheet (because they are functions of world sheet coordinates τ and σ), though the tips of the vectors would, in general, be off of the world sheet. There would be two component vectors composing $j_\mu^\alpha(\tau, \sigma)$ at every point on the world sheet, $j_\mu^\tau(\tau, \sigma) = j_\mu^0(\tau, \sigma) = \mathcal{P}_\mu^\tau(\tau, \sigma)$ and $j_\mu^1(\tau, \sigma) = j_\mu^1(\tau, \sigma) = \mathcal{P}_\mu^\sigma(\tau, \sigma)$. Each of those component vectors has 4 components μ as seen in the 4D spacetime Minkowski space, and they add vectorially to give $j_\mu^\alpha(\tau, \sigma)$, which itself has 4 components for $\mu = 0, 1, 2, 3$.

We show the conserved quantity $\int \mathcal{P}_\mu d\sigma = \int j_\mu^\tau d\sigma = \int j_\mu^0 d\sigma = \int \mathcal{P}_\mu^\tau d\sigma = p_\mu$ is actually physical 4-momentum using the general formula in the last column above that relates conjugate momentum to physical momentum. (See Klauber, Vol. 1, pg. 23, (B2-2.3).) Generally, $\pi_\mu \neq \mathcal{P}_\mu$, but in this particular case, they are equal.

Zwiebach shows p_μ is physical 4-momentum in a different way, via 3 steps, as follows. 1) Finding conserved quantity p_μ (without knowing what it is physically) in the static gauge, 2) showing p_μ is the same in any gauge, and 3) showing p_μ is actually physical 4-momentum in the static gauge, so therefore p_μ is physical 4-momentum in any gauge (and is gauge invariant.)

Note that the physical 4-momentum density of the string equals $\mathcal{P}_\mu = \pi_\mu = j_\mu^\tau = j_\mu^0 = \mathcal{P}_\mu^\tau$, and has nothing to do with $j_\mu^\sigma = j_\mu^1 = \mathcal{P}_\mu^\sigma$. Thus, the total 4-momentum p_μ has nothing to do with $j_\mu^\sigma = j_\mu^1 = \mathcal{P}_\mu^\sigma$.

We know that the 4-momentum density has form $\mathcal{P}_\mu = \rho_m \frac{\partial X_\mu}{\partial \tau_{proper}}$, where ρ_m is the rest mass per unit σ length parameter. The derivative of any 4D position vector coordinates of a point with respect to proper time is tangent to the world line of the point. For a string, it would be tangent to the world sheet at each point on the world sheet. Thus, the $\mathcal{P}_\mu (= \pi_\mu = j_\mu^\tau = j_\mu^0 = \mathcal{P}_\mu^\tau)$ vectors at each point are tangent to the world sheet. Unless the world sheet happens to be flat, they will point in directions off of the world sheet, though they are actually vectors confined to the world sheet. One can also show the $j_\mu^\sigma = j_\mu^1 = \mathcal{P}_\mu^\sigma$ vector is everywhere tangent to the world sheet, since it ends up having form X'_μ , where the prime indicates derivative with respect to σ . Such a derivative is tangent to constant τ line and that line lies in the world sheet.

10. Relativistic Point Particle vs Relativistic String Solutions Light-Cone Gauge & Light-Cone Coordinates: Classical Mechanics

As an aid for Zwiebach (text which equation numbers and pages below reference) Robert D Klauber Jan 26, 2024

	<u>Point Particle</u>		<u>Open String Field</u>	
Independent variables	x^I, x_0^-, p^I, p^+	(11.25) [220]	$X^I, x_0^-, \mathcal{P}^{\tau I}, p^+$	(12.5) [237] $\mathcal{P}^\tau =$ momentum density
Full description of motion	$x^\mu(\tau) = (x^+, x^-, x^I)$		$X^\mu(\tau, \sigma) = (X^+, X^-, X^I)$	
Eq of motion	$\dot{p}_\mu = \frac{dp_\mu}{d\tau} = m^2 \ddot{x}_\mu = 0$	m^2 instead of m gives unitless τ (nat units)	$\ddot{X}^\mu - X''^\mu = 0$	(9.39) [183]
General solution	$x^\mu = x_0^\mu + \frac{p^\mu}{m^2} \tau$	Motion of point particle with no external force	$X^\mu = x_0^\mu + \underbrace{\sqrt{2\alpha'} \alpha_0^\mu}_{2\alpha' p^\mu} \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma$	(9.56), p^μ from (9.52) Neumann B.C.s Wave motion of string with no external force
Transverse motion (given)	$x^I = x_0^I + \frac{p^I}{m^2} \tau$	(11.15) [218]	$X^I = (9.56)$ above with $\mu = I$	(9.69) [188]
Dependent variables	$x^+ = \frac{p^+}{m^2} \tau \quad (x_0^+ = 0)$ $x^- = x_0^- + \frac{p^-}{m^2} \tau$	← Light-cone gauge cond. (11.7) [217] & (11.29) [221] (11.14)[218] & (11.30)[221]	$X^+ = 2\alpha' p^+ \tau = \sqrt{2\alpha'} \alpha_0^+ \tau$ ($\mu = +, x_0^+ = \alpha_n^+ = 0$ in (9.56)) $X^- = (9.56)$ with $\mu = -$	← Light-cone gauge cond. (9.70) [188] (9.72) [188]
Auxiliary to get above	$p^- = \frac{1}{2p^+} (p^I p^I + m^2)$ (From $p^2 = m^2$)	(11.12) [217] & (11.31) [221]	$\sqrt{2\alpha'} \alpha_n^- = \frac{1}{p^+} L_n^\perp \quad L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I \alpha_p^I$ $\sqrt{2\alpha'} \alpha_0^- = 2\alpha' p^- = \frac{1}{p^+} L_0^\perp \quad L_0^\perp = 2\alpha' p^+ p^-$ $L_0^\perp = \frac{1}{2} \underbrace{\alpha_0^I \alpha_0^I}_{\alpha' p^I p^I} + \sum_{p=1} \alpha_p^{I\dagger} \alpha_p^I = \alpha' (p^I p^I + M^2)$ (From $p^2 = m^2$)	(9.77) derived [189] $L_n^\perp =$ transverse Viasoro mode (only used for α_n^-) (9.78) [189] (p^- from point particle) M^2 from (9.83) [190]
Momenta density	N/A		$\mathcal{P}^\tau = \mathcal{P}^{\tau\mu} = (\mathcal{P}^{\tau+}, \mathcal{P}^{\tau-}, \mathcal{P}^{\tau I}) \quad \mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu$	
Momenta of motion	$p^\mu = (p^+, p^-, p^I)$		$p^\mu = (p^+, p^-, p^I) \quad p^\mu = \int \mathcal{P}^{\tau\mu} d\sigma$	
Hamiltonian	$H = \frac{p^+ p^-}{m^2} = \frac{(p^I p^I + m^2)}{2m^2}$	(11.34) [222] Not energy, but τ translation operator (τ gauge dependent)	$H = 2\alpha' p^+ p^- = L_0^\perp$	(12.16) [239] Not energy, but τ translation operator (τ gauge dependent)
Valuable relation			$\dot{X}^- \pm X'^- = \frac{1}{\beta\alpha'} \frac{1}{2p^+} \underbrace{(\dot{X}^I \pm X'^I)^2}_{=0 \text{ in this gauge family}} = \frac{1}{p^+} L_n^\perp e^{-in(\tau \pm \sigma)}$	(9.79) [190] $\beta = 2$ for open (9.77) [189] above

11. Generators of Translation, Rotation, and Boost

See Zwiebach, Sects. 11.5 and 11.6, pgs 226-233 and Klauber, Vol. 2, Wholeness Charts 2-2, pgs. 20-21; 6-3, pg. 172

Robert D. Klauber Feb 28, 2023

	Translation		Lorentz Transformations (Rotations and Boosts)	
	Particle – 1 Spatial D	Field – 4D	Particle – 4D	Field – 4D
Dependent variable	$x = x(t) \quad p = p(t)$	$\phi^\mu = \phi^\mu(t, \mathbf{x}) \quad \pi^\nu = \pi^\nu(t, \mathbf{y})$	$x^\mu = x^\mu(\tau) \quad p^\nu = p^\nu(\tau)$	$\phi^\mu = \phi^\mu(t, \mathbf{x}) \quad \pi^\nu = \pi^\nu(t, \mathbf{x})$
Classical		$\delta = \delta(\mathbf{x} - \mathbf{y})$ below to save space		
Poisson brackets	$\{u, v\} = \left\{ \frac{\partial u}{\partial x} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial x} \right\}$	$\{u, v\} = \left\{ \frac{\partial u}{\partial \phi^r} \frac{\partial v}{\partial \pi_r} - \frac{\partial u}{\partial \pi_r} \frac{\partial v}{\partial \phi^r} \right\} \delta$	As 2 columns to left	As 2 columns to left
Special case	$\{x, p\} = \left\{ \frac{\partial x}{\partial x} \frac{\partial p}{\partial p} - \frac{\partial x}{\partial p} \frac{\partial p}{\partial x} \right\} = 1$	$\{\phi^s, \pi_t\} = \left\{ \frac{\partial \phi^s}{\partial \phi^r} \frac{\partial \pi_t}{\partial \pi_r} - \frac{\partial \phi^s}{\partial \pi_r} \frac{\partial \pi_t}{\partial \phi^r} \right\} \delta$ $= \delta_r^s \delta_t^r \delta(\mathbf{x} - \mathbf{y}) = \delta_t^s \delta(\mathbf{x} - \mathbf{y})$	$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ Lorentz charges, Zwiebach (11.76), x^μ and p^ν satisfy $\{x, p\}$ relations	$\mathcal{M}^{\mu\nu} = \phi^\mu \pi^\nu - \phi^\nu \pi^\mu$ $M^{\mu\nu} = \int \mathcal{M}^{\mu\nu} dx$ ϕ^μ and π^ν satisfy $\{\phi^\mu, \pi_\nu\}$ relations
Transformation	$x' = x + \varepsilon \quad \delta x = \varepsilon$	$\phi'^\mu = \phi^\mu + \varepsilon^\mu \quad \delta \phi^\mu = \varepsilon^\mu$	$x'^\mu = \Lambda_\nu^\mu x^\nu$ small $\rightarrow \delta x^\mu = \varepsilon^{\mu\nu} x_\nu$	$\phi'^\mu(x'^\alpha) = \Lambda_\nu^\mu \phi^\nu(\Lambda_\beta^\alpha x^\beta) \rightarrow \delta \phi^\mu = \varepsilon^{\mu\nu} \phi_\nu$
Via Poisson brackets	$\delta x = \{x, \varepsilon p\} = \varepsilon \{x, p\} = \varepsilon$	$\delta \phi^\mu = \int \{ \phi^\mu, \varepsilon^\nu \pi_\nu \} d\mathbf{y} = \varepsilon^\nu \delta_\nu^\mu = \varepsilon^\mu$	$\delta x^\mu = \left\{ x^\mu, -\frac{1}{2} \varepsilon_{\alpha\beta} M^{\alpha\beta} \right\} = \varepsilon^{\mu\nu} x_\nu$	$\delta \phi^\mu = \int \left\{ \phi^\mu, -\frac{1}{2} \varepsilon_{\alpha\beta} M^{\alpha\beta} \right\} d\mathbf{y} = \varepsilon^{\mu\nu} \phi_\nu$
Transform operator	p (via Poisson bracket)	π_ν (via Poisson bracket)	$-\frac{1}{2} M^{\alpha\beta}$ (via Poisson bracket)	$-\frac{1}{2} M^{\alpha\beta}$ (via Poisson bracket)
Quantum				
Commutators	$[x, p] = i \quad (\hbar = 1)$	$[\phi^\mu, \pi_\nu] = i \delta_\nu^\mu \delta \rightarrow [\phi^\mu, \pi^\nu] = i g^{\mu\nu} \delta$	As 2 columns to left	As 2 columns to left
Transformation	$x' = x + \varepsilon \quad \delta x = \varepsilon$	As at left for x^μ indep variable	$x'^\mu = \Lambda_\nu^\mu x^\nu$ small $\rightarrow \delta x^\mu = \varepsilon^{\mu\nu} x_\nu$	As at left for x^μ indep variable
Via commutators	$\delta x = [x, -i\varepsilon p] = -i\varepsilon [x, p] = \varepsilon$	N/A: x^μ indep, not dep variable	$\delta x^\mu = \left[x^\mu, -\frac{1}{2} \varepsilon_{\alpha\beta} M^{\alpha\beta} \right] = \varepsilon^{\mu\nu} x_\nu$ Zwiebach (11.79)	N/A: x^μ indep, not dep variable
Transform operator	$p = -i \frac{d}{dx}$ (via commutator)	N/A for x^μ	$-\frac{1}{2} M^{\alpha\beta}$ (via commutator)	N/A for x^μ
ϕ transformation	$\phi'(t, x) = \phi(t, x + \varepsilon)$ $\phi = Ae^{-i(Et - px)}$ $\rightarrow \phi' = Ae^{-i(Et - p(x + \varepsilon))}$	$\phi'^\mu = \phi^\mu + \varepsilon^\mu = \phi^\mu + \delta \phi^\mu$ $\delta \phi^\mu = \int [\phi^\mu, -i\varepsilon^\nu \pi_\nu] d\mathbf{y} = \delta_\nu^\mu \varepsilon^\nu = \varepsilon^\mu$ Mirrors $\delta \mathbf{x}$ for particle	Scalar ϕ unchanged under Lorentz transf. Ditto for $Et - px$	$\phi'^\mu = \phi^\mu + \delta \phi^\mu \quad \delta \phi^\mu = \varepsilon^{\mu\nu} \phi_\nu$ $\delta \phi^\mu = \int [\phi^\mu, -\frac{1}{2} \varepsilon_{\alpha\beta} M^{\alpha\beta}] d\mathbf{y} = \varepsilon^{\mu\nu} \phi_\nu$
ϕ transform operator	$T_\varepsilon = e^{i\varepsilon p}$	$-i\pi_\nu$ (via field commutator)	Identity operator	$-\frac{1}{2} M^{\alpha\beta}$ (via field commutator)
	$\phi'(t, x) = T_\varepsilon \phi(t, x)$ $= T_\varepsilon Ae^{-i(Et - px)} = e^{i\varepsilon p} Ae^{-i(Et - px)}$ $= Ae^{-i(Et - p(x + \varepsilon))} = \phi(t, x + \varepsilon)$	N/A	$\phi'(x') = \phi(x)$	N/A

Notes

Recall from Klauber Vol. 1, Chaps. 1 and 2, that a basic postulate for quantization (going from classical theory to quantum theory) is the taking of the classical Poisson brackets over into commutators (with an extra factor of i and $\hbar = 1$). This is what we do in this chart. It is virtually never noted in texts that the generator of translation, so often referred to quantum theory, has a direct analogue in classical theory. The difference is simply that for one we use commutators, and for the other, Poisson brackets. The parallel between the classical and quantum realms extends beyond merely translation to the general Lorentz (including rotation) transformation.

For the 1d (x and t) particle case, in quantum theory, the ϕ translation operator comprises a Lie group, with continuous parameter ε . As there is only one parameter in this case, it is a $U(1)$ group. The operator p is then a generator of the associated Lie algebra.

Note that in $3d$, since $[p_i, p_j] = 0$ the Lie algebra generators for different spatial dimensions ($i = 1,2,3$) all commute. So, they don't collectively form a higher degree Lie algebra. There are simply three different, independent $U(1)$ Lie groups/algebras (for translation), each acting on its own without regard to the others. We will see this is not the case for rotation, or for Lorentz boosts. The operators there do not commute, and their non-zero commutation relations lead to higher degree Lie groups.

Similar logic applies to 4D fields in translation. Each of the four components of a field may each be translated independent of the others.

For Lorentz transformations for a particle, there are six independent $M^{\mu\nu}$, three for boosts and three for rotations. $M^{\mu\nu}$ is antisymmetric, so it has 6 independent parameters. Various values for these parameters determine the degree of rotation or boost the particle undergoes during transformation.

We know rotations do not commute, so it should be no surprise that the different components of $M^{\mu\nu}$ do not generally commute. Thus, unlike translation, each $M^{\mu\nu}$ (for given μ and ν) does not form an independent Lie group. The commutation relations between the $M^{\mu\nu}$ give rise to higher degree Lie groups. The rotation subgroup, for example, is $SO(3)$, which should be no surprise. $M^{\mu\nu}$ is Hermitian.

Note that for $i,j = 1,2,3$, $M^{ij} = x^i p^j - x^j p^i$ is angular momentum in the direction perpendicular to the i - j plane.

In the quantum realm, the commutator of $M^{\mu\nu}$ with 4D position vector generates the change in that vector under a Lorentz transformation (including rotations), as shown in the chart.

In Zwiebach (11.80), pg. 230, $[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho} M^{\nu\sigma} - i\eta^{\nu\rho} M^{\mu\sigma} + i\eta^{\mu\sigma} M^{\rho\nu} - i\eta^{\nu\sigma} M^{\rho\mu}$ (which can be proven via substitution). This commutator defines the Lorentz Lie algebra. Any quantum theory one poses must satisfy this commutation relation in order to be Lorentz covariant. The commutator is a constraint any potential theory must meet to be viable.

All of this is background for Zwiebach taking these results into the light-cone gauge and light-cone coordinates. See pg. 4 for the world sheet coordinates as a 4D field dependent on parameters τ, σ on the world sheet.

With specific regard to $M^{\mu\nu}$, in the covariant gauge with light-cone coordinates, the above commutation relation holds, as Zwiebach shows on pg.233. Since the commutator is a 4D covariant relationship, it should, of course, remain valid under a change of coordinates.

In the light -cone gauge, however, one must modify the $M^{\mu\nu}$ carefully in order to have it satisfy the commutator above and also, to be Hermitian (which is necessary for any generator of a Lie Algebra).

Showing Lorentz Transformation Generation via Poisson Bracket

$$\delta x^\rho = \varepsilon^{\rho\nu} x_\nu \stackrel{?}{=} \left\{ x^\rho, -\frac{1}{2} \varepsilon_{\mu\nu} M^{\mu\nu} \right\} \quad (1)$$

$$\left\{ x^\rho, -\frac{1}{2} \varepsilon_{\mu\nu} M^{\mu\nu} \right\} = -\frac{1}{2} \varepsilon_{\mu\nu} \left\{ x^\rho, (x^\mu p^\nu - x^\nu p^\mu) \right\} \quad (2)$$

$$\begin{aligned} &= -\frac{1}{2} \varepsilon_{\mu\nu} \left(\left\{ x^\rho, x^\mu p^\nu \right\} - \left\{ x^\rho, x^\nu p^\mu \right\} \right) \\ &= -\frac{1}{2} \varepsilon_{\mu\nu} \left(\frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial (x^\mu p^\nu)}{\partial p_\alpha} - \frac{\partial x^\rho}{\partial p_\alpha} \frac{\partial (x^\mu p^\nu)}{\partial x^\alpha} - \frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial (x^\nu p^\mu)}{\partial p_\alpha} + \frac{\partial x^\rho}{\partial p_\alpha} \frac{\partial (x^\nu p^\mu)}{\partial x^\alpha} \right) \quad (3) \\ &= -\frac{1}{2} \varepsilon_{\mu\nu} \left(\delta_\alpha^\rho x^\mu g^{\alpha\nu} - (0) \frac{\partial (x^\mu p^\nu)}{\partial x^\alpha} - \delta_\alpha^\rho g^{\alpha\mu} x^\nu + (0) \frac{\partial (x^\nu p^\mu)}{\partial x^\alpha} \right) = -\frac{1}{2} \varepsilon_{\mu\nu} (x^\mu g^{\rho\nu} - g^{\rho\mu} x^\nu) \end{aligned}$$

In the row below, we make use of the anti-symmetry of $\varepsilon^{\rho\mu}$.

$$(3) = -\frac{1}{2} \varepsilon_\mu{}^\rho x^\mu + \frac{1}{2} \varepsilon^\rho{}_\mu x^\mu = -\frac{1}{2} \varepsilon^{\mu\rho} x_\mu + \frac{1}{2} \varepsilon^{\rho\mu} x_\mu = \varepsilon^{\rho\mu} x_\mu = \varepsilon^{\rho\nu} x_\nu \quad (4)$$

Continuation of Chart on Page 1

	Translation		Lorentz Transformations (Rotations and Boosts)			
	Particle – 1d	Field – 4D	Particle – 4D	Field – 4D	World Sheet Coordinates X^μ as Field Covariant Fields, Light-Cone Coordinates	Light Cone gauge & Coordinates
Dependent variable	See pg. 1 ↓	See pg. 1 ↓	See pg. 1 ↓	See pg. 1 ↓	$\phi^\mu(t, \mathbf{x}) \rightarrow X^\mu(\tau, \sigma)$ $\pi^\mu(t, \mathbf{x}) \rightarrow \mathcal{P}^\nu(\tau, \sigma)$	As at left, but light-cone gauge
Classical						
Poisson brackets					Like 1 st & 3rd boxes to left in “Field 4D” column. $\{u, v\} = \left\{ \frac{\partial u}{\partial X^r} \frac{\partial v}{\partial \mathcal{P}_r} - \frac{\partial u}{\partial \mathcal{P}_r} \frac{\partial v}{\partial X^r} \right\} \delta(\sigma' - \sigma)$	
Special case					$\mathcal{M}^{\mu\nu} = X^\mu \mathcal{P}^\nu - X^\nu \mathcal{P}^\mu$ $M^{\mu\nu} = \int \mathcal{M}^{\mu\nu} d\mathbf{x}$ X^μ and \mathcal{P}^ν satisfy $\{X^\mu, \mathcal{P}_\nu\}$ Poisson bracket relations	Must modify definition of Lorentz generators $M^{\mu\nu}$ to keep correct commutation relations for $M^{\mu\nu}$ (which are needed to keep Lorentz invariance)
Transformation					$X'^\mu(x'^\alpha) = \Lambda_\nu^\mu X^\nu(\Lambda_\beta^\alpha x^\beta) \rightarrow \delta X^\mu = \varepsilon^{\mu\nu} X_\nu$	The above restricts the theory to D=26 and leads to unstable tachyon scalars.
Via Poisson brackets					$\delta X^\mu = \int \left\{ X^\mu, -\frac{1}{2} \varepsilon_{\alpha\beta} M^{\alpha\beta} \right\} d\mathbf{y} = \varepsilon^{\mu\nu} X_\nu$	See Zwiebach pgs. 260-262.
Transform operator					$-\frac{1}{2} M^{\alpha\beta}$ (via Poisson bracket)	
Quantum						
Commutators					Like 1 st & 3rd boxes to left in “Field 4D” column. $[X^\mu(\tau, \sigma), \mathcal{P}_\nu(\tau, \sigma')] = i\delta_\nu^\mu \delta(\sigma - \sigma')$ $\rightarrow [X^\mu, \mathcal{P}^\nu] = ig^{\mu\nu} \delta(\sigma - \sigma')$	
Transformation					Similar to left for τ, σ indep variables, but τ & σ transfs not discussed	
Via commutators					N/A: τ, σ indep, not dep variable	
Transform operator					N/A for τ, σ	
ϕ (= X^μ here) transformtn					$X'^\mu(x'^\alpha) = \Lambda_\nu^\mu X^\nu(\Lambda_\beta^\alpha x^\beta) \rightarrow \delta X^\mu = \varepsilon^{\mu\nu} X_\nu$ $\delta X^\mu = \int [X^\mu, -\frac{1}{2} \varepsilon_{\alpha\beta} M^{\alpha\beta}] d\sigma' = \varepsilon^{\mu\nu} X_\nu$	
ϕ transform operator					$-\frac{1}{2} M^{\alpha\beta}$ (via field commutator)	
					N/A	

12. Deducing D = 26: A Summary of and Little Different Wrinkle from Zwiebach

Ref: Zwiebach, Sect. 12.4 pgs 250-253 and Sect. 12.5, pgs 259-262. Robert D. Klauber May 2, 2023

The present writer has big problems with (12.110) (he doesn't believe it and will never be convinced), so we ignore that result herein. It is not needed to form a meaningful theory.

The Hamiltonian of (12.101) is (where p has nothing to do with momentum and is simply an index label for the p^{th} mode, and \mathbb{Z} comprises all real integers, positive, negative, and zero)

$$H = L_0^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{-p}^I \alpha_p^I = \frac{1}{2} \alpha_0^I \alpha_0^I + \frac{1}{2} \sum_{p=1}^{\infty} \alpha_{-p}^I \alpha_p^I + \frac{1}{2} \sum_{p=1}^{\infty} \alpha_p^I \alpha_{-p}^I, \quad (12.101)$$

where the last term is not normal ordered. Zwiebach converts that term to a normal ordered terms plus an infinite constant, which converts (12.101) to (12.103), where D is the dimension of spacetime, i.e.,

$$H = L_0^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{-p}^I \alpha_p^I = \frac{1}{2} \alpha_0^I \alpha_0^I + \sum_{p=1}^{\infty} \alpha_{-p}^I \alpha_p^I + \frac{1}{2} (D-2) \sum_{p=1}^{\infty} p. \quad (12.103)$$

The last term is reminiscent of the infinite zero point energy (ZPE) of quantum field theory (QFT). That term is ignored in QFT, though its meaning is an unsettled issue with field theorists. So, we choose to follow tradition and simply ignore it here, without trying to force it into something it is not.

Jumping ahead to pg. 260, where the Lorentz generators $M^{\mu\nu}$ are discussed, we see that their commutation relations in the light-cone gauge must be parallel to the covariant formulation of the same relations, in order for Lorentz covariance to hold. In particular,

$$[M^{-I}, M^{-J}] = 0, \quad (12.148)$$

is true in the covariant formulation, so it must be true in the light-cone gauge formulation.

It turns out (it is not actually said in the text) that, for string theory as developed herein in the light-cone gauge, the LHS of (12.148) does not equal zero. It cannot be made to equal zero without redefining something in that theory.

The redefinition that makes (12.148) hold is the following, where a is, at this point, an undetermined constant.

$$\text{Old definition } p^- = \frac{1}{2\alpha' p^+} L_0^\perp \Rightarrow \text{new definition } p^- = \frac{1}{2\alpha' p^+} (L_0^\perp + a) \quad (1)$$

Using this in (12.150) give us the M^{-I} of (12.151). When we plug that into (12.148), we get (12.152) on the RHS.

The only way that can equal zero is if $D = 26$, and $a = -1$.

Using the new definition in (1) with this value of a affects the mass squared, as shown in (12.108).

$$\text{Old relation } M^2 = -p^2 = 2p^+ p^- - p^I p^I = \frac{1}{\alpha'} L_0^\perp - p^I p^I = \frac{1}{\alpha'} \sum_{n=1}^{\infty} n \alpha_n^{I\dagger} \alpha_n^I \quad (2)$$

$$\text{New relation } M^2 = -p^2 = 2p^+ p^- - p^I p^I = \frac{1}{\alpha'} (L_0^\perp + a) - p^I p^I = \frac{1}{\alpha'} \left(a + \sum_{n=1}^{\infty} \alpha_n^{I\dagger} \alpha_n^I \right) = \frac{1}{\alpha'} \left(a + \sum_{n=1}^{\infty} n \alpha_n^{I\dagger} \alpha_n^I \right) \quad (12.108)$$

$$\text{Number operator } N^\perp = \sum_{n=1}^{\infty} \alpha_n^{I\dagger} \alpha_n^I = \sum_{n=1}^{\infty} n \alpha_n^{I\dagger} \alpha_n^I \quad (\text{sum on } I)$$

$$n = \text{string mode number} \quad \underbrace{\alpha_n^{I\dagger} \alpha_n^I}_{\text{(when not } = 0, \text{ typically } = 1)}} = \text{number of strings in } n\text{th mode in } I \text{ direction} \quad (12.164)$$

$\alpha_n^{I\dagger} \alpha_n^I$ is equal to one or zero throughout Chap. 12 and virtually throughout the book.

Bottom line: To keep Lorentz invariance (maintain (12.148) in the light-cone gauge, we needed to introduce a constant in the definition of p^- , as in (1). The Lorentz invariance constraint of (12.148) forced two things upon us. 1) a must equal -1 , and 2) the dimension of spacetime must equal 26. That, in turn, forced a shift (a 3rd thing) in the mass squared operator by $a/\alpha' = 1/\alpha'$.

13 Open vs Closed Relativistic String Solutions

Light-Cone Gauge & Light-Cone Coordinates: Classical Mechanics

As an aid for Zwiebach (text which equation numbers below reference)

Robert D Klauber May 3, 2023

	<u>Open String Field</u> $\beta = 2$	<u>Closed String Field</u> $\beta = 1$
Indep variables	$X^I, x_0^-, \mathcal{P}^{\tau I}, p^+$	Same $X^I, x_0^-, \mathcal{P}^{\tau I}, p^+$
Motion descrip	$X^\mu(\tau, \sigma) = (X^+, X^-, X^I)$	Same $X^\mu(\tau, \sigma) = (X^+, X^-, X^I)$
Eq motion	$\ddot{X}^\mu - X''^\mu = 0$	Same $\ddot{X}^\mu - X''^\mu = 0$
General sol	$X^\mu =$ $x_0^\mu + \underbrace{\sqrt{2\alpha'}\alpha_0^\mu}_{2\alpha'p^\mu} \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma$ (9.56), p^μ from (9.52) Neumann B.C.s Wave motion of string with no external force	$X^\mu = X_L^\mu(u) + X_R^\mu(v) = \downarrow \quad (u = \tau + \sigma \quad v = \tau - \sigma)$ $x_0^\mu + \underbrace{\sqrt{2\alpha'}\alpha_0^\mu}_{\alpha'p^\mu} \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n} \left(\underbrace{\alpha_n^\mu e^{in\sigma}}_R + \underbrace{\bar{\alpha}_n^\mu e^{-in\sigma}}_L \right)$ (13.9), (13.24), & (13.22) No B.C. but 2π identification Wave motion of string with no external force
Transverse	$X^I = (9.56)$ above with $\mu = I$	$X^I = (13.24)$ above with $\mu = I$
Dependent variables	$X^+ = \beta\alpha' p^+ \tau = 2\alpha' p^+ \tau = \sqrt{2\alpha'}\alpha_0^+ \tau$ $(\mu = +, x_0^+ = \alpha_n^+ = 0 \text{ in (9.56) above})$ $X^- = (9.56)$ above with $\mu = -$ \leftarrow Light-cone gauge (9.70)	$X^+ = \beta\alpha' p^+ \tau = \alpha' p^+ \tau = \sqrt{2\alpha'}\alpha_0^+ \tau$ $(\mu = +, x_0^+ = \alpha_n^+ = 0 \text{ in (13.24) above})$ $X^- = (13.24)$ above with $\mu = -$ \leftarrow Light-cone gauge (9.70) (Closed $\alpha_0^+ = \text{half of open } \alpha_0^+$. Same symbol, diff meaning)
Auxiliary for \uparrow	$\sqrt{2\alpha'}\alpha_n^- = \frac{1}{p^+} L_n^\perp \quad L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I \alpha_p^I$ $\sqrt{2\alpha'}\alpha_0^- = 2\alpha' p^- = \frac{1}{p^+} L_0^\perp \quad L_0^\perp = 2\alpha' p^+ p^-$ $L_0^\perp = \underbrace{\frac{1}{2}\alpha_0^I \alpha_0^I}_{\alpha' p^I p^I} + \sum_{p=1} \alpha_p^{I\dagger} \alpha_p^I = \alpha'(p^I p^I + M^2)$ (9.77) Derived [189] $L_n^\perp =$ transverse Viasoro modes (9.78) M^2 from (9.83)	$\sqrt{2\alpha'}\alpha_n^- = \frac{2}{p^+} L_n^\perp \quad \text{Same } L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I \alpha_p^I$ $\sqrt{2\alpha'}\bar{\alpha}_n^- = \frac{2}{p^+} \bar{L}_n^\perp \quad \bar{L}_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \bar{\alpha}_{n-p}^I \bar{\alpha}_p^I$ $\sqrt{2\alpha'}\alpha_0^- = \alpha' p^- = \frac{2}{p^+} L_0^\perp \quad L_0^\perp = \frac{1}{2} \alpha' p^+ p^- = \bar{L}_0^\perp$ $L_0^\perp = \frac{\alpha'}{4} p^I p^I + \sum_{p=1} \alpha_p^{I\dagger} \alpha_p^I$ or in (13.24), $\sqrt{\frac{\alpha'}{2}}\alpha_n^- = \frac{1}{p^+} L_n^\perp$ (13.40) & (13.37) (13.40) and (13.41) ($\bar{L}_0^\perp = L_0^\perp$) (13.43 in (13.42), same form for \bar{L}_0^\perp)
Hamiltonian	$H = 2\alpha' p^+ p^- = L_0^\perp$	$H = \alpha' p^+ p^- = L_0^\perp + \bar{L}_0^\perp$
Valuable relation	$\dot{X}^- \pm X'^- = \frac{1}{\beta\alpha'} \frac{1}{2p^+} (\dot{X}^I \pm X^{I'})^2$ This yields $L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I \alpha_p^I$ (9.65) $\beta = 2$ for open (9.77) above	$\dot{X}^- \pm X'^- = \frac{1}{\beta\alpha'} \frac{1}{2p^+} (\dot{X}^I \pm X^{I'})^2$ This yields $L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I \alpha_p^I \quad \bar{L}_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \bar{\alpha}_{n-p}^I \bar{\alpha}_p^I$ (13.35) $\beta = 1$ for closed (13.37) above

14 Lagrangians: A Summary of Different Systems

An aid for Zwiebach (which plain eq nums reference – “K” eq num refs to Klauber, Vol. 1)
[xx] reference page number in the text

Robert D Klauber March 6, 2023

	<u>Non-Relativistic</u>				<u>Relativistic</u>			
	Free		Interacting		Free		Interacting	
Particle, $L=$	$T = \frac{1}{2}m\dot{x}^2$ $= \frac{p^2}{2m}$	(4.22) [78] with $V=0$	$T - V =$ $\frac{1}{2}m\dot{x}^2 - V(x)$	(4.22) [78]	$T = -mc^2\sqrt{1 - v^2/c^2}$ t system $= -mc^2$ τ system	(5.8) [92] (5.7) [92]	$T - V = -mc^2 + \frac{q}{c}A_\mu \frac{dx^\mu}{d\tau}$ τ system	(5.33) [97]
Harmonic oscill	$\frac{1}{2n}\dot{q}_n^2 - \frac{n}{2}q_n^2$	(12.68) [247]	Not common		Not common		Not common	
Field, $\mathcal{L}=$					t system below		t system below	
Scalar	$\mathcal{T} - \mathcal{U} =$ $\frac{1}{2}\rho\dot{x}^2 - \mathcal{U}$	\mathcal{U} = internal pot energy	$\mathcal{T} - \mathcal{U} - \mathcal{V} =$ $\frac{1}{2}\rho\dot{x}^2 - \mathcal{U} - \mathcal{V}$	\mathcal{V} = external pot energy	$\partial_\alpha\phi^\dagger \partial^\alpha\phi - \mu^2\phi^\dagger\phi$	K(3-32) [49]	$\partial_\alpha\phi^\dagger \partial^\alpha\phi - \mu^2\phi^\dagger\phi - \mathcal{V}$	
Spinor	Not common		Not common		$\bar{\psi}(i\gamma^\alpha\partial_\alpha - m)\psi$	K(4-60) [104]	$\bar{\psi}(i\gamma^\alpha\partial_\alpha - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu$	K(7-20) [186]
Photon	N/A		N/A		$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$	$\leftarrow A_\mu = 0 \text{ in } \rightarrow$	$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + e\bar{\psi}\gamma^\mu\psi A_\mu$	K(11-7) [288]
String, $\mathcal{L}=$					τ, σ parameter space below			
General	$\frac{1}{2}\mu_0(\dot{y})^2$ $-\frac{1}{2}T_0(y')^2$	(4.35) [81]	$\frac{1}{2}\mu_0(\dot{y})^2$ $-\frac{1}{2}T_0(y')^2 - \mathcal{V}$		$-\frac{T_0}{c}\sqrt{-\gamma} =$ $-\frac{T_0}{c}\sqrt{(\dot{X}\cdot X')^2 - (\dot{X})^2(X')^2}$	(6.44) & (6.46) [112] Min area of world sheet		
Static gauge	N/A		N/A		$-\frac{T_0}{c}\sqrt{-(\dot{X})^2(X')^2} = -T_0\frac{\partial\mathbf{x}}{\partial\sigma}$ $= -T_0\frac{ds}{d\sigma}\sqrt{1 - v_\perp^2/c^2}$	(6.66) [118] (6.89) [123]		
\uparrow & new σ parametrization	N/A		N/A		As above			
$n\cdot X = \alpha' n\cdot p\tau$ $n\cdot \mathcal{P}^\tau = \frac{\beta}{2\pi} n\cdot p$	N/A		N/A		As above			
Light cone gauge	N/A		N/A		As above			
Simple surrogate	N/A		N/A		$\frac{1}{4\pi\alpha'}(\dot{X}^I\dot{X}^I - X'^I X'^I)$	(12.81) [248]		
World sheet fermions	N/A		N/A		$\mathcal{L}_\psi = \frac{1}{2\pi} \left\{ \begin{aligned} &\psi_1^I (\partial_\tau + \partial_\sigma)\psi_1^I \\ &+ \psi_2^I (\partial_\tau - \partial_\sigma)\psi_2^I \end{aligned} \right\}$	(14.10) [310]		

15 Number Operators in QFT vs String Theory

Note: We ignore the ordering factor (ZPE in QFT, a in string theory)

Robert D Klauber May 3, 2023

	Quantum Field Theory (Eq & pg nums, Klauber, Vol. 1)		String Theory (Eq & pg nums from Zwiebach)	
Commutator	$[a_{\mathbf{k}}, a_{\mathbf{k}}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$	(3.41) [51]	$[a_m^I, a_n^J] = \delta_{mn} \eta^{IJ}$	(12.64) [245]
Creation & Destruction Operators	$n_{\mathbf{k}}$ is the number of particles of 3-momentum \mathbf{k} $a_{\mathbf{k}}^\dagger 0\rangle = n_{\mathbf{k}} = 1\rangle$ $a_{\mathbf{k}}^\dagger n_{\mathbf{k}}\rangle = \sqrt{n_{\mathbf{k}} + 1} n_{\mathbf{k}} + 1\rangle$ $a_{\mathbf{k}} n_{\mathbf{k}}\rangle = \sqrt{n_{\mathbf{k}}} n_{\mathbf{k}} - 1\rangle$	(3-81) [59]	\hat{n}_n^I is number of strings in mode n in I th direction $a_n^{I\dagger} 0\rangle = \hat{n}_n^I = 1\rangle$ $a_n^{I\dagger} \hat{n}_n^I\rangle = \sqrt{\hat{n}_n^I + 1} \hat{n}_n^I + 1\rangle$ $a_n^I \hat{n}_n^I\rangle = \sqrt{\hat{n}_n^I} \hat{n}_n^I - 1\rangle$	(12.65) [246], more or less, but for alpha $\alpha_n^{I\dagger} = \sqrt{n} a_n^{I\dagger}$
Number Operators	$N_{\mathbf{k}} = a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ $N_{\mathbf{k}} n_{\mathbf{k}}\rangle = a_{\mathbf{k}}^\dagger a_{\mathbf{k}} n_{\mathbf{k}}\rangle = n_{\mathbf{k}} n_{\mathbf{k}}\rangle$ $N = \sum_{\mathbf{k}'} N_{\mathbf{k}'} = \sum_{\mathbf{k}'} a_{\mathbf{k}'}^\dagger a_{\mathbf{k}'}$ $N n_{\mathbf{k}}\rangle = \sum_{\mathbf{k}'} a_{\mathbf{k}'}^\dagger a_{\mathbf{k}'} n_{\mathbf{k}}\rangle = n_{\mathbf{k}} n_{\mathbf{k}}\rangle$	(3-56) [54]	$N_n^I = a_n^{I\dagger} a_n^I$ $N_n^I \hat{n}_n^I\rangle = a_n^{I\dagger} a_n^I \hat{n}_n^I\rangle = \hat{n}_n^I \hat{n}_n^I\rangle$ $N^I = \sum_{I,n} N_n^I = \sum_{I,n} a_n^{I\dagger} a_n^I$ $N^I \hat{n}_n^I\rangle = \sum_{I,n} a_n^{I\dagger} a_n^I \hat{n}_n^I\rangle = \hat{n}_n^I \hat{n}_n^I\rangle$	Not shown.
Eigenvalue	$n_{\mathbf{k}}$ is the number of particles of 3-momentum \mathbf{k} If only one or no particles, $n_{\mathbf{k}} = 1$ or 0		\hat{n}_n^I is number of strings in mode n in I th direction If only one or no strings, $\hat{n}_n^I = 1$ or 0	
Other number operator	$\tilde{N}_{\mathbf{k}} = \mathbf{k} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ $\tilde{N}_{\mathbf{k}} n_{\mathbf{k}}\rangle = \mathbf{k} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} n_{\mathbf{k}}\rangle = \mathbf{k} n_{\mathbf{k}} n_{\mathbf{k}}\rangle$ $\tilde{N} = \sum_{\mathbf{k}'} \tilde{N}_{\mathbf{k}'} = \sum_{\mathbf{k}'} \mathbf{k}' a_{\mathbf{k}'}^\dagger a_{\mathbf{k}'}$ $\tilde{N} n_{\mathbf{k}}\rangle = \sum_{\mathbf{k}'} \mathbf{k}' a_{\mathbf{k}'}^\dagger a_{\mathbf{k}'} n_{\mathbf{k}}\rangle = \mathbf{k} n_{\mathbf{k}} n_{\mathbf{k}}\rangle$	Not shown	$N_n^{\perp I} = n a_n^{I\dagger} a_n^I$ $N_n^{\perp I} \hat{n}_n^I\rangle = n a_n^{I\dagger} a_n^I \hat{n}_n^I\rangle = n \hat{n}_n^I \hat{n}_n^I\rangle$ $N^\perp = \sum_{I,n'} n' N_{n'}^I = \sum_{I,n'} n' a_{n'}^{I\dagger} a_{n'}^I$ $N^\perp \hat{n}_n^I\rangle = \sum_{I,n'} n' a_{n'}^{I\dagger} a_{n'}^I \hat{n}_n^I\rangle = n \hat{n}_n^I \hat{n}_n^I\rangle$	(12.164) [264]
Eigenvalue	$\mathbf{k} n_{\mathbf{k}}$ is the number of particles of 3-momentum \mathbf{k} , $n_{\mathbf{k}}$, times the 3-momentum \mathbf{k} If only one or no particles, $\mathbf{k} n_{\mathbf{k}} = \mathbf{k}$ or 0		$n \hat{n}_n^I$ is number of strings in mode n in I th direction, \hat{n}_n^I , times the mode number n If only one or no strings, $n \hat{n}_n^I = n$ or 0	
Conclusion	For single particle state, eigenvalue of $\tilde{N}_{\mathbf{k}}$ is \mathbf{k}	Not shown	For a single string in a single mode n , the eigenvalue of N^\perp is n	above (12.169) [264]
	QFT states can be single particle or multiparticle		String states studied in Zwiebach are single strings	
	As an aside, $\tilde{N}_{\mathbf{k}}$ is actually the 3-momentum operator P	(3-101) [64]	For a single string with modes in more than one direction I , see Zwiebach, bottom of page 264	[264]

16 Superstrings: Summary of Zwiebach Chapter 14 and Helpful Notes

Robert D. Klauber www.quantumfieldtheory.info January 26, 2024

Section 14.3, pgs 309-312

Instead of employing ψ_1 and ψ_2 of (14.10) to (14.23) to deduce Ψ of (14.23) to (14.26) and the rest of the chapter, start with an action for Ψ , rather than one for both ψ_1 and ψ_2 as in (14.10).

$$S_\Psi = \frac{1}{2\pi} \int d\tau \int_{-\pi}^{+\pi} d\sigma \Psi^I (\partial_\tau + \partial_\sigma) \Psi^I . \quad (1)$$

Demanding a stationary action in the usual way, we get the same thing as (14.15), but with $\psi_2 = 0$, $\psi_1 = \Psi$, and our boundary limits from $-\pi$ to $+\pi$. The equation of motion and its solution form are thus

$$(\partial_\tau + \partial_\sigma) \Psi^I = 0 \quad \rightarrow \quad \Psi^I = \Psi^I(\tau - \sigma) . \quad (2)$$

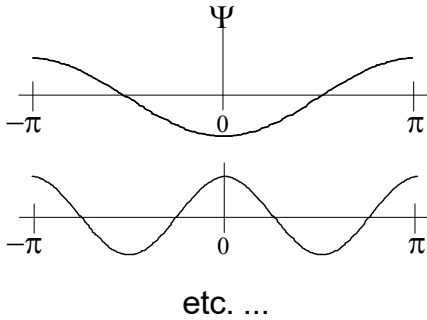
The boundary condition is

$$\Psi^I \delta \Psi^I \Big|_{-\pi}^{+\pi} = 0 \quad \rightarrow \quad \Psi^I(\tau, \pi) \delta \Psi^I(\tau, \pi) = \Psi^I(\tau, -\pi) \delta \Psi^I(\tau, -\pi) . \quad (3)$$

Since $\delta \Psi$ and Ψ have the same sign, as shown in (14.25) and (14.26),

$$\Psi^I(\tau, \pi) = \pm \Psi^I(\tau, -\pi) . \quad (4)$$

The meaning of (14.26) [(4) above] can be understood pictorially, as shown below.



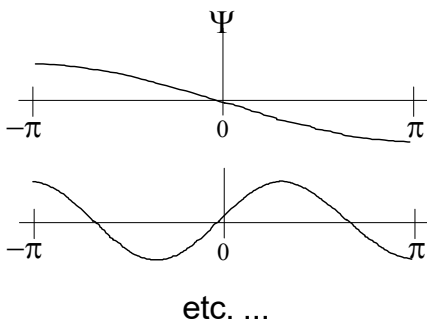
Ramond boundary condition

$$\Psi^I(-\pi) = +\Psi^I(\pi) \quad \tau = 0 \text{ for illustration}$$

$$\text{wavelengths } 2\pi, \frac{2\pi}{2}, \frac{2\pi}{3}, \dots$$

$$\text{frequencies } f \text{ proportional to inverse of above } \frac{1}{2\pi}, \frac{2}{2\pi}, \frac{3}{2\pi}, \dots$$

$$\text{mode numbers } n = 0, 1, 2, 3, \dots$$



Neveu-Schwartz boundary condition

$$\Psi^I(-\pi) = -\Psi^I(\pi)$$

$$\text{wavelengths } 4\pi, \frac{4\pi}{3}, \frac{4\pi}{5}, \dots$$

$$\text{frequencies } f \text{ proportional to inverse of above } \frac{1}{2\pi}, \frac{3}{2\pi}, \frac{5}{2\pi}, \dots$$

$$\text{mode numbers } r = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$$

World Sheet Bosons/Fermions vs Spacetime Bosons/Fermions

It is key to keep in mind the distinction between world-sheet fermions (bosons) and spacetime fermions (bosons).

$|\text{NS}+\rangle$ are world sheet bosons and spacetime bosons

$|\text{NS}-\rangle$ are world sheet fermions and spacetime bosons (including tachyon)

$|\text{R}+\rangle$ are world sheet bosons and spacetime fermions

$|\text{R}-\rangle$ are world sheet fermions and spacetime fermions

BUT ALL employ anti-commuting creation and destruction operators on the world sheet.

(continued on next page)

Spacetime States for Open and Closed Strings

Open strings: For spacetime, truncate both NS and R to $|NS+\rangle$ (bosons) and $|R-\rangle$ (fermions) \rightarrow same number at each M^2 .

Massless: 8 bosons and 8 fermions.

Don't use $|NS-\rangle$, since it has a tachyon. Other combinations not SUSY.

Closed strings:

$$\text{Type IIA - L sector } \begin{Bmatrix} NS+ \\ R- \end{Bmatrix}, \text{ R sector } \begin{Bmatrix} NS+ \\ R+ \end{Bmatrix};$$

$(NS+, NS+)(R-, R+) =$ spacetime bosons, $(NS+, R+)(R-, NS+) =$ spacetime fermions

Massless: $8 \times 8 = 64$ of each of above combinations \rightarrow 128 bosons and 128 fermions

Same result if exchange $R-$ and $R+$ above.

$$\text{Type IIB - L sector } \begin{Bmatrix} NS+ \\ R- \end{Bmatrix}, \text{ R sector } \begin{Bmatrix} NS+ \\ R- \end{Bmatrix};$$

$(NS+, NS+)(R-, R-) =$ spacetime bosons, $(NS+, R-)(R-, NS+) =$ spacetime fermions

Massless: $8 \times 8 = 64$ of each of above combinations \rightarrow 128 bosons and 128 fermions

Same result if use $R+$ instead of $R-$ above.

Heterotic O(32) Summary: Problem 14-5.

Note: I believe $R'+$ is a spacetime boson as stated below, though not said explicitly in Zwiebach. That sector comes from 26D spacetime bosonic strings, so this should be correct. It is the only way the tensor products below make sense.

$NS'+$ and $R'+$ (left moving) are spacetime bosons (because they come from 26D bosonic string)

$NS+$ (right moving) are spacetime bosons (as they are in open superstring theory)

$R-$ (right moving) are spacetime fermions (as they are in open superstring theory)

$NS'+ \otimes NS+$ (spacetime boson times spacetime boson) is a spacetime boson

$R'+ \otimes NS+$ (spacetime boson times spacetime boson) is a spacetime boson

$NS'+ \otimes R-$ (spacetime boson times spacetime fermion) is a spacetime fermion

$R'+ \otimes R-$ (spacetime boson times spacetime fermion) is a spacetime fermion

17 Branes and Open Bosonic Strings: An Overview of Chap 15 Zwiebach

Robert D. Klauber August 8, 2023 www.quantumfieldtheory.info

Section, equation, and page numbers herein are with reference to Zwiebach, *A First Course in String Theory*.

1 Dd Branes (Space filling, $p = d$) - 1st row of Wholeness Chart 1 herein

$$X^\mu(\tau, \sigma) = X^+, X^-, X^I \quad I = 2, \dots, p \quad p = d \quad d+1 = D \quad (1)$$

$$X^\mu = x_0^\mu + \underbrace{\sqrt{2\alpha'} \alpha_0^\mu}_{2\alpha' p^\mu} \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos n\sigma \quad (9.56) [186], p^\mu \text{ from } (9.52) [185]$$

$$M^2 = \underbrace{\frac{1}{\alpha'} \left(a + \sum_{n=1} n a_n^{I\dagger} a_n^I \right)}_{a=-1 \text{ and } d=25 \text{ for Lorentz invariance}} = \frac{1}{\alpha'} \left(\sum_{n=1} n a_n^{I\dagger} a_n^I - 1 \right) = \frac{1}{\alpha'} (N^\perp - 1) \quad (12.108) [252] (12.155), (12.156) [262]$$

2 Dp branes and boundary conditions (Sect. 15.1)

Consider branes with spatial dimension p , where $p < 25$ embedded in the full $d = 25$ spatial dimension space. Strings vibrate in all 25 dimensions, but they have endpoints with Neumann B.C.s in the Dp brane and Dirichlet B.C.s with all dimensions outside the Dp brane. Take the coordinates μ to i , where i represents directions tangent to the Dp brane. For coordinates normal to the Dp brane, use coordinate symbol a for the D (Dirichlet) string coordinates normal to the Dp brane.

In the light cone gauge, the string coordinates are

$$X^\mu(\tau, \sigma) = X^+, X^-, X^i, X^a \quad i = 2, \dots, p \quad a = p+1, \dots, d \quad p < d \quad (15.7) [333]$$

3 Open Strings on Dp-branes (Sect. 15.2) - 2nd row of Wholeness Chart 1

X^i of (15.7) is simply (9.56) above with $\mu = i$ and the p value of (15.7). The coordinates normal to the Dp brane are

$$X^a(\tau, \sigma) = \tilde{x}^a + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a e^{-in\tau} \sin n\sigma, \quad (15.20) [335]$$

where we note there is no $p^a \tau$ term, since the overall (average) momentum of the string in a DD direction is zero. Such fields can't be Maxwell fields, which must always have momentum.

$$M^2 = \frac{1}{\alpha'} \left(-1 + \underbrace{\sum_n \sum_{i=2}^p n a_n^{i\dagger} a_n^i + \sum_m \sum_{a=p+1}^d m a_m^{a\dagger} a_m^a}_{N^\perp} \right) = \frac{1}{\alpha'} (N^\perp - 1) \quad (15.27) [336]$$

4 Open Strings between Parallel Dp-Branes (Sect 15.3) – 3rd row of Wholeness Chart 1

3rd row of Wholeness Chart 1 for visual image of two parallel branes showing different symbol meanings. $[ij]$ represent string oriented from i th to j th brane. i and j called Chan-Paton indices. Strings with ends on same brane (symbols [11] and [22]) are like row 2 of Wholeness Chart 1.

Strings stretching from brane 1 to brane 2

$$X^a(\tau, \sigma) = \tilde{x}_1^a + (\tilde{x}_2^a - \tilde{x}_1^a) \frac{\sigma}{\pi} + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^a e^{-in\tau} \sin n\sigma \quad (\alpha_n^a \text{ here different from } \alpha_n^a \text{ of single brane}) \quad (15.45) [340]$$

$$M^2 = \left(\frac{\tilde{x}_2^a - \tilde{x}_1^a}{2\pi\alpha'} \right)^2 + \frac{1}{\alpha'} (N^\perp - 1) \quad N^\perp \text{ same form as } (15.27) \text{ above} \quad (15.11) [34]$$

For two separate but coincident branes (i.e., $\tilde{x}_2^a - \tilde{x}_1^a = 0$ [343]), and $N^\perp = 1$, there are 4 massless gauge ($U(2)$ Yang-Mills) fields. For N coincident branes, there are N^2 massless gauge ($U(N)$ Yang-Mills) fields.

5 Strings between Parallel Dp and Dq Branes, $p > q$ (Sect. 15.4) – 4th row of Wholeness Chart 1

See the figure in 4th row of Wholeness Chart 1 for visual image of two parallel branes of different dimensions.

In the light-cone gauge, coordinates (note particular symbols) are

$$X^\mu(\tau, \sigma) = X^+, X^-, X^i, X^r, X^a \quad i = 2, \dots, q \quad r = q+1, \dots, p \quad a = p+1, \dots, d \quad (15.63) [347]$$

Coordinates for strings stretching from brane 1 to brane 2 in r (ND) direction are

$$X^r(\tau, \sigma) = \tilde{x}_2^r + i\sqrt{2\alpha'} \sum_{n \in \mathbb{Z}_{\text{odd}}} \frac{2}{n} \alpha_{n/2}^r e^{-i\frac{n}{2}\tau} \cos \frac{n\sigma}{2}. \quad (15.72) [348]$$

$$M^2 = \left(\frac{\tilde{x}_2^a - \tilde{x}_1^a}{2\pi\alpha'} \right)^2 + \frac{1}{\alpha'} \left(N^\perp - 1 + \frac{1}{16}(p-q) \right) \quad (15.84) [350]$$

$$N^\perp = \underbrace{\sum_n \sum_{i=2}^q n a_n^{i\dagger} a_n^i}_{\text{tangent (NN)}} + \underbrace{\sum_{k \in \mathbb{Z}_{\text{odd}}} \sum_{r=q+1}^p \frac{k}{2} a_{k/2}^{r\dagger} a_{k/2}^r}_{\text{mixed (ND)}} + \underbrace{\sum_m \sum_{a=p+1}^d m a_m^{a\dagger} a_m^a}_{\text{normal (DD)}} \quad (15.85) [350]$$

Wholeness Chart 1. Overview of Branes and Open Strings

	<u>Visually in Low Dimensions</u>	<u>Ground States</u>	<u>Tachyons</u>	<u>1 Tangent</u>	<u>1 Normal</u>
Dp brane in all d (p=d) spatial dimensions		$ p^+, \vec{p}\rangle$ $\vec{p} = (p^2, \dots, p^d)$ (12.159) [263]	$N^\perp = 0$ $n = 0$ For $D = 26$, $M^2 = -\frac{1}{\alpha'}$ Lorentz scalar	$N^\perp = 1$ $n = 1$ For $D = 26$ $M^2 = 0$ Maxwell field, 24 components	No such animal (All string oscillations inside p=d)
Dp brane in d (p<d) spatial dimensions		$ p^+, \vec{p}\rangle$ $\vec{p} = (p^2, \dots, p^p)$ (15.82) [336] no \vec{p} outside brane	$N^\perp = 0$ $n=m=0$ For $D = 26$ $M^2 = -\frac{1}{\alpha'}$ Lorentz scalar, like above	$N^\perp = 1$ $n = 1, m = 0$ For $D = 26$ $M^2 = 0$ Maxwell field, p-1 components	$N^\perp = 1$ $n = 0, m = 1$ For $D = 26$ $M^2 = 0$ Massless scalar each a direction
2 Dp branes in d (p<d) spatial dimensions	<p>Figure 15.1 [339] strings vibrate in all d directions</p>	$ p^+, \vec{p}; [11]\rangle$ $ p^+, \vec{p}; [22]\rangle$ $ p^+, \vec{p}; [12]\rangle$ $ p^+, \vec{p}; [21]\rangle$ (15.54) [341] [11] and [22] like row above	[12] and [21] $N^\perp = 0$ $n=m=0$ $D = 26, M^2 =$ $\left(\frac{\tilde{x}_2^a - \tilde{x}_1^a}{2\pi\alpha'} \right)^2 - \frac{1}{\alpha'}$ M^2 neg, zero, or pos Lorentz scalar Tachyon if <0	[12] and [21] $N^\perp = 1$ $n = 1, m = 0$ For $D = 26$ $M^2 = \left(\frac{\tilde{x}_2^a - \tilde{x}_1^a}{2\pi\alpha'} \right)^2$ Massive vector (not Maxwell). One of scalars (at right) added for p components	[12] and [21] $N^\perp = 1$ $n = 0, m = 1$ For $D = 26$ $M^2 = \left(\frac{\tilde{x}_2^a - \tilde{x}_1^a}{2\pi\alpha'} \right)^2$ Massive scalars. Scalar pointing between branes added to vector at left. So d-p-1 scalars here.
Parallel Dp and Dq branes (p<d, q<d, p+q=d)	<p>Figure 15.3 [346]</p>	$ p^+, \vec{p}; [11]\rangle$ $ p^+, \vec{p}; [22]\rangle$ $ p^+, \vec{p}; [12]\rangle$ $ p^+, \vec{p}; [21]\rangle$ $\vec{p} = (p^2, \dots, p^q)$ (15.86) [350] [11] and [22] like two rows above	[12] and [21] $N^\perp = 0$ $n=k=m=0$ $D = 26, M^2 =$ $\left(\frac{\tilde{x}_2^a - \tilde{x}_1^a}{2\pi\alpha'} \right)^2 - \frac{1}{\alpha'}$ M^2 neg, zero, or pos Lorentz scalar Tachyon if <0	[12] and [21] $N^\perp \geq 1$ For $D = 26$ $M^2 > 0$ since $p > q$. No massless gauge fields.	As at left

18 How Strings Give Rise to Fields Like Maxwell Fields

Robert D. Klauber March 13, 2024 www.quantumfieldtheory.info

1 Quantization: The Big Picture

Recall from QFT that when we quantize a classical field, the classical field is typically a displacement vector. Think of an elastic solid or a fluid continuum, where the classical field is comprised of the vector displacement in space of the continuum at every point, and that displacement is generally a function of time. We can symbolize the displacement by X^μ , so

$$X^\mu = X^\mu(t, \mathbf{x}) = X^\mu(x^\alpha), \quad \text{a classical displacement field.} \quad (1)$$

Upon quantization, i.e., upon invoking the canonical commutation relations, the classical field becomes a quantum field, which is not a displacement typically, but an operator (that creates and destroys states). We can symbolize the quantum field by A^μ , so

$$X^\mu = X^\mu(x^\alpha) \xrightarrow{\text{quantization}} A^\mu = A^\mu(x^\alpha), \quad \text{a quantum creation/destruction operator field.} \quad (2)$$

2 QFT Review

2.1 Maxwell Fields in QFT

In 4D, if A^μ has two independent components (two transverse fields in 3D space) plus is massless, it can be a Maxwell field, i.e., a photon field. That is, for a Maxwell field, the number of independent components in 4D is $4-2=2$, or $D-2$.

We can generalize to higher dimensions D , where d is the number of spatial dimensions.

$$\text{independent components} = D-2 = d-1 \quad \text{necessary for a Maxwell field.} \quad (3)$$

Thus, it is necessary that a Maxwell field must be i) massless plus have ii) $d-1$ independent components, but that is not enough (not sufficient). A candidate field must also satisfy Maxwell's equation.

Bottom line #1: A Maxwell field i) is massless, ii) has $d-1$ independent components, iii) has D total components, and iv) satisfies Maxwell's equation. These are necessary and sufficient conditions for (2) to be a Maxwell field.

2.2 Massive Fields in QFT

In 4D, massive vector fields in QFT, such as the W s and Z of electroweak theory, have three independent components, not two. (See Klauber, *Student Friendly QFT, Vol. 2, The Standard Model*, Sect. 5.4, pgs. 154-157.) That is, massive 4D boson vector fields have $4-1=D-1=d=3$ independent components.

We can generalize to higher D , again.

$$\text{independent components} = D-1 = d \quad \text{necessary for a massive boson field.} \quad (4)$$

Bottom line #2: A massive vector field has i) mass, ii) has d independent components, iii) has D total components, and iv) satisfies the Proca equation [(5-80), pg. 157 in above reference Klauber.] These are necessary and sufficient conditions for (2) to be a massive vector field.

2.3 Scalar Fields in QFT

In 4D, a scalar field has a single component. It is not a vector with D components of any mix of independent and dependent components.

Generalizing to any D , we have

$$\text{independent components} = 1 \quad \text{necessary for a scalar field for any } D. \quad (5)$$

Bottom line #3: A scalar field, whether massive or massless, has i) one independent component, and ii) satisfies the Klein-Gordon equation. These are necessary and sufficient conditions for (2) to be a scalar field (for no component index μ) or a collection of scalar fields where each value of μ represents a different independent scalar field (not a component of a vector field).

3 QFT Boson Equations in Light-Cone Coordinates

3.1 Scalar Fields in Light-Cone Coordinates

The Klein-Gordon equation, in the usual spacetime coordinates, is

$$(\partial_\mu \partial^\mu - m^2)\phi(x^a) = 0 \quad (10.14) [197]. \quad (6)$$

For light-cone coordinates, we have

$$\partial_\mu \partial^\mu = -2 \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} + \frac{\partial}{\partial x^I} \frac{\partial}{\partial x^I} \quad \text{parallel to (2.52) [23] for } x_\mu x^\mu. \quad (7)$$

The K-G equation, in those coordinates, becomes

$$\left(-2 \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} + \frac{\partial}{\partial x^I} \frac{\partial}{\partial x^I} - m^2 \right) \phi(x^+, x^-, x^I) = 0. \quad (8)$$

We can Fourier transform ϕ in (8), just in the x^- and x^I parts, so, where we note that p^+ is the conjugate momentum for x^- ,

$$\phi(x^+, x^-, x^I) = \int e^{-ix^- p^+ + ix^I p^I} \phi(x^+, p^+, p^I) \frac{dp^+}{2\pi} \frac{d^{D-2} p^I}{(2\pi)^{D-2}} \quad (10.28) [200]. \quad (9)$$

The K-G equation (8) then becomes

$$\left(-2 \frac{\partial}{\partial x^+} (-ip^+) - p^I p^I - m^2 \right) \phi(x^+, p^+, p^I) = 0 \quad (10.29) [200], \quad (10)$$

or

$$\left(i \frac{\partial}{\partial x^+} - \frac{1}{2p^+} (p^I p^I - m^2) \right) \phi(x^+, p^+, p^I) = 0 \quad (10.30) [200] \& (12.192) [270]. \quad (11)$$

Here x^+ is an independent variable upon which ϕ depends. In the light-cone gauge and coordinates, the dependent X^+ is a function of p^+ and τ , i.e.,

$$X^+ = 2\alpha' p^+ \tau \quad \text{for open strings, in light cone gauge (9.70) [188].} \quad (12)$$

Since X^μ are the coordinates on the string worldsheet, and we want to know the form of the K-G equation if it were confined to the worldsheet, we can take our independent coordinate x^+ of (11) (which normally spans all space) as the dependent coordinate X^+ (which spans only the worldsheet), That is

$$x^+ \text{ on the worldsheet} = X^+, \quad (13)$$

where, from (12),

$$\frac{\partial}{\partial x^+} = \frac{\partial}{\partial X^+} = \frac{1}{2\alpha' p^+} \frac{\partial}{\partial \tau}. \quad (14)$$

With (14) into (11), we have

$$\left(i \frac{\partial}{\partial \tau} - \alpha' (p^I p^I - m^2) \right) \phi(x^+, p^+, p^I) = 0 \quad \text{K-G eq on worldsheet, open string. (12.194) [270]} \quad (15)$$

3.2 Photon Fields in Light-Cone Coordinates

Taking $m = 0$ and $\phi \rightarrow A^\nu$ in (6), we get Maxwell's equation in the Lorenz gauge,

$$\partial_\mu \partial^\mu A^\nu(x^a) = 0. \quad (16)$$

Following the same procedure that got us from (6) to (15), we find, parallel to (15), for the transverse (to the x^+ and x^- axes) photon field

$$\left(i \frac{\partial}{\partial \tau} - \alpha' p^I p^I \right) A^I(x^+, p^+, p^I) = 0 \quad \text{Maxwell eq on worldsheet, open string (12.194) [270].} \quad (17)$$

I believe here we are thinking of the photon as traveling in the x^I direction, where it has no longitudinal component. It also has no time direction component, as the only photon field components in 4D are the spatial ones transverse to the photon velocity direction. Thus, since A^+ and A^- are each a combination of A^0 and A^I , which equal zero for this photon orientation, then $A^+ = A^- = 0$.

However, if the photon were not so judiciously aligned, it seems (17) should hold for all components, i.e.,

$$\left(i \frac{\partial}{\partial \tau} - \alpha' p^I p^I \right) A^V(x^+, p^+, p^I) = 0 \quad v = +, -, I. \quad (18)$$

Note that the components of the photon field can, in general, be in any direction in all of spacetime, including not tangent to the worldsheet, even though the string itself is confined to the worldsheet, i.e., the independent coordinates $x^\mu = x^+, x^-, x^I$ or x^0, x^I are confined to the world sheet.

(15) and (18) can be re-arranged to be equations of the Schrödinger type form.

$$\begin{aligned} i \frac{\partial}{\partial \tau} \phi &= \alpha' (p^I p^I - m^2) \phi \\ i \frac{\partial}{\partial \tau} A^V &= \alpha' p^I p^I A^V \end{aligned} \quad (19)$$

4 String Field Equations

4.1 Basic String Equation of Motion in Independent Parameters τ and σ

The basic string equation of motion for the chosen gauge family (in either Minkowski or light-cone coordinates) is

$$\left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2} \right) X^V(\tau, \sigma) = 0 \quad v = +, -, I \quad \text{or} \quad = 0, 1, I \quad (9.39) [183], (20)$$

with open string solution

$$X^V(\tau, \sigma) = x_0^V + \frac{\sqrt{2\alpha'} \alpha_0^V}{2\alpha' p^V} \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^V e^{-in\tau} \cos n\sigma \quad (9.56) [186]. \quad (21)$$

Upon quantization, the field X^V is sometimes represented as ψ^V in Zwiebach (12.185),

$$X^V(\tau, \sigma) \xrightarrow[\text{notation in Zwiebach}]{\text{after quantization}} \psi^V(\tau, \sigma) \quad [269]. \quad (22)$$

4.2 Photon Field Directly from Basic String Field Equation of Motion

The question becomes “is (20) equivalent to (16)?”

Consider the static gauge in (20) where $\tau = t$ and σ represents the physical length along the string s . To keep things simple, work in Minkowski coordinates (v of the RH of (20)), though the same conclusion can be drawn with light-cone coordinates. We then have

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} \right) X^V = 0. \quad (23)$$

The physical length s along a vector obeys

$$\frac{\partial^2}{\partial s^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \dots = \partial_\mu \partial^\mu, \quad (24)$$

which turns (23) into

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \dots \right) X^v = 0 \quad \rightarrow \quad \partial_\mu \partial^\mu X^v = 0. \quad (25)$$

(25) equals Maxwell's equation (16) and also, (25) equals (23). (23) equals (20) in this gauge. The equation of motion (20) is invariant under change of gauge.

Conclusion

So, in general it seems, (20), the string equation of motion for zero mass, can equal the photon equation of motion, Maxwell's equation (16).

Caveat: For somewhat complicated reasons we won't get into, $d\sigma$ only equals ds for a massless string. The following section shows another way to consider relativistic strings equivalent to Maxwell fields.

4.3 Another Way to Make the Connection between Strings and Photons

The gauge family of choice is defined by, where $\beta = 1$ for closed strings and 2 for open string,

$$n_\mu X^\mu = \beta \alpha' (n_\mu p^\mu) \tau = \beta \frac{1}{2\pi T_0} (n_\mu p^\mu) \tau \quad (26)$$

$$n_\mu \mathcal{P}^{\tau\mu} = \frac{\beta}{2\pi} n_\mu p^\mu \quad (27)$$

where

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \frac{\partial X^\mu}{\partial \tau}. \quad (28)$$

The σ Gauge Condition (27)

Consider (28) into (27), where to simplify, a and b are the only non-zero components of the unit vector,

$$n_\mu = (a, b, 0, 0, \dots), \quad (29)$$

$$n_\mu \frac{1}{2\pi\alpha'} \frac{\partial X^\mu}{\partial \tau} = \frac{\beta}{2\pi} n_\mu p^\mu \quad \rightarrow \quad a \frac{\partial X^0}{\partial \tau} - b \frac{\partial X^1}{\partial \tau} = \alpha' \beta (ap^0 - bp^1). \quad (30)$$

Since a and b can be varied, the only way (30) can be true is if the quantities multiplied by a are equal, and the quantities multiplied by b are equal. Also, we consider Minkowski coordinates where $X^0 = t$ on the string worldsheet, and $X^1 = x$ the distance in the x direction on the world sheet, both as seen by an observer in the t - x system. Then, (30) gives us

$$\begin{aligned} a \frac{\partial X^0}{\partial \tau} &= a\alpha'\beta p^0 & b \frac{\partial X^1}{\partial \tau} &= b\alpha'\beta p^1 \\ \frac{\partial t}{\partial \tau} &= \alpha'\beta E & \frac{\partial x}{\partial \tau} &= \alpha'\beta p^1 \end{aligned} \quad (31)$$

Now, dividing the bottom two relations in (31), we get

$$\frac{\partial x}{\partial t} = \frac{p^1}{E} \quad \rightarrow \quad v = \frac{p^1}{E} \quad v = \text{velocity of string as seen by observer}. \quad (32)$$

For massless particles $|p| = E$, and here for us, $|p| = p^1$. Thus,

$$v = 1 \quad \text{for a massless particle. In natural units, this is the speed of light.} \quad (33)$$

Strings are wavelike. (Their equation of motion is a wave equation.) And if they are massless, they have speed equal to that of light. Hence, they can represent photons, or any other massless elementary particle.

Note further, that if $p^1 = E$, then from the second row of (31), $x = t$. The string travels the edge of the light cone, as objects traveling at the speed of light must do.

The τ Gauge Relation (26)

We can get the same result using the gauge relation (26) for τ .

$$aX^0 - bX^1 = \beta \frac{1}{2\pi T_0} (aE - bp^1) \tau \quad at = a\beta \frac{1}{2\pi T_0} E\tau \quad bx = b\beta \frac{1}{2\pi T_0} p^1 \tau \quad (34)$$

$$\frac{x}{t} = \frac{p^1}{E} = v \quad \text{for massless, } p^1 = E \quad v = 1 \quad (\text{the speed of light}) \quad (35)$$

Conclusion

A massless string satisfies a wave equation and moves at the speed of light, just as photons do. The massless string field can be considered a Maxwell field.

4.4 Photon Field from String Field via Schrödinger Type Equation

A different approach, used in Zwiebach, pgs. 268-270, is to express the equation of motion for $X^V = \psi^V$ of (20) to (22) in a form similar to (18), i.e., in a Schrödinger equation type form. To do that, Zwiebach focuses on the transverse components of the string field, labeled with a subscript I .

$$\text{string field } \psi^V(\tau, p^+, p^I) \xrightarrow{\text{for transverse components only}} \psi_I(\tau, p^+, p^I). \quad (36)$$

He then looks at the Fourier transform of (36)

$$\psi_I(x^+, x^-, x^I) = \int e^{-ix^- p^+ + ix^I p^I} \psi_I(x^+, p^+, p^I) \frac{dp^+}{2\pi} \frac{d^{D-2} p^I}{(2\pi)^{D-2}}. \quad \text{imbedded in (12.185) [269].} \quad (37)$$

FIRST QUESTION: HOW DOES THE ARGUMENT OF (37) (x^+, x^-, x^I) ARISE FROM THE ARGUMENT (τ, σ) OF (20) to (22)??

Zwiebach then states, pg. 269 above (12.186), that “The Schrödinger equation [type form] satisfied by the general case (12.183) [our (37)] is”

$$i \frac{\partial}{\partial \tau} \psi_I = H \psi_I. \quad (38)$$

SECOND QUESTION: WHERE DOES HE GET THIS FROM? IS HE NOT ASSUMING WHAT HE SET OUT TO PROVE?

He then notes that, in string theory,

$$H = L_0^\perp - 1 = \alpha' p^I p^I + N^\perp - 1 = \alpha' (p^I p^I + M^2) \quad (12.187) [269], \quad (39)$$

so, (38) becomes, for photons with $M = 0$,

$$i \frac{\partial}{\partial \tau} \psi_I = \alpha' p^I p^I \psi_I. \quad (40)$$

Conclusion

Since the string equation of motion (40) expressed in the light-cone gauge is the same as the photon equation of motion (Maxwell's equation) (19) in the light-cone gauge, strings can manifest as photons.

PROBLEM?: Did he assume what he wanted to prove.

This page intentionally left blank for best arrangement of subsequent pages.

19. Kalb-Ramond String Charge vs Electromagnetic Charge

Robert D. Klauber www.quantumfieldtheory.info March 10, 2024

See Zwiebach, Chap. 16, for equations and page numbers referenced below. This chart summarizes Sect. 16.1, pgs 356 to 362.

“0” subscripts on Lagrangians indicate free particles or fields; “*P*” subscripts indicate interaction Lagrangians. Note that in 4D, $dV = d^3x dt$ is invariant, so rest frame $dV_0 = d^3x_0 d\tau = dV = d^3x dt$ in any frame (where subscript “0” here means rest frame). Similarly, for D dimensions, $dV = d^d x dt = d^D x$ (where $d = D - 1$) is invariant, so in any dimensions, $dV_0 = dV = d^D x$ in any frame.

Entity	Electromagnetism	Strings	
Field	A_μ (Maxwell field = vector potential)	$B_{\mu\nu}$ (Kalb-Ramond field = tensor potential; antisymmetric)	
Tensor of measurables	E and B fields components of $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$	K-R measurable fields components of $H_{\mu\nu\rho} \equiv \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$	Both antisym, any 2 indices
Action, Charged object, D dimens	$S = \underbrace{-m \int_{\text{path}} d\tau}_{\text{free charged point particle}} + \underbrace{q \int_{\text{path}} A_\mu \frac{dx^\mu}{d\tau} d\tau}_{\text{particle-}A_\mu\text{ interaction}} - \underbrace{\frac{1}{4\kappa_0^2} \int F_{\mu\nu} F^{\mu\nu} d^D x}_{\text{free photon field}} \quad (16.2)$ $- \int_{\text{path}} m \frac{1}{\gamma} dt + \underbrace{q \int_{\text{path}} A_\mu \frac{dX^\mu}{d\tau} d\tau}_{\text{string-}B_{\mu\nu}\text{ interaction}} - \frac{1}{4\kappa_0^2} \int F_{\mu\nu} F^{\mu\nu} d^d x dt \quad [356]$ $+ \int_{\text{path}} q A_\mu u^\mu \gamma dt + \int_{\text{path}} q A_\mu dx^\mu$	$S = \underbrace{S_{str}}_{\text{free string}} + \underbrace{S_B}_{\text{string-}B_{\mu\nu}\text{ interaction}} + \underbrace{S_H}_{\text{free K-R field}} \quad [357]$ $= S_{str} - \int B_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} d\sigma d\tau - \frac{1}{6\kappa^2} \int H_{\mu\nu\rho} H^{\mu\nu\rho} d^D x \quad S_B = (16.3)$ $= S_{str} - \frac{1}{2} \int B_{\mu\nu} \frac{\partial X^{[\mu}}{\partial \tau} \frac{\partial X^{\nu]}}{\partial \sigma} d\sigma d\tau - \frac{1}{6\kappa^2} \int H_{\mu\nu\rho} H^{\mu\nu\rho} d^D x \quad (16.4)$	KR charge taken as 1 X^μ designates spacetime path; x^μ , fixed coordinate system grid
<i>L</i>	$L_0^{\text{particle}} = -\frac{m}{\gamma} \quad L_I^{e/m} = q A_\mu u^\mu \gamma \quad L_0^{\text{photon}} = -\frac{1}{4\kappa_0^2} \int F_{\mu\nu} F^{\mu\nu} d^d x$	Not relevant for us.	
Action, charged region, D dimens	$S = \underbrace{- \int \rho_{0m} d^d x_0 d\tau}_{\text{free mass, charged region}} + \underbrace{\int A_\mu u^\mu \rho_{0e/m} d^D x}_{\text{charged region-}A_\mu\text{ interaction}} - \underbrace{\frac{1}{4\kappa_0^2} \int F_{\mu\nu} F^{\mu\nu} d^D x}_{\text{free photon field}}$ $- \int \rho_{0m} d^D x \quad S_I^{e/m}$ $\mathcal{L}_0^{\text{particle}} = -\rho_{0m} \quad \mathcal{L}_I^{e/m} = A_\mu \rho_{0e/m} u^\mu = A_\mu j^\mu \quad \mathcal{L}_0^{\text{photon}} = -\frac{1}{4\kappa_0^2} F_{\mu\nu} F^{\mu\nu}$	The string is a charged region (charge spread over the string length).	

Define current	Maxwell current = $j^\mu = \rho_{0\ e/m} u^\mu = \rho_{0\ e/m} \frac{\partial x^\mu}{\partial \tau} \Big _{\text{path}} = \rho_{0\ e/m} \frac{\partial X^\mu}{\partial \tau}$	K-R current = $j^{\mu\nu} = \frac{1}{2} \int \delta^D(x-X) \left(\frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} - \frac{\partial X^\nu}{\partial \tau} \frac{\partial X^\mu}{\partial \sigma} \right) d\sigma d\tau$ (16.11)	$j^{\mu\nu}$ antisym $\rightarrow j^{00} = 0$
S_I	$S_I^{e/m} = \int A_\mu j^\mu d^D x$ Only non-zero in region where charge density is non-zero.	$S_I^{K-R} = S_B = -\frac{1}{2} \int B_{\mu\nu} j^{\mu\nu} d^D x$ (16.10) [358] $\delta^D(x-X)$ in $j^{\mu\nu}$ confines action to string world sheet. Zero elsewhere.	$j^{\mu\nu}$ only non-zero along string
$\delta S \rightarrow$ eq of motion	$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = j^\mu$ (16.7) [358] Gauss and Ampere equations in Maxwell's equations. $j^\mu = e/m$ source	$\frac{1}{\kappa^2} \frac{\partial H^{\mu\nu\rho}}{\partial x^\rho} = j^{\mu\nu}$ (16.14) [359] K-R charged string (represented in $j^{\mu\nu}$) source of K-R field.	
Conservation of charge	$\partial_\mu j^\mu = 0$ (16.15) [359]	$\partial_\mu j^{\mu\nu} = 0$ (16.16) [359]	
Charge density	j^0	j^{0k} $k = 1, 2, 3$ ($j^{0\mu}$ really, but we have $j^{00} = 0$) (16.17) [359]	
Divergence comparison	Electrostatics where $\frac{\partial j^0}{\partial t} = 0$ from (16.15) $\rightarrow \partial_i j^i = 0$ ($\nabla \cdot \mathbf{j} = 0$)	$j^{00} = 0$ with (16.16) $\rightarrow \partial_k j^{k0} = 0$ ($\nabla \cdot \mathbf{j}^0 = 0$) (16.19) [359]	At left, string conserve of charge relation
Charge	Scalar $Q = \int j^0 d^D x$	3-vector $\mathbf{Q} = \int \mathbf{j}^0 d^D x$ (16.20) [360]	
String charge direction	N/A	$\mathbf{j}^0(\mathbf{x}, t) = \frac{1}{2} \int \delta(\mathbf{x} - \mathbf{X}(t, \sigma)) \frac{\partial \mathbf{X}(t, \sigma)}{\partial \sigma} d\sigma$ (16.23) [360] Charge density \mathbf{j}^0 is tangent to string at every point on string. Points in $+\sigma$ direction. Unoriented strings have no K-R field and don't carry K-R (string) charge.	(16.11) [358] in static gauge, $\tau = t$, with $\nu = 0$
K-R field variation vector potential	N/A	$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu$ (16.43) [366] Λ_ν different from $B_{\mu\nu}$ as tensor potential for $H_{\mu\nu\rho}$. (See 1 st and 2 nd rows at top of chart.)	

20 How Maxwell Charge Is Located at Endpoints of Open Strings

Robert D. Klauber www.quantumfieldtheory.info

March 10, 2024

Chapter, section, equation and page references are to Zwiebach.

Chap 16. String Charge and Electric Charge [356-375]

Pg. 356, Chapter Abstract: “If a point particle couples to the Maxwell field, then that particle carries electric charge. ... string endpoints carry electric charge ...”

Sect. 16.1 [356-362]

1st part of section shows e/m point particle Lagrangian and how that has coupling to Max field.

[357] “Can a relativistic string be charged? The above argument makes it clear that Maxwell charge is naturally carried by points. ... It is therefore plausible that the endpoints of open strings carry electric Maxwell charge. We will show later that this is indeed the case.”

Sect. 16.3 [365-370]

[365] “If D-branes have Maxwell fields [living on them], is there any object that carries electric charge for these fields?” (Note this is related to Kalb-Ramond fields $B_{\mu\nu}$.)

“... the realization that the ends of the open string behave as electric point charges! They are charged under the Maxwell field that lives on the D-brane where the string ends.” (As an aside, the electric field lines of the point charges carry K-R string charge, even though they carry no electric charge. This is needed to conserve K-R charge.)

Elaboration of pages [366-368]

In varying $B_{\mu\nu}$, the Kalb-Ramond [tensor] field, we find we can express that variation in terms of a parameter Λ_μ (the Kalb-Ramond [vector] potential, a vector in D dimensions), as

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \quad (16.43) [366]$$

In varying S_B , the action for the Kalb-Ramond field¹, we find (after some effort)

$$\delta S_B = \int \Lambda_m \frac{\partial X^m}{\partial \tau} d\tau \Bigg|_{\sigma=0}^{\sigma=\pi} \neq 0. \quad m = 0, 1, \dots, p \text{ for } Dp\text{-brane} \quad (16.53) [368]$$

Since the action is not zero under variation (due here to the B.C.’s at the string end points), the Kalb-Ramond charge is not conserved. We can cause it to be conserved if we redefine the action for the K-R field as

$$\hat{S}_B = S_B + \int A_m \frac{\partial X^m}{\partial \tau} d\tau \Bigg|_{\sigma=0}^{\sigma=\pi} \quad (16.55) [368]$$

where we define the transformation under variation for the field A_m as

$$\delta A_m = -\Lambda_m. \quad (16.56) [368]$$

Then, for the redefined string action, the variation, using (16.55), (16.53), and (16.56), and understanding that variations are done only for fields $B_{\mu\nu}$ and A_m , not X^μ , τ , or σ , is

$$\begin{aligned} \delta \hat{S}_B &= \delta S_B + \delta \left(\int A_m \frac{\partial X^m}{\partial \tau} d\tau \Bigg|_{\sigma=0}^{\sigma=\pi} \right) = \int \Lambda_m \frac{\partial X^m}{\partial \tau} d\tau \Bigg|_{\sigma=0}^{\sigma=\pi} + \int (\delta A_m) \frac{\partial X^m}{\partial \tau} d\tau \Bigg|_{\sigma=0}^{\sigma=\pi} = 0 \\ &= \int \Lambda_m \frac{\partial X^m}{\partial \tau} d\tau \Bigg|_{\sigma=0}^{\sigma=\pi} + \int (-\Lambda_m) \frac{\partial X^m}{\partial \tau} d\tau \Bigg|_{\sigma=0}^{\sigma=\pi} = 0 \end{aligned} \quad (1)$$

Since (1) equals zero, we have, by Noether’s theorem, a conserved K-R charge, i.e., conserved string charge.

¹ $S_B = -\int B_{\mu\nu} \frac{\partial X^\mu}{\partial \tau} \frac{\partial X^\nu}{\partial \sigma} d\tau d\sigma$ (16.3)[357] = $-\frac{1}{2} \int B_{\mu\nu} \frac{\partial X^{[\mu}}{\partial \tau} \frac{\partial X^{\nu]}}{\partial \sigma} d\tau d\sigma$ (middle term in (16.4) [357]).

But what about electric charge? To answer that question, start by noting from (16.2) [356] that the action for a charge located at a point is

$$S'_{e/m \text{ interact point charge}} = q \int_{-\infty}^{+\infty} A_{\mu}(x(\tau)) \frac{dx^{\mu}(\tau)}{d\tau} d\tau \quad \left(= q \int_{\mathcal{P}} A_{\mu}(x) dx^{\mu} \quad \mathcal{P} = \text{path of point charge in spacetime} \right), (2)$$

where x^{μ} are the coordinates in spacetime of the point particle, i.e., as τ evolves, the particle worldline is the locus of points $x^{\mu}(\tau)$.

For the action confined to a string, we take $x^{\mu} \rightarrow X^{\mu}$, where X^{μ} represents the spacetime coordinates along the string where a point charge exists. So, for a string with a point charge located at X^{μ} , (2) becomes

$$S'_{\text{interaction point charge on a string}} = q \int A_{\mu}(X(\tau)) \frac{dX^{\mu}(\tau)}{d\tau} d\tau. (3)$$

The worldline of a point charge on the string is the locus of points $X^{\mu}(\tau)$, where σ for the point charge may, or may not, change over time. (3) implies we are following the point charge in its route through spacetime, but that route is confined to being somewhere on the string.

For a charge located at the end of the string where $\sigma = \pi$, (3) becomes

$$S'_{\text{interaction point charge at } \sigma=\pi \text{ end}} = q \int A_{\mu}(X(\tau)) \frac{dX^{\mu}(\tau)}{d\tau} d\tau \Big|_{\sigma=\pi}. (4)$$

For the other string end, we just take $\sigma = \pi \rightarrow \sigma = 0$, so

$$S'_{\text{interaction point charge at } \sigma=0 \text{ end}} = q \int A_{\mu}(X(\tau)) \frac{dX^{\mu}(\tau)}{d\tau} d\tau \Big|_{\sigma=0}. (5)$$

Now, if we, by convention, take $q = \pm 1$, we represent a positive charge at $\sigma = \pi$ and a negative charge at $\sigma = 0$ with terms in the action

$$S'_{\text{interaction + charge at } \sigma=\pi \text{ - charge at } \sigma=0} = \int A_{\mu} \frac{dX^{\mu}}{d\tau} d\tau \Big|_{\sigma=\pi} - \int A_{\mu} \frac{dX^{\mu}}{d\tau} d\tau \Big|_{\sigma=0} = \int A_m \frac{\partial X^m}{\partial \tau} d\tau \Big|_{\sigma=0}^{\sigma=\pi}. (6)$$

But, the terms in (6) are the same ones we added to (16.55) to get an invariant \hat{S}_B , i.e., to get a conserved K-R (string) charge.

Bottom line: By redefining the action in order to get a conserved string charge, we have to add terms that represent electric point charges on the ends of the string.

With our convention the $\sigma = 0$ end has negative charge, and the $\sigma = \pi$ end has positive charge.

For Closed Strings

Note that for closed strings, the B.C. condition in (16.53) is zero. That is

$$\delta S_B = \delta \int A_m \frac{\partial X^m}{\partial \tau} d\tau \Big|_{\sigma=0}^{\sigma=2\pi} = 0. (7)$$

=0 for 0 and 2π
the same point

Hence, 1) the Kalb-Ramond action is invariant, so the K-R (string) charge is conserved, and 2) we have no terms in the action that lead to point charges on the closed string.

Bottom line: Closed strings carry no electric charge. They can carry K-R (string) charge.

21. Electric Charge on Compact D-Branes

Robert D. Klauber August 24, 2023

www.quantumfieldtheory.info

This is an elaboration of material found in Zwiebach, Sect. 16.4 [pgs 370-373]. Equations cited with “(16.X)” and page numbers are with reference to that text.

1 Background

The e/m interaction part of the action for an electric point charge q is

$$S_I^{e/m} = \int A_\mu(x^\alpha(\tau)) j_{e/m, \text{point}}^\mu(\tau) d\tau = \int A_\mu(x^\alpha(\tau)) q u^\mu(\tau) d\tau = q \underbrace{\int A_\mu(x^\alpha(\tau)) \frac{dx^\mu(\tau)}{d\tau} d\tau}_{\text{Zwiebach (16.1) [356]}} = q \int A_\mu(x^\alpha) dx^\mu, \quad (1)$$

where $x^\mu(\tau)$ is the path of the particle through spacetime, and the 4D source current is

$$j_{e/m, \text{point}}^\mu(\tau) = q \frac{dx^\mu(\tau)}{d\tau}. \quad (2)$$

As an aside, for an electric distributed charge it is

$$S_I^{e/m} = \int A_\mu(\tau, x') j_{e/m}^\mu(\tau, x') d^4x' = \int A_\mu(\tau, x') \rho_{e/m}(\tau, x') u^\mu(\tau, x') d^4x'. \quad (3)$$

For a point particle located at x ,

$$\rho_{e/m}(\tau, x') = q \delta(x' - x), \quad (4)$$

and (3) reduces to the quantity after the second equal sign in (1).

Bottom line: If in string theory, we find a relationship like (1) arising at any point, it would represent the action for a charged point particle in a Maxwell field.

Note that all of the above reasoning is not restricted to 4D spacetime, but to spacetimes of any number of spatial dimensions.

2 Higher Order Tensor Fields than Maxwell's

2.1 The Second Order Field

2.1.1 The Action

The Maxwell field $A^\mu(x^\alpha)$ is a vector field, a first order tensor. In his Chap. 16, Zwiebach extrapolates this analysis to a possible 2nd order tensor field $B^{\mu\nu}(X^\alpha)$, where the field acts over the string world sheet coordinates $X^\alpha(\tau, \sigma)$ rather than the particle worldline $x^\alpha(\tau)$. It is reasonable, therefore, to deduce that the action in this case parallels (1), and we find

$$S_I^B = k_B \int B_{\mu\nu}(X^\alpha(\tau, \sigma)) \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\sigma} d\tau d\sigma, \quad (16.3) [357]$$

where Zwiebach takes $k_B = -1$, by convention. $B^{\mu\nu}$ is an anti-symmetric tensor and is known as a Kalb-Ramond (K-R hereafter) field (as opposed to a Maxwell field with only one index.)

In subsequent sections, Zwiebach discusses the K-R field on an open string and its associated source, the string charge. We do not address that herein, but note that the string charge is a vector quantity, whereas the electric charge is a scalar. As we progress to higher order tensor fields, the associated charge will have one index less than the field it is the source for.

2.1.2 Open vs Closed Strings

Open Strings

Additionally, which is elaborated on in Chap. 20 herein, the end points of the open string are shown to have point electric charges on them, due to the K-R field being coupled to the Maxwell field. So, an open string has

- i) electric point charges at its ends (negative at $\sigma = 0$ and positive at $\sigma = 2\pi$), and
- ii) string (K-R) charge (a vector) all along its length.

Closed Strings

As shown in that same chapter, a closed string carries

- i) no electric charge (as there are no ends), and
- ii) (I believe) string charge (a vector) all along its length.

2.2 Higher Order Anti-symmetric Tensor Fields

2.2.1 Extrapolating the Action to Any Order

In (16.3), the K-R field acts over a two-dimensional world sheet, which we can think of as inside of a 4D spacetime world, or as inside of a $D = d + 1$ spacetime, where d , the number of space dimensions, is arbitrary.

A string is considered a 2D brane in spacetime, a Dp -brane, where $p = 1$, i.e., a D1-brane. Spacetime generally has more dimensions than the string brane, i.e., $d > p$ (exception = space filling brane where $p = d$).

Parallel to (16.3), the action for tensor fields higher than second order acting on a Dp brane, where $p \leq d$, can be crafted as (where our S_{I_p} equals Zwiebach's S_p)

$$S_{I_p} = - \int A_{\mu\mu_1\mu_2\dots\mu_p} \left(X^\alpha(\tau, \sigma^1, \sigma^2, \dots, \sigma^p) \right) \frac{dX^\mu}{d\tau} \frac{dX^{\mu_1}}{d\sigma^1} \frac{dX^{\mu_2}}{d\sigma^2} \dots \frac{dX^{\mu_p}}{d\sigma^p} d\tau d\sigma^1 d\sigma^2 \dots d\sigma^p \quad (16.62) [371]$$

$$\alpha = 0, 1, 2, \dots, p, \dots, d \quad \mu = 0, 1, 2, \dots, p, \dots, d \quad \mu_k = 0, 1, 2, \dots, p, \dots, d$$

In (16.3), the brane (world sheet there) has $p = 1$ spatial dimension, though the spacetime has d spatial dimensions. Here, in (16.62), we have p parameters σ_i and each index $\mu, \mu_1, \mu_2, \dots, \mu_p$ runs over $0, 1, 2, \dots, d$. There are p parameters and d spatial dimensions.

2.2.2 New Kinds of Charge

One can presume that (16.62) leads to new kinds of charges in branes of higher dimension than that of the string world sheet. Just as the K-R two index field led to a K-R (string) charge of one index, and the single index Maxwell field led to electric charge (a scalar) with no index, it should follow that a tensor field of order $p + 1$, such as that of (16.62), would have its own associated source charge of p indices. (Charge is a tensor of order p .)

Zwiebach does not discuss this, and we won't discuss it further here, but it can help to keep this in mind in order to distinguish these higher order charges from electric charge.

2.2.3 Open Strings

A Dp -brane traces out a $p + 1$ dimension world volume. In parallel with our analysis of (16.3) showing the endpoints of the open string (which trace out edges of the world sheet) are electrically charged, higher dimensional world volumes of p spatial dimensions will carry electric charge if coupled to a massless anti-symmetric tensor field with $p+1$ indices.

2.2.4 Closed Strings

Bosonic strings

It turns out that for the closed bosonic string, there are no other antisymmetric tensor fields (in any number of dimensions) besides the K-R field. Thus, the closed bosonic string can never carry electric charge.

It seems this conclusion arises from the fact that the bosonic string is a $D1$ -brane, i.e., $p = 1$, so in (16.62), we must have a field with two indices. For the K-R field, we take our generalized symbol $A_{\mu\mu_1}$ in (16.62) to the specific K-R field symbol $B_{\mu\nu}$ in (16.3). For a two-index field, there is only one formulation that is antisymmetric. This would be true no matter how many dimensions μ and ν run over. So, we conclude that only one field, the K-R field, can be antisymmetric in μ and ν . (Since the components of $B_{\mu\nu}$ can vary, in a given problem/situation, there can be different fields $B_{\mu\nu}$, but

they are just different K-R fields. This is analogous to the fact that the e/m field A_μ can take on different component values, but these are just different values for the Maxwell field A_μ .

Additionally, and not clear in Zwiebach, I believe for any closed brane of any d , the K-R field is the only one allowed.

Superstrings

It turns out (analysis is extensive and not shown here or in Zwiebach) that Type IIA and IIB superstrings permit additional antisymmetric tensor fields beyond the K-R field (called Ramond-Ramond fields). For further discussion, see pg. 371 in Zwiebach.

2.3 Electric Charge Arising from Compact Superstring Branes

As noted, a string is a $D1$ -brane (it is one dimensional in space and two dimensional in spacetime). A string is just a special case brane. There are closed strings, i.e., closed $D1$ -branes, and there are closed higher dimensional branes.

We can consider a Dp -brane that is compacted in m dimensions, where $m < p$. The compacting in each m dimension is considered to be circular and such that it yields an m dimensional torus. The surface of such a torus is m dimensional. A circle is a 1-torus. A doughnut is a 2-torus.

We further consider superstring, rather than purely bosonic string, theory, so additional antisymmetric fields other than the K-R field can exist for a given brane. Thus, we consider (16.62) for a Dp -brane. And, we analyze such a brane that is compact in all p dimensions. (A brane can't be compact in $p + 1$ dimensions, because it can't be compacted in time. If it were, part of the compactification would have to travel backwards in time.)

Coordinates of the brane, in each of its compactified dimensions, are thus

$$\begin{aligned} X^1 &= R^1 \sigma^1 & X^2 &= R^2 \sigma^2 & \dots & X^p &= R^p \sigma^p \quad (\text{no sum on } p) \\ 0 \leq \sigma^1 &< 2\pi & 0 \leq \sigma^2 &< 2\pi & & 0 \leq \sigma^p &< 2\pi. \end{aligned} \quad (5)$$

Using (5) in (16.62), we find

$$S_{I_p} = -\int A_{\mu_1 \mu_2 \dots \mu_p} (X^\alpha(\tau, \sigma^1, \sigma^2, \dots, \sigma^p)) \frac{dX^\mu}{d\tau} R^1 R^2 \dots R^p d\tau d\sigma^1 d\sigma^2 \dots d\sigma^p, \quad (6)$$

where integration over each σ^k ($k = 1, 2, \dots, p$) is from 0 to 2π . And, again, μ and α both = $0, 1, 2, \dots, p, \dots, d$.

The radii R^k are considered extremely small, so the antisymmetric field $A_{\mu_1 \mu_2 \dots \mu_p}$ will not vary much as one travels in the σ^k direction. Thus, we can assume an essentially constant value in that direction for the field, and thus,

$$A_{\mu_1 \mu_2 \dots \mu_p} (\text{varies over } \sigma^k) \rightarrow A_{\mu_1 2 \dots p} (\text{taken effectively constant over } \sigma^k). \quad (7)$$

Alternatively, we could consider $A_{\mu_1 2 \dots p}$ as representing the average value over each σ^k . In doing (7), $A_{\mu_1 2 \dots p}$ loses its dependence on the σ^k upon which X^α is dependent.

$$A_{\mu_1 \mu_2 \dots \mu_p} (X^\alpha(\tau, \sigma^1, \sigma^2, \dots, \sigma^p)) \rightarrow A_{\mu_1 2 \dots p} (X^\alpha(\tau)). \quad (8)$$

With (8), (6) becomes

$$S_{I_p} = -(2\pi R^1)(2\pi R^2) \dots (2\pi R^p) \int A_{\mu_1 2 \dots p} (X^\alpha(\tau)) \frac{dX^\mu}{d\tau} d\tau. \quad (9)$$

The X^α has some components on (tangent to) the compactified space of the brane and some components outside (normal to) that space. We break these up, notation-wise, as follows.

$$X^\alpha \text{ where } \alpha = k \text{ or } m \quad k = 1, 2, \dots, p \text{ (tangent to brane) and } m = 0, p+1, \dots, d \text{ (normal to brane)} \quad (10)$$

The components tangent to the brane are not of concern to an observer outside the brane, as the brane is in such a compactified space that it appears effectively as a point. We are deducing what the compactified brane, with its

antisymmetric field(s) looks like to that outside observer (in a lower dimension space than d). So, we will concern ourselves with the m components, as defined in (10), and express (9) for those. For superscript n ranging over the same space normal to the brane (just as m represents),

$$S_{I_p}^{normal} = -(2\pi R^1)(2\pi R^2) \dots (2\pi R^p) \int A_{m12\dots p}(X^n(\tau)) \frac{dX^m}{d\tau} d\tau. \quad (11)$$

Recall from (1) where $x^\mu(\tau)$ represents the coordinates of path of a particle in spacetime, here X^m (and X^n) represents the path of the brane (which looks like a point) through the spacetime external (normal) to the brane. So, we can take upper case X to lower case x in (11), but keep in mind that our integral is over the path of the point-like brane in the external spacetime.

Recall we use upper case X as the coordinate of the string/brane displacement (which oscillates in time and spacetime and is a dependent variable), but lower case x as the spacetime coordinate (which is fixed, does not oscillate, and is an independent variable). Even though x is independent, when we want an integral to be over the world line of a particle with coordinates x , we imply this by writing $x(\tau)$, to imply the integral is over that world line (which evolves with τ).

With this, and realizing that the factors in front of the integral sign in (11), for a p -torus, equal the volume of the p -torus, we have

$$S_{I_p}^{normal} = -V_p \int A_{m12\dots p}(x(\tau)) \frac{dx^m}{d\tau} d\tau = -V_p \int A_{m12\dots p}(x) dx^m, \quad (12)$$

which looks strikingly like (1), the action for a electrically charged point particle as the source of a Maxwell field. This leads one to suspect the compact brane acts in the non-compact space like an electrically charged particle, which is the source of a Maxwell field $A_{m12\dots p}(x)$. Indeed the field here has components in the space normal to the brane and is a vector (has a single index m , since $1, 2, \dots, p$ are fixed). This conjecture works, but by convention is altered a bit by defining a new field as follows.

$$A_{m12\dots p}(x(\tau)) = \frac{1}{(\alpha')^{p/2}} \tilde{A}_m(x(\tau)) \quad (16.70) [372]$$

With (16.70) in (12), we have

$$S_{I_p}^{normal} = -\frac{V_p}{(\alpha')^{p/2}} \int \tilde{A}_m(x(\tau)) \frac{dx^m}{d\tau} d\tau = -\frac{V_p}{(\alpha')^{p/2}} \int \tilde{A}_m(x(\tau)) dx^m = -\frac{V_p}{(l_s)^p} \int \tilde{A}_m(x(\tau)) dx^m. \quad (16.71) [372]$$

In analogy to (1), the compactified brane appears in the uncompactified space as a point particle with electric charge

$$Q = \frac{V_p}{(\alpha')^{p/2}} = \frac{V_p}{(l_s)^p}, \quad (16.72) [373]$$

interacting with a Maxwell field \tilde{A}_m . Note that in natural units, charge is unitless, and (16.72) is, indeed, unitless. In addition, for a dimensionless action (16.71), \tilde{A}_m has units of $1/L = M$, which is the same as that for an e/m field. Getting unit to work out is the main reason, the new field was defined in (16.70).

22. Moduli

Robert D. Klauber Feb 4, 2024 www.quantumfieldtheory.info

Equations numbered such as (21.68) refer to Zwiebach. The symbol [483] means page 483 in Zwiebach.

1 Oscillations of Compact Dimensions and Moduli

In string theory with compact spaces, such as a $6d$ torus, or a $6d$ Calabi-Yau space, oscillations arise in the fabric of space itself, as opposed to oscillations of a string, or even of a brane, located in that space. This may seem strange at first, but in general relativity we have gravitational waves, which are nothing but oscillations in the fabric of spacetime itself. This concept is extrapolated to the compact spaces of string theory.

In string theory before considering space oscillations, one took a classical oscillating string field and quantized it, to make it a field. Oscillations of classical string field displacement X^μ became quantum fields that created and destroyed string states, which one presumes are particle states. A similar thing happens with oscillations of compact dimensions. The dimension itself, such as the radius R in one dimension of a torus, becomes a field that creates and destroys states, which one presumes are particles. A vector displacement like X^μ becomes a quantum vector field; a scalar like a radius R becomes a quantum scalar field.

Consider a circular compact spatial dimension of radius R , for which the radius can oscillate in various modes.

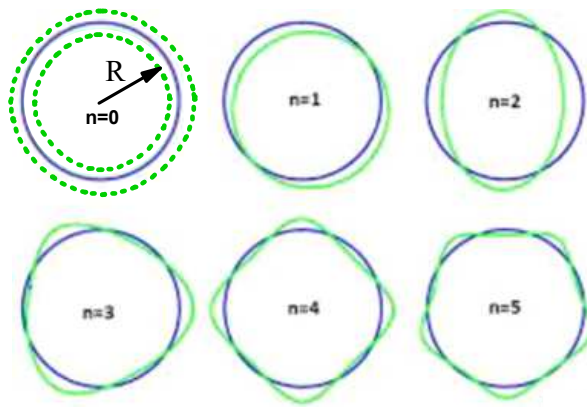


Figure 1. Various Modes of Oscillation of a Compact Circular Dimension

Violet circular line is static (non-oscillating) dimension. In the first figure, green dashed lines are max and min displacements. In other figures, the non-circular green line shows one extreme of the oscillation mode. Image credit: IET Research Hub - Wiley

For modes of $n > 0$, the oscillations can move circumferentially, i.e., they can comprise a wave moving in the circumferential direction with variations in R , i.e., variation transverse to the direction of travel.

For a 2-torus, we would have a second radius and a space with oscillations normal to each compact circumference. And so on, for higher dimensional compact spaces.

The radius R can vary with location in 4D (non-compact or reduced) spacetime. That is, variations in R can propagate as waves in 3D. Thus, where we denote 4D reduced spacetime position with shorthand notation x (which = x^μ),

$$\text{Displacement of compact radial dimension} = R(x) \quad \text{which varies with 4D location } x \quad (1)$$

Just as quantizing string displacements X^μ resulted in a quantum field, displacement $R(x)$ of a compact dimension results in a field, here a scalar field.

$$\text{Displacement of radial dimension } R(x) \quad \xrightarrow{\text{quantization}} \quad \text{scalar field } R(x) \quad (2)$$

Upon quantization, $R(x)$ comprises a scalar field with various wave modes. As a quantum field, it creates and destroys a scalar state, which, as with the rest of QFT, carries the properties (quantum numbers like 3-momentum, charge, etc.) of the associated field that creates it. In practice then, one can think of $R(x)$ as the state of the compact space, or alternatively as a classical field representing the radius of that space, with numerical radius $R(x)$ at 3D location x^i at time x^0 .

2 Compact Dimension Spaces: Their Potential and Stability

2.1 The Potential for a Dimension Space

We generally think of a potential as associated with some field, such as the electric field in electrostatics or the gravity field in gravitational theory. The motion of objects (waves and particles) is affected by the potential.

In general relativity, curved spacetime gives rise to what we otherwise think of as a potential. It affects the motion of objects and waves. More curvature means more effect. This curvature is represented in the metric $g_{\mu\nu}$, from which we can deduce the Riemann

curvature tensor. For circularly curved spaces with radii R_i in each curved dimension, the metric is a function of the R_i . And thus, the effective potential is a function of the R_i . And the motion of the dimensions themselves (think R_i) would depend on that potential (which is itself a function of the R_i).

Consider a compact space comprised of two dimensions, where $R_1 = R_2 = R$. With the theory of general relativity, we can deduce what the effective potential in that space would be. We don't do that here (and neither does Zwiebach, but see the Appendix herein for a bit more detail)), and instead just state the result, which is the (where we will explain the various symbols)

$$\text{effective potential for compact 2D circular space } V(R) = -a_g \frac{\chi}{R^4} \quad \chi = 2 - 2g. \quad (21.68) [483]$$

g is non-negative integer known as the genus and takes different values for differently curved spaces. χ is known as the Euler number. a_g is a positive constant.

2.2 Dimensional Stability and the Potential

For a two-sphere, net curvature is positive and we get $g = 0$, so the potential is negative. This leads to a force

$$F_R = -\frac{\partial V}{\partial R} = -\frac{\partial \left(-a_g \frac{2}{R^4} \right)}{\partial R} = -4a_g \frac{2}{R^3} < 0. \quad (3)$$

A force in the minus R direction will compress that circular dimension to zero. Any non-zero modulus for a two-sphere is unstable. Such a space cannot, therefore, exist. If it is not already of zero size ($R \neq 0$), it is unstable. One can get the same results by simply looking at the potential V (21.68) [483] and noting what R will make it a minimum. This is displayed for the 2-sphere visually in the left most diagram of Fig. 1.

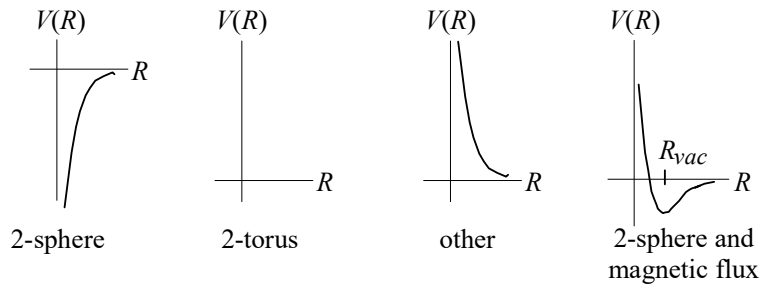


Figure 1. Potentials for Various Compact Spaces

For a two-torus (see Fig. 1), the net curvature is zero and we get $g = 1$, so the potential is zero. This means there is zero force in the R directions, and R is a modulus, which can take on any value, in principle.

For spaces with $g > 1$ (“other” in Fig. 1) the net curvature is negative, the potential is positive, and we get a force in the positive R direction that drives the space to infinite size in the R directions. (The potential seeks a minimum.) Such a space for finite R is unstable.

We discuss the right-most diagram in Fig. 1 later.

When the field $R(x)$ is not unstable, it is called a modulus, as for example, with the 2-torus. In principle, it can take on different values. The set of all possible $R(x)$ is called moduli space.

Wholeness Chart 1. Some Compact 2d Spaces and Stability

<u>Space</u>	<u>Net Curvature</u>	<u>Genus g</u>	χ	<u>Potential</u>	F_R	<u>Effect</u>	<u>Stability?</u>	<u>Is R a Modulus?</u>	<u>Massless?</u>
2-sphere	positive	0	2	$\sim -1/R^4$	negative	$R \rightarrow 0$	unstable	no	yes
2-torus, $R_1 = R_2$	zero	1	0	0	0	any R	neutral (constant R)	yes, but arbitrary	yes
Other	negative	> 1	< 0	$\sim +1/R^4$	positive	$R \rightarrow \infty$	unstable	no	yes

Regarding the massless column, recall from QFT that a massive scalar has a term in the Lagrangian of form

$$m^2\phi^\dagger\phi \text{ (complex) or } \frac{1}{2}m^2\phi\phi \text{ (real)}. \quad (4)$$

The Lagrangian for the compact space is comprised of a kinetic term (in derivatives of the field), the potential term (21.68) [483], and presumably interaction terms. None of these has the form of (4), i.e., there is no $\frac{1}{2}m^2RR$ term, so, we conclude the field is massless, for all three rows of the Wholeness Chart 1.

We can't have an unstable compact space, so at this point (ignoring the right most diagram of Fig. 1 for now) we are limited to a 2-torus with equal radii in both directions. And for that, with no other contributions to the Lagrangian, we can have any value for the continuous parameter R , the radius, which in this case is a modulus.

3 Stabilizing R for the 2-sphere and Making it a Modulus

(21.68) [483] represents the gravitational potential part of the Lagrangian for the compact space. It is possible that magnetic fields could exist, and if they are coupled to gravity, we would have additional terms in the gravitational Lagrangian carrying this coupling. Since, in our examples, the radius is the essential parameter in determining the spatial curvature, it must play the key role in gravitational effects. So, a magnetic field coupled to gravity would have to have the geometric properties of the space inherent to it. For the spaces we are considering, the key geometric property is the radius R .

As discussed in Zwiebach [483], with reference to his Chap. 19, magnetic flux is quantized on a compactified space comprising a torus. A similar effect occurs with a 2-sphere, with magnetic potential (as found by extensive analysis not shown here), where n is an integer and a_f is a positive constant, of

$$a_f \frac{n^2}{R^6}. \quad (5)$$

If we add that to (21.68) [483], we get a total gravitational plus magnetic potential of

$$V(R) = -a_g \frac{\chi}{R^4} + a_f \frac{n^2}{R^6} \quad \chi = 2 - 2g = 2 \quad n = \text{an integer}. \quad (21.69) [483]$$

This is displayed in the right-most diagram of Fig. 1. From (21.69), we can find a minimum via

$$\frac{\partial V}{\partial R} = 0, \quad (6)$$

so

$$\frac{\partial V}{\partial R} = a_g \frac{8}{R^5} - a_f \frac{6n^2}{R^7} = 0 \quad \rightarrow \quad R^{vac} = n \sqrt{\frac{4a_g}{3a_f}}. \quad (7)$$

So, R no longer goes to zero with time. It is stable. Therefore, under these conditions, R for a 2-sphere can be a modulus. And its value is fixed via (21.69) [483] and (7). This is an example of moduli stabilization via flux compactification.

This analysis and the right-most diagram of Fig. 1 should remind us of Higgs symmetry breaking, where the universe ended up in the true vacuum when the potential found its minimum. Compact spaces, along with the reduced 4D spacetime, with no strings comprise the vacuum. Just as the Higgs potential stabilized at its minimum, which is the vacuum, compact space potential stabilizes at its minimum, which is the string theory vacuum. R^{vac} is the compact space radius for that vacuum.

For different values of n , we get different moduli, and as long as n can take on any positive integer values, there are an infinite number of possible forms for the potential and thus an infinite number of possible moduli (possible R^{vac}).

4 Fixing the Modulus R for the 2-torus

For the 2-torus, R is not unstable. But it is not stable either. It can take on any value. There are ways, however, to fix the value of R , which can be advantageous in theory building.

To fix R in this case, we would need two additional terms in (21.68) [483], since the term therein is zero for the 2-torus with $\chi = 0$. We can get a positive term similar to the second term in (21.69) [483] for the 2-sphere, which here we label $f_+(n, R)$. But, we also need a negative potential like we have in (21.69).

Getting this entails the use of orientfolds, which we do not delve further into here. Simply understand that orientfolds introduce a negative term into the potential so we end up with a potential of form (where f_{orient} is a positive function related to an orientfold and depends on an integer m)

$$V(R) = f_+(n, R) - f_{orient}(m, R) \quad f_+ \text{ behaves similar to } a_f \frac{n^2}{R^6}; \quad f_{orient} > 0; \quad n, m = \text{integers}. \quad (8)$$

For given m and n , (8) has a minimum and via (6), one can find the value for R at which it occurs. Employing orbifolds (or possibly other means) leading to relations like (8) is known as moduli fixing. As different m and n lead to different minima, there are many different R_i^{vac} values, an infinite number in principle, if there are no upper limits on m and n .

5 Higher Dimensional Spaces

For compact spaces of higher dimensions than two, we have additional contributions via general relativity to the potential, can have additional fluxes quantized via other integers, such as m, k, l , etc. and have additional moduli. For each different set of integers (m, n, k, \dots) we would have a different set of moduli R_i^{vac} .

Such spaces are the background for string oscillations. They exist without strings (particles, as our theory presumes). So, such spaces comprise different possible vacua. There are many possible vacua in string theory.

6 Calabi-Yau Spaces

For $6d$ Calabi-Yau space, the potential can depend on hundreds of different integers, leading to many different possible moduli, where each such set of moduli comprises a different string vacuum.

7 The String Vacuum

The form of the compact $6d$ space varies in the five different string theories (e.g., 6-torus, Calabi-Yau space). The vacuum in each case would comprise curled up dimensions with moduli, such as R above for each circular dimension or other parameters (such as those associated with Calabi-Yau space), which can vary considerably in number, type, and magnitude. The potential for the space would then vary with all of these parameters and the integers by which they are quantized.

As noted above, a compact space without strings is a vacuum, but still has moduli. For different moduli, there are different vacua, known as the string vacua. There are an extraordinarily large number of possible ones, and the collection of all of these possibilities is called the string landscape. Each possible vacuum in the string landscape space comprises a local minimum of the potential in that space, and the moduli take on vacuum expectation values at that minimum.

Since compact spaces can have hundreds of different integers characterizing them, there are an enormous number of vacua in the landscape. Theorists have determined this number exceeds a mind-boggling 10^{500} . It is presumed that our universe (i.e., its vacuum) is one of these possibilities, but no one has yet found the particular one. On pages 484 to 490, Zwiebach discusses the likelihood that our universe is the landscape, given certain things we know now, such as the energy content of dark energy, and assuming such energy is vacuum energy.

8 Whence the Magnetic Field?

One could ask, as I did when first studying moduli, if we are analyzing the vacuum, then where do the magnetic fields come from? If there are magnetic fields, then the vacuum isn't really a vacuum.

The answer, I've surmised, is that, as discussed in Zwiebach Chap. 19, rotated D-branes manifest as magnetic fields. So, one can presume, the vacua associated with magnetic fields can instead be considered vacua with rotated D-brane coordinate spaces.

9 Massless vs Massive Moduli

Recall from Higgs symmetry breaking that the potential in terms of the high-energy Higgs field ϕ can be re-expressed in terms of a low-energy Higgs field σ , where v is the vacuum expectation value of ϕ . See Klauber, Vol. 2, Fig. 7-2 [209],(7-55) [222] and the Higgs terms in (7-58) [223]. In Fig. 2, we compare the true vacuum Higgs field σ to the vacuum modulus field for the 2-sphere with magnetic flux compact space. (Note we don't distinguish in this case, the true from false vacuum for the modulus, as there is no false vacuum for the 2-sphere.)

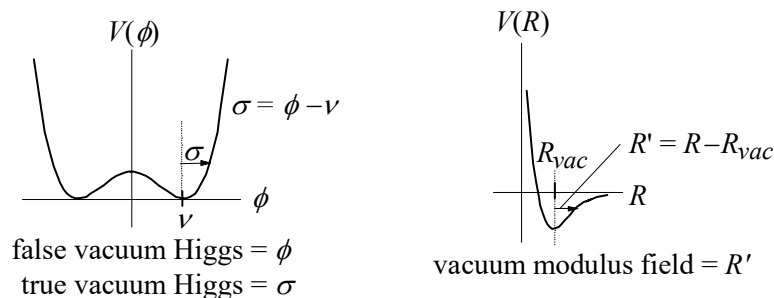


Figure 2. The Higgs Field Vacuum vs a Modulus Field Vacuum
(for a 2-Sphere Compact Space with Magnetic Flux)

The true vacuum Higgs σ can oscillate around the false vacuum Higgs VEV ν as ϕ varies. The shape of the potential curve near ν has mathematical form, in the potential (and thus, in the Lagrangian), of

$$\frac{1}{2} m_H^2 \sigma^2. \tag{9}$$

The form (9) is the hallmark of a massive scalar, where m_H is the mass. The vacuum modulus field R' has similar graphical shape about the R field VEV of R_{vac} as σ does about ν , and R' can be visualized as oscillating about that VEV. An increase in R' results in a restoring force leftward pushing R' back toward zero. Similarly, a decrease in R' results in a restoring force rightward, pushing R' back toward zero.

Mathematically, one can find a term in the potential, and thus, the Lagrangian, similar to (9), where the coefficient of an effective $\frac{1}{2}R'^2$ is the mass squared of the vacuum modulus R' . Then, the field R' is a massive field, whereas R without a magnetic field (for the 2-sphere) was massless (and unstable).

10 Summary of Spaces with and without Magnetic Flux

Wholeness Chart 2 summarizes this article.

Wholeness Chart 2. Some Compact 2d Spaces with and without Magnetic Flux

<u>Space</u>	<u>Curvature</u>	<u>g</u>	<u>χ</u>	<u>Potential V</u>	<u>V_{min}?</u>	<u>F_R</u>	<u>Effect</u>	<u>Stable?</u>	<u>Is R a Modulus?</u>	<u>Multiple R^{vac}?</u>
2-sphere	positive	0	2	$-a_g 2/R^4$	no	negative	$R \rightarrow 0$	unstable	no	none
2-sphere with flux	same \uparrow	same \uparrow	same \uparrow	$-a_g 2/R^4 + a_f n^2/R^6$	yes	forces R to V_{min}	$R \rightarrow R^{vac}$	stable	yes	one for each n
2-torus, $R_1 = R_2$	zero	1	0	0	no	0	any R	neutral	yes, but arbitrary	none (or maybe all R)
2-torus with flux	same \uparrow	same \uparrow	same \uparrow	$f_+(n,R)$ (similar to $a_f n^2/R^6$)	no	positive	$R \rightarrow \infty$	unstable	no	none
Tori or C-Y spaces with flux & orientfold(s)	same \uparrow	same \uparrow	same \uparrow	$f_+(n,R) - f_{orient}(m,R)$	yes	forces R to V_{min}	$R \rightarrow R^{vac}$	stable	yes	one for each set of n,m

11 Appendix: Oscillating Moduli and the Gravitational Constant

Recall the relation between the gravitational constant G ($= G^{(4)}$ below) we are familiar with in the reduced 4D spacetime we are familiar with and the higher dimensional gravitational constant (where V_C below is the volume of the compact space).

$$\frac{G^{(D)}}{G^{(4)}} = \frac{G^{(D)}}{G} = V_C \quad \text{or} \quad \frac{G^{(D)}}{V_C} = G. \tag{3.117} [67]$$

In our 4D macro universe, G is constant. But if the compact space has a viable modulus R , that modulus oscillates, so the volume of the compact space varies. This fact is used by theorists as part of the general relativity calculation that gives us (21.68) [483].

23. Polyakov String Action

Robert D. Klauber www.quantumfieldtheory.info March 3, 2024

Everything herein is in the covariant (not light-cone) formulation.

<u>Context</u>	<u>String Action</u>	<u>Zwiebach</u>	<u>Comment</u>
Nambu-Goto Action	$S = -T_0 \int \sqrt{-\gamma} d\tau d\sigma = -\frac{1}{2\pi\alpha'} \int \sqrt{-\gamma} d\tau d\sigma$ $= -\frac{1}{2\pi\alpha'} \int \sqrt{(\partial_\tau X^\mu \partial_\sigma X_\mu)^2 - (\partial_\tau X^\mu)^2 (\partial_\sigma X_\mu)^2} d\tau d\sigma$	(6.44) [112] & (6.39) [111]	Natural units
Action equivalent to Nambu-Goto, easier to work with	$S = \frac{1}{4\pi\alpha'} \int (\partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu) d\tau d\sigma$	(24.4) [569]	
Re-expressing above	<p>For $\hat{\eta}^{\alpha\beta} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ $\alpha, \beta = 0(\text{for } \tau), 1(\text{for } \sigma)$</p> $S = -\frac{1}{4\pi\alpha'} \int \hat{\eta}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu d\tau d\sigma$	(24.65) [583]	Zwiebach uses η for our $\hat{\eta}$, but that can be confused with 4D $\eta_{\mu\nu}$
Re-expressed again	<p>$\tau = \xi^0$ $\sigma = \xi^1$ α, β taking values ξ^0 and ξ^1</p> $S = -\frac{1}{4\pi\alpha'} \int \hat{\eta}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu d\xi^0 d\xi^1 \quad (\text{A})$	Not used in Zwiebach	
Try another action with unknown $h^{\alpha\beta}$	$S = -\frac{1}{4\pi\alpha'} \int \sqrt{-h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu d\xi^0 d\xi^1 \quad (\text{B})$ <p>$h = \text{Det}(h_{\alpha\beta})$ $h_{\alpha\beta}$ is inverse of $h^{\alpha\beta}$; $h^{\alpha\delta} h_{\delta\beta} = \delta^\alpha_\beta$.</p>	(24.70) [583]	This is <u>Polyakov action</u> .
Try a particular form for $h^{\alpha\beta}$	$h^{\alpha\beta} = \rho^2 (\xi^0, \xi^1) \hat{\eta}^{\alpha\beta} \quad h_{\alpha\beta} = \frac{1}{\rho^2} \hat{\eta}_{\alpha\beta} \quad (\text{C})$ $h = \text{Det}(h_{\alpha\beta}) = \frac{1}{\rho^4} \text{Det}(\hat{\eta}_{\alpha\beta}) = -\frac{1}{\rho^4}$	(24.92) [586]	ρ is real; ρ^2 is positive
Use (C) in (B)	$S = -\frac{1}{4\pi\alpha'} \int \frac{1}{\rho^2} \rho^2 \hat{\eta}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu d\xi^0 d\xi^1$ $= -\frac{1}{4\pi\alpha'} \int \hat{\eta}^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu d\tau d\sigma \quad (\text{D})$		(D) = (A), so with form (C) chosen for $h^{\alpha\beta}$, (B) = (A)
1st Conclusion	For $h^{\alpha\beta} = \rho^2 (\xi^0, \xi^1) \hat{\eta}^{\alpha\beta}$, S of (B) is identical to (A), so we can use (C) in (B) if it is advantageous.		(C) is conformal gauge.
Try another form for $h^{\alpha\beta}$	$h^{\alpha\beta} = \frac{1}{f^2(\xi)} \gamma^{\alpha\beta} \quad h^{\alpha\delta} h_{\delta\beta} = \gamma^{\alpha\delta} \gamma_{\delta\beta} = \delta^\alpha_\beta \rightarrow h_{\alpha\beta} = f^2 \gamma_{\alpha\beta} \quad (\text{E})$ $\gamma = \text{Det}(\gamma_{\alpha\beta}) \quad h = \text{Det}(h_{\alpha\beta}) = \frac{1}{f^4} \gamma$	(24.86) & (24.87) [585]	
Use (E) in (B)	$S = -\frac{1}{4\pi\alpha'} \int \sqrt{-h} h^{\alpha\beta} \underbrace{\partial_\alpha X^\mu \partial_\beta X_\mu}_{\gamma_{\alpha\beta}} d\xi^0 d\xi^1 \quad (\text{F})$ $= -\frac{1}{4\pi\alpha'} \int \sqrt{-\frac{1}{f^4} \gamma} f^2 \underbrace{\gamma^{\alpha\beta} \gamma_{\alpha\beta}}_2 d\tau d\sigma = -\frac{1}{2\pi\alpha'} \int \sqrt{-\gamma} d\tau d\sigma$	(24.88) to (24.90) [586]	Same as first row at top, original form of Nambu-Goto action.
2nd Conclusion	For $h^{\alpha\beta} = \frac{1}{f^2(\xi)} \gamma^{\alpha\beta}$, S of (B) is equivalent to Nambu-Goto action, which is equivalent to (A).		
Final Conclusion	Can use $h^{\alpha\beta}$ of (C) or (E), as convenient \rightarrow get same theory		

This page intentionally left blank.

24. Interacting Strings: An Intro

Robert D. Klauber www.quantumfieldtheory.info March 10, 2024

Equation and page references are to Zwiebach, *A First Course in String Theory*

1 Preliminary Concepts

1.1 Degenerate Polygons

Degenerate polygons play a key role in interacting string theory, but they are neither polygons nor degenerate, in the usual physics sense of that word. In QM, for example, “degenerate” implies more than one eigenvector (wavefunction) has the same eigenvalue (such as energy). The meaning in the term “degenerate polygon” is quite different.

Fig. 1 may help in visualizing what is meant by a degenerate polygon. To turn a true polygon into a degenerate one, we move the point C to the line connecting points A and B.

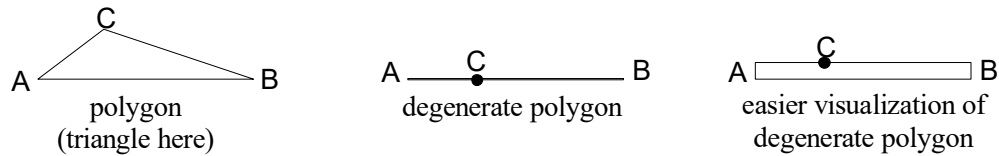


Figure 1. Typical Polygon vs Degenerate Polygon

Note that the degenerate polygon in the middle of Fig. 1 is not what we would consider a polygon, but a straight line (overlapping lines here, actually). The drawing on the far right is a common way of representing such a degenerate polygon, as it shows the overlapping lines of the polygon clearly, though in reality that drawing is a true polygon (rectangle). In that righthand drawing A and B appear as lines, but each line is merely a symbol for a point in the actual degenerate polygon.

Fig. 2 shows another, more general type, polygon and one way it can be converted (by moving points) to a degenerate polygon. Point B in the far right drawing is actually a single point represented as a short line to aid in visualization.

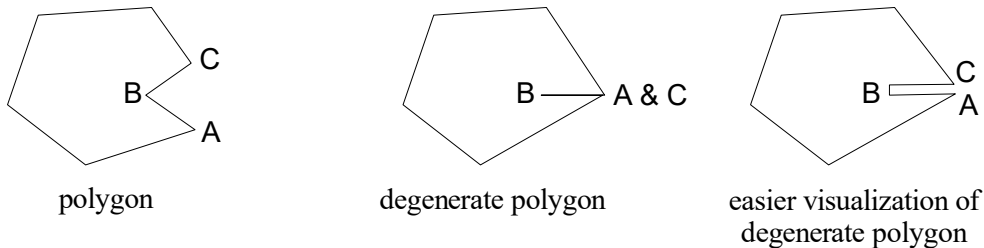


Figure 2. Another Example of a True Polygon vs a Degenerate Polygon

We can consider the straight-line segments of a polygon as directed. That is, we can think of moving along successive lines from one end to the other, in one direction or the other around the edge. See arrows in Fig. 3 indicating orientation in one direction. We can then define a turning angle α_i as the angle made at a point i on the polygon to the subsequent line in the polygon, as shown in Fig. 3. Positive turning angle is ccw. In the figure, α_A is positive; α_B is negative.

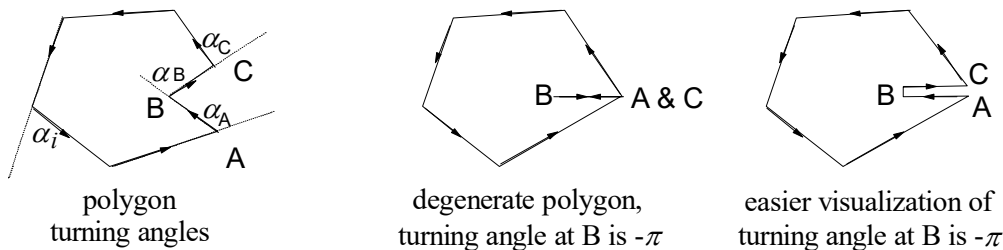


Figure 3. Polygons, Degenerate Polygons, and Turning Angles

Note that for the degenerate polygon, we have a point (B in Fig. 3), where the turning angle is $-\pi$. In general, for degenerate polygons, we will have at least one turning angle of either π or $-\pi$. In fact, that can be used a definition for degenerate polygon.

1.2 String Interactions on the Worldsheet (τ - σ Plane, not X^μ Plane)

In QFT, a single particle splitting into two particles is represented by a Feynman diagram like Fig. 4.



Figure 4. QFT Feynman Diagram for a Photon Splitting into a Charged Fermion and Anti-fermion

The comparable string interaction (where our theory presumes strings are standard model particles) can be shown in the τ - σ worldsheet (of parameter space, rather than the X^μ worldsheet of spacetime) as in Fig. 25.6 [605] of Zwiebach, a simulation of which is Fig. 5 herein below. In the left diagram of that figure, a single string, represented by the vertical line at P_1 splits at point Q into two strings, represented by the vertical lines P_2 and P_3 . The worldsheet diagram can be interpreted as a degenerate polygon, as shown on the right side of Fig. 5.

Note the first string (on the left end) extends from the τ axis up to a particular value of σ . That particular value, at the end of the string, has heretofore (in free string theory) been taken to be π (for an open string). However, in interaction theory, it turns out to be more advantageous to take that value of σ as follows, where the beginning of the string is at $\sigma = 0$, and momentum in the $+$ direction of light-cone coordinates is p^+ .

$$\text{Length of incoming string} = \sigma_{1\text{end}} = 2\pi\alpha' p_1^+, \text{ and} \tag{1}$$

$$\text{length of 2nd string} = 2\pi\alpha' p_2^+ \qquad \text{length of 3rd string} = 2\pi\alpha' p_3^+. \tag{2}$$

Since momentum is conserved, i.e.,

$$p_1^+ = p_2^+ + p_3^+, \tag{3}$$

the total length of the final two strings together equals the length of the initial string. Note the “length” here is not the actual physical length of the string. It is only the value of the parameter σ that we choose to make analysis easier.

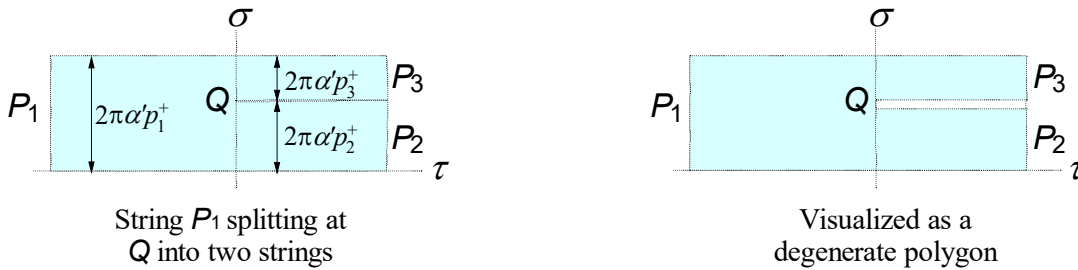


Figure 5. One String Splitting into Two, Construed as Degenerate Polygon

1.3 Degenerate Polygons on the Worldsheet

The string P_1 is considered to be incoming from $\tau = -\infty$, and the two outgoing strings going to $\tau = +\infty$. Thus, the strings at those times (τ values, really) are an infinite number of times farther away horizontally than vertically. This is illustrated in Fig. 6, where we can think of the left and right ends extending out to infinity.

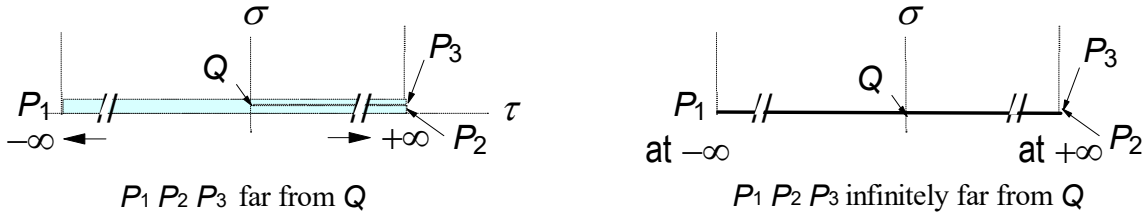


Figure 6. The String Interaction Degenerate Polygon Shape for Incoming and Outgoing Strings at Infinity

In such case, the turning angle at P_1 is π , at P_2 , π , at Q , $-\pi$, and at P_3 , π . In essence, with such turning angles, the degenerate polygon, for string P_1 at minus infinity, and P_2 and P_3 at plus infinity, looks like a straight line on the real axis of the τ - σ space. P_1 , P_2 , and P_3 have effectively become vertex points (rather than lines) on a degenerate polygon in the τ - σ space.

2 The Worldsheet and Riemann Surfaces

2.1 What is a Riemann Surface?

A Riemann surface is a two-dimensional surface which, roughly speaking, has a two-parameter grid on it wherein one parameter is real and one is imaginary. The complex plane where $z = x + iy$ is an example of a Riemann surface. We will soon see other examples.

A more precise definition involves complex variable theory and can be found in Zwiebach on page 599.

2.2 The Worldsheet as a Riemann Surface

It turns out that a consistent string theory can be structured by considering worldsheets like that of Figs. 5 and 6 as Riemann surfaces. In doing so, σ is conventionally taken as the imaginary parameter and τ as the real one. We represent any point on the (Riemann surface) worldsheet by w .

$$w = \tau + i\sigma = r_w e^{i\phi_w}. \quad (4)$$

2.3 Mapping to a Different Representation of the Riemann Surface

A Riemann surface can be mapped one-to-one onto another surface. If such mapping is constrained to be done in a certain way (read “analytic”, for readers who know what that means¹) the original surface and the surface it is mapped to are considered equivalent Riemann surfaces. Each is simply a different representation of the same entity.

For example, a complex space (Riemann surface) with points represented by z ,

$$z = x + iy = r_z e^{i\phi_z}, \quad (5)$$

can be mapped from w space (the Riemann surface of Figs. 5 and 6) to z space,

$$z = z(w) \quad \text{with inverse mapping} \quad w = w(z). \quad (6)$$

2.3.1 One Particular Mapping

For example, consider the particular mapping from w space (the Riemann surface of Figs. 5 and 6) to z -space, where we have a single string with momentum in the $+$ direction (along the light cone edge) of p^+ ,

$$z = e^{\frac{w}{2\alpha'p^+}} = e^{\frac{\tau}{2\alpha'p^+}} e^{i\frac{\sigma}{2\alpha'p^+}} = r_z e^{i\phi_z} \quad r_z = e^{\frac{\tau}{2\alpha'p^+}} \quad \phi_z = \frac{\sigma}{2\alpha'p^+}. \quad (7)$$

Note we can find the inverse of (7) by taking its natural logarithm.

$$\ln z = \frac{w}{2\alpha'p^+} \quad \rightarrow \quad w(z) = 2\alpha'p^+ \ln z. \quad (8)$$

¹ Zwiebach notes he will often use the word “conformal” for “analytic”, and he is doubtless not alone in this practice. Analytic maps are conformal, but conformal maps are not necessarily analytic.

2.3.2 Another Particular Mapping

Zwiebach [601-602] also shows another possible mapping to a surface with points η ,

$$\eta(w) = \frac{1+iw}{1-iw} \quad \text{or from } z\text{-space} \rightarrow \quad \eta(z) = \frac{1+iz}{1-iz}, \quad (9)$$

but we need not put any more attention now on this particular mapping. At this point, it is just an example.

2.3.3 Why Do We Need to Map?

It turns out in string theory that different interactions can be mapped from w -space to z -space and compared most efficiently in z -space to one another, as one sees after working with the theory for some time.

2.3.4 Visualizing the First of These Mappings

Zwiebach [601-602] explores the first mapping above of Sect. 2.3.1 and displays the results in Figs. 25.6 [600] and the left side of Fig. 25.7 [601], which we mimic in Fig. 7 herein below.

To get a feeling for the z -space representation of the string in the righthand diagram of Fig. 7, in Table 1, we calculate a few points in that space using the right-most part of (5).

Table 1. Finding Points in z -space from Points in w -space

<u>Point in w-space</u>	<u>$z = r_z e^{i\phi_z} = e^{\frac{\tau}{2\alpha'p^+}} e^{i\frac{\sigma}{2\alpha'p^+}}$</u>	<u>x and y</u>
$\tau = -\infty \quad \sigma = 0$	$r_z = 0 \quad \phi_z = 0$	$x = y = 0$
$\tau = -\infty \quad \sigma = 2\pi\alpha'p^+$	$r_z = 0 \quad \phi_z = \pi$	$x = y = 0$
$\tau = -\infty \quad \sigma = \pi\alpha'p^+$	$r_z = 0 \quad \phi_z = \pi/2$	$x = y = 0$
F τ negative $\sigma = 0$	between zero and A on real axis	$0 < x < 1, y = 0$
E τ negative $\sigma = 2\pi\alpha'p^+$	between zero and D on real axis	$-1 < x < 0, y = 0$
A $\tau = 0 \quad \sigma = 0$	$r_z = 1 \quad \phi_z = 0$	$x = 1, y = 0$
D $\tau = 0 \quad \sigma = 2\pi\alpha'p^+$	$r_z = 1 \quad \phi_z = \pi$	$x = -1, y = 0$
$\tau = 0 \quad \sigma = \pi\alpha'p^+$	$r_z = 1 \quad \phi_z = \pi/2$	$x = 0, y = 1$
B τ positive $\sigma = 0$	between A and $+\infty$ on real axis	$1 < x < +\infty, y = 0$
C τ positive $\sigma = 2\pi\alpha'p^+$	between D and $-\infty$ on real axis	$-\infty < x < -1, y = 0$
$\tau = +\infty \quad \sigma = 0$	$r_z = \infty \quad \phi_z = 0$	$x = +\infty, y = 0$
$\tau = +\infty \quad \sigma = 2\pi\alpha'p^+$	$r_z = \infty \quad \phi_z = \pi$	$x = -\infty, y = 0$
$\tau = +\infty \quad \sigma = \pi\alpha'p^+$	$r_z = \infty \quad \phi_z = \pi/2$	$x = 0, y = \infty$

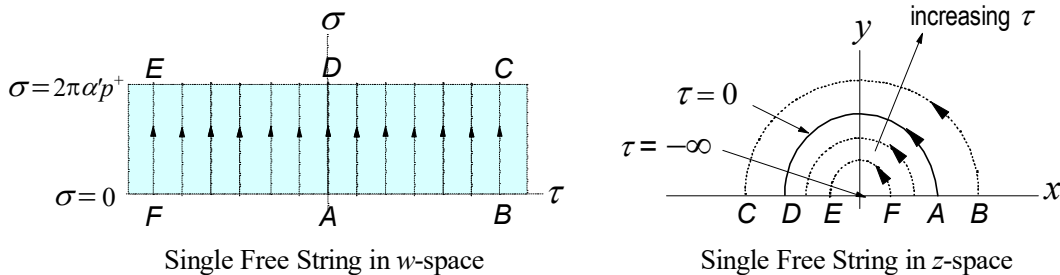


Figure 7. A Single Free String Viewed on the τ - σ Worldsheet (w -space) and in z -space
 (The solid line in the right diagram is the unit circle, so A is at $x = +1$ and D is at $x = -1$.)

Note that in z -space, the ends of the string in the infinite past are on the real line at $x = 0$, i.e., at $z = 0$. From (8) we can see that the log of zero is $-\infty$, so for $z = x = 0$, where $w = \tau = -\infty$. Had we taken $\ln(z - 1)$, we would find that for $z = x = 1$, we would have $w = \tau = -\infty$, i.e., the ends of the string at $\tau = -\infty$ on the worldsheet, would be at $x = 1$ on the real axis in z -space. This understanding will help us in what is to come.

2.3.5 Definitions

What we have been calling z -space is typically labeled $\bar{\mathbb{H}}$, and we will refer to it by that label henceforth.

If we are dealing with polygons in w -space (the worldsheet), then the transformation from $\bar{\mathbb{H}}$ to w -space is known as the Schwartz-Christoffel map $w(z)$. In particular, it takes points z_i in $\bar{\mathbb{H}}$ over to vertices P_i in polygons in w . Since we deal with degenerate polygons representing string interactions on the worldsheet, this is a map we will need to be using.

Note that the inverse of $w(z)$, i.e., $z(w)$ is also called a Schwartz-Christoffel map.

In Fig. 7 on the righthand side, we see that the origin, while a point in $\bar{\mathbb{H}}$, is at infinity in w -space. Such points, which represent infinity on the worldsheet, are called punctures. We actually have one puncture in $\bar{\mathbb{H}}$ in Fig. 7 at the origin and one at infinity (which is not actually a point). Punctures are on the boundary of a given space and not actually included as part of that space (since they are infinite). They “puncture” the space, in a sense.

3 The Three String Interaction and the Schwartz-Christoffel Map

In general, finding an equation $w(z)$ for the Schwartz-Christoffel map is not an easy task. We merely state here a result (10) derived by Zwiebach [606-607] for the case of the three-string interaction of Figs. 5 and 6.

3.1 A Particular Mapping Equation That Works

With a view to (7), (8), and the point made just after Fig. 7, one can glean a bit of the logic for the commonly employed Schwartz-Christoffel map for the three-string interaction,

$$w = -2\alpha' p_2^+ \ln(z + 1) - 2\alpha' p_3^+ \ln(z - 1) \quad \text{Zwiebach (25.46) [607].} \quad (10)$$

Note what we get if we take the real number point $z = -1$. Then,

$$w = -2\alpha' p_2^+ \ln(0) - 2\alpha' p_3^+ \ln(-2) = -2\alpha' p_2^+ (-\infty) - 2\alpha' p_3^+ \ln(-2) = +\infty \quad (11)$$

On the worldsheet this is a point on the real axis at infinity, where $w = \tau = +\infty$, so $z = -1$ is a puncture in $\bar{\mathbb{H}}$.

For $z = 1$,

$$w = -2\alpha' p_2^+ \ln(2) - 2\alpha' p_3^+ \ln(0) = -2\alpha' p_2^+ \ln(2) - 2\alpha' p_3^+ (-\infty) = +\infty. \quad (12)$$

This point is also on the worldsheet real axis at infinity, where $w = \tau = +\infty$, so $z = +1$ is also a puncture in $\bar{\mathbb{H}}$.

From the right side of Fig. 6, we might want to consider these two points in $\bar{\mathbb{H}}$ ($z = -1$ and $z = 1$ as P_2 and P_3 ($w = +\infty$ for both) in the worldsheet. And we will.

Note that the constant “1” used in two places in (10) is taken as such by convention. It places two punctures in $\bar{\mathbb{H}}$ on the real axis at -1 and $+1$. Due to the minus sign in (10), these punctures represent the infinite future for the two strings. Compare with (8), with the opposite sign, that led to a puncture representing the infinite past.

Consider the point in $\bar{\mathbb{H}}$ $z = +\infty$. Then (10) maps that to

$$w = -2\alpha' p_2^+ \ln(\infty) - 2\alpha' p_3^+ \ln(\infty) = -\infty. \quad (13)$$

We could consider this point in $\bar{\mathbb{H}}$ ($z = +\infty$) as P_1 ($w = -\infty$) on the worldsheet. And we do.

It looks like (10) makes sense as a good map to use for the three-string interaction, but before being sure, we need to investigate what it does with the interaction point Q . Before doing that, we need to state, without proof, an important rule for Schwartz-Christoffel mapping.

Before doing that, we note that (10) maps points on the real line in $\bar{\mathbb{H}}$ (z -space) to points on the real line of the worldsheet Riemann surface (w -space). This is important, as the degenerate polygon in Fig. 6 has the incoming string, the interaction point Q , and the two outgoing strings all on the real axis (τ axis).

3.2 Ordering Rule for Schwartz-Christoffel Maps

It is well known in mathematical circles that the order of vertex points under a Schwartz-Christoffel map is not altered. That is, if we have points ordered from left to right in $\overline{\mathbb{H}}$, say $P_1P_2P_3$, then those points, when mapped to the (degenerate) polygon vertices on the worldsheet must have order along the boundary of the polygon of $P_1P_2P_3$ or $P_2P_3P_1$ or $P_3P_1P_2$, but not, for example, $P_2P_1P_3$. And vice versa. Vertex points along the polygon boundary must, under mapping to $\overline{\mathbb{H}}$, be ordered along the real axis in the same cyclical sequence as their corresponding points along the polygon boundary.

Note that we have the proper order for the three P_i points in our Schwartz-Christoffel map (10). That is, along the polygon on the worldsheet, we have order $P_1P_2P_3$, and along the real line in $\overline{\mathbb{H}}$, we have $P_2P_3P_1$. P_2 at $z = -1$; P_3 at $z = 1$; P_1 at $z = \infty$.

3.3 The Interaction Point Q

Since Q (interaction point) in Fig. 6 is between P_2 and P_3 on the worldsheet polygon, it must be between P_2 and P_3 on the real axis in $\overline{\mathbb{H}}$. This is shown pictorially in Figure 25.9 [605] of Zwiebach, which we mimic below as our Fig. 8.

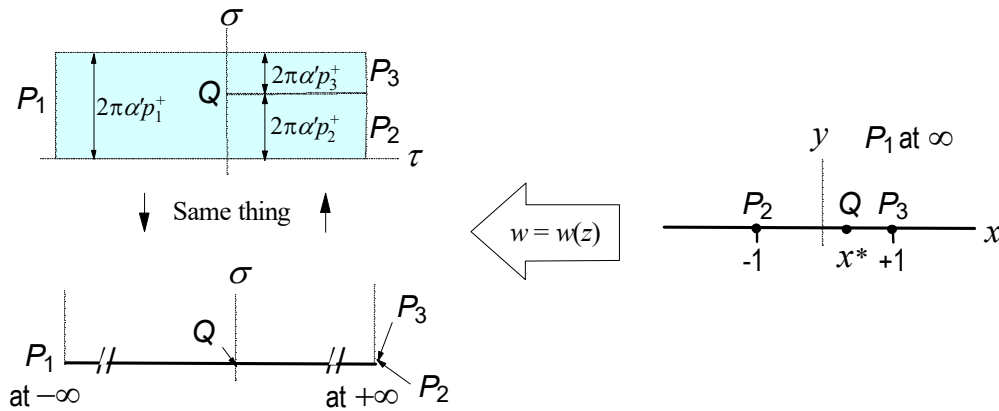


Figure 8. The Schwartz-Christoffel Map from z -space (RHS) to w -space (LHS)

3.4 Points Other than Endpoints on Strings

The points P_2 and P_3 shown on the right side of Fig. 8 really represent just the string endpoints in $\overline{\mathbb{H}}$ (z -space) in the infinite future. At those times the end points of each string collapse to the same point in $\overline{\mathbb{H}}$. However, at other times, the strings, represented in z -space are not points. Figure 25.11 in Zwiebach [608] shows this, and we mimic that figure below in our Fig. 9. (See the appendix for details on P_1 .)

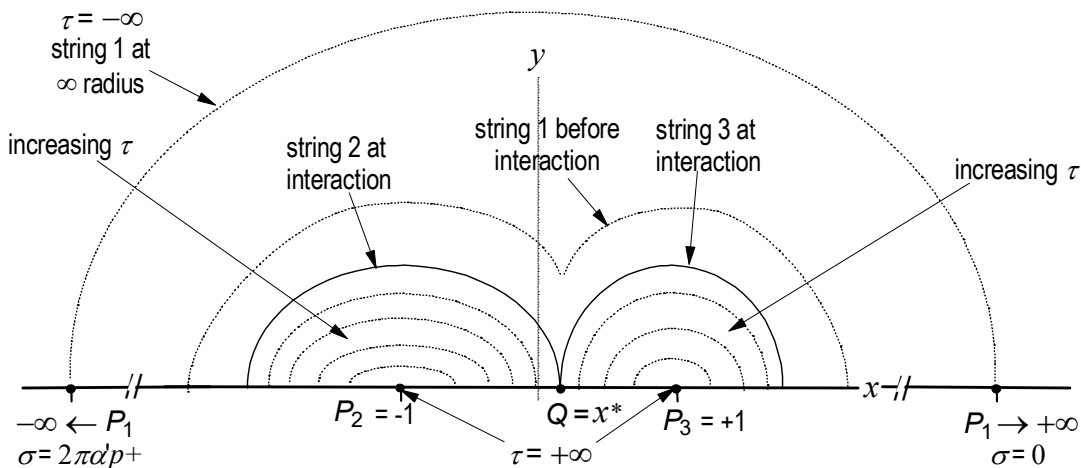


Figure 9. $\overline{\mathbb{H}}$ Riemann Surface for Three String Interaction Evolution with τ

3.5 Finding x^* for Point Q

We can find x^* , the location on the real axis in $\overline{\mathbb{H}}$ for point Q by using the derivative of (10) with respect to z .

$$\frac{dw}{dz} = -\frac{2\alpha' p_2^+}{z+1} - \frac{2\alpha' p_3^+}{z-1} \quad (14)$$

Locally, at the point Q , z is real ($=x$), and at its mapped location in w -space, σ remains unchanged as one progresses along the degenerate polygon, i.e., w changes equal τ changes. Thus,

$$\frac{dw}{dz} = \frac{d\tau}{dx} = -\frac{2\alpha' p_2^+}{x+1} - \frac{2\alpha' p_3^+}{x-1} \quad \text{near } Q. \quad (15)$$

In proceeding from P_2 to P_3 , we can see from Fig. 8 that τ is at a local minimum at point Q . Thus, we can set (15) equal to zero when $x = x^*$.

$$\left. \frac{d\tau}{dx} \right|_Q = -\frac{2\alpha' p_2^+}{x^*+1} - \frac{2\alpha' p_3^+}{x^*-1} = 0. \quad (16)$$

So,

$$\frac{p_2^+}{p_3^+} = -\frac{x^*+1}{x^*-1} = \frac{1+x^*}{1-x^*} \quad \rightarrow \quad x^* = \frac{p_2^+ - p_3^+}{p_2^+ + p_3^+}. \quad (17)$$

4 General Rules for Any Interaction

From the three interacting strings example and (10), we can glean general rules for relating w to z for any interaction of any number of strings, as follows.

To find w from z , use the general relation

$$w = \pm 2\alpha' p_1^+ \ln(z - z_1) \pm 2\alpha' p_2^+ \ln(z - z_2) \pm \dots \pm 2\alpha' p_n^+ \ln(z - z_n) \quad n+1 \text{ punctures,} \quad (18)$$

with these rules.

1. Add a logarithm, one for each string i except the one with a turning point mapped to infinity in the upper half plane of $\overline{\mathbb{H}}$.
2. The argument of each logarithm is $z - z_i$, where z_i is the puncture point in $\overline{\mathbb{H}}$ (z -space) corresponding to a string at $\tau = +\infty$ or $-\infty$ in w -space.
3. The pre-factor of each logarithm is $2\alpha' p_i^+$.
4. The sign on each logarithm term is negative when, in w -space, σ increases when crossing the turning point in the degenerate polygon. The sign is positive when σ decreases when crossing the turning point.

5 The Open Four-String Interaction

5.1 The Riemann Sphere

5.1.1 Mapping to the Riemann Sphere

To help in analyzing the 4-string interaction, we need to first consider another type of Riemann space, the Riemann sphere. The term Riemann surface pertains specifically to a plane, whereas the Riemann sphere, as the name implies is a Riemann space comprising the surface of a sphere. The surface of a complex plane carries the mathematical label \mathbb{C} , whereas the surface of a sphere having complex coordinates is labeled $\hat{\mathbb{C}}$.

We need a mapping to convert from the plane to the sphere, and vice versa. That is obtained via a projection as shown in Figure 10.

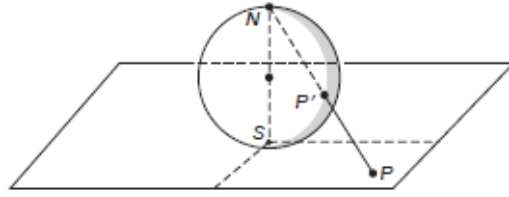


Figure 10. Mapping from the Riemann Plane to the Riemann Sphere

Consider a straight line from any point P on the plane to the north pole of the sphere. That point is considered mapped to the point P' on the sphere surface, where the line intersects that surface. We can consider the points on the sphere to carry the same labels as the points on the plane, i.e., $z = x + iy$, and talk about the points on the sphere, yet since it is easier, display those points on a plane (with orthogonal grid lines x and y). Note that the Riemann sphere is sometimes called the extended Riemann surface.

We don't need to know the mathematical relation for this mapping (it is complicated). We just need to know that we can carry it out, and that it is one-to-one. Note, all of infinity in z maps to the north pole, as single point, on the sphere.

5.1.2 Mapping from One Riemann Sphere to Another

Now, the mathematicians tell us, and we will simply accept it, that mapping from one Riemann sphere to another can be carried out with the following mapping.

$$w = \frac{az + b}{cz + d} \quad a, b, c, d \text{ real, } ad - bc > 0. \quad (19)$$

By convention, in string theory, we take

$$ad - bc = 1. \quad (20)$$

5.1.3 Proving Something We'll Need Using Riemann Spheres

We need (19) to prove one thing. That if a series of points on the real line of z -space for a Riemann surface (a plane) has a certain order along that line, the mapped points in the Riemann surface w -space (a plane) have the same order. We do this by using (19) to draw a conclusion about Riemann sphere to Riemann sphere mappings, and then from the projection mapping of Fig. 10, presume the same conclusion applies to Riemann surface to Riemann surface mappings. We do this because it is easier to prove using the mapping of (19) than the mapping of (18).

We start by taking the derivative of (19), where we assume z is real, i.e., $z = x$. Note the last step below results from the last part of (19).

$$\frac{dw}{dz} = \frac{a}{cz + d} - \frac{(az + b)c}{(cz + d)^2} = \frac{a(cz + d) - (az + b)c}{(cz + d)^2} = \frac{acz + ad - azc - bc}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2} > 0. \quad (21)$$

If we are moving in an increasing z direction along the real axis in z -space from one point to another, from (21), we must also be increasing in the w direction along the real axis in w -space from the mapping of the first point to the second. In other words, the order of points along the real axis must remain the same under the map.

We then use the mapping of Fig. 10 to conclude that the same result must hold for Riemann surfaces as we found here for Riemann spheres. For points along the real axis, the cyclic ordering of points must remain unchanged. By way of example, for three such points being mapped from z to w ,

$$P_1 P_2 P_3 \text{ must map to cyclic ordering } P_1 P_2 P_3 \text{ or } P_2 P_3 P_1 \text{ or } P_3 P_1 P_2, \text{ but not } P_1 P_3 P_2 \text{ (which is not cyclic)}. \quad (22)$$

If the cyclic order changes, the two Riemann surfaces are not equivalent. There are only two different cyclic orders for three points, so there are two different Riemann surfaces for which the three points are on the real axis.

We call the abstract set where each member of the set is a different (inequivalent) Riemann surface, modulus space. In this case, there are two discrete members of modulus space for w -spaces, the two inequivalent Riemann surfaces for which the three P_i have different cyclic order. We label that space as \mathcal{N}_3 where the "3" means three points.

Note that things are always easier when we take our points in z -space as on the real axis. Real numbers are always easier to handle than complex numbers and the mapping (19) tells us that if the z_i are real, then the mapped w_i are also. In string theory, we stick to real numbers whenever we can.

Bottom line: Using the Riemann sphere, we have proven that the cyclic order of points on the real axis in one Riemann surface (plane) must be preserved under mapping to a second Riemann surface in order for the surfaces to be equivalent. Inequivalent Riemann surfaces are members of (points in) an abstract set (space) called modulus space.

We have shown the cyclic order is maintained under mapping for points lying on the real axis, but the principle applies, in general, to the boundary of any space. The real axis is the boundary of $\overline{\mathbb{H}}$ (z -space, for example). In such spaces, there are no points below the real axis, i.e., no points with negative imaginary parts.

2nd bottom line: The cyclic ordering of points on the boundary of a Riemann surface is maintained under analytic mapping.

5.2 z -space to w -space Map

We now examine the string interaction shown in Fig. 11, where two strings are incoming and two are outgoing. The numbering may seem a little strange, as in QFT, one usually numbers the two incoming as 1 and 2; and the two outgoing as 3 and 4, but we follow Zwiebach on the numbering.

In w -space, the points P_i , Q_A , and Q_B appear in Fig. 11 to have non-zero σ values, but because they all have τ values of magnitude infinity, they effectively live on the real axis (τ axis) and can be treated as lying on the τ axis in that space, i.e., treated as real.

The order of points along the boundary of the degenerate polygon on the left side of Fig. 11 is that of the arrows indicating orientation of that boundary. For example, Q_A in w -space is between P_1 and P_2 in that space, so, as we learned in the prior section, it must lie between P_1 and P_2 in z -space. So, in fact, the ordering of points as shown in w -space of fig. 11 is the same as the ordering of points in z -space of that figure.

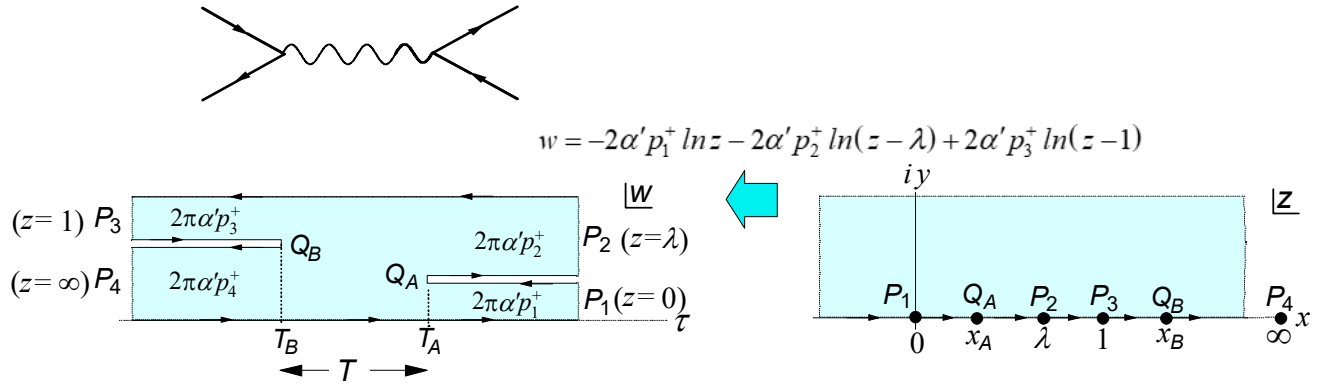


Figure 11. Map Between z -space and w -space

We know the location of the four punctures in w -space. Recall that we can choose three of the puncture locations in z -space however we wish. By convention, these are taken to be

$$P_1: z_1 = 0 \quad (w_1 = +\infty) \quad P_3: z_3 = 1 \quad (w_3 = +\infty) \quad P_4: z_4 = +\infty \quad (w_4 = -\infty). \quad (23)$$

We know that P_2 has $w_2 = -\infty$, but we are not free to specify z_2 , so we label its value as λ , an unknown at this point. We do note that, in order to maintain cyclic order, its value must lie between that of P_1 and P_3 , i.e.,

$$z_2 = \lambda \quad 0 < \lambda < 1. \quad (24)$$

Now, we don't use the sphere-to-sphere mapping of (19), since it leads to bizarre results due to all the infinities in w -space (which you can check by substitution into (19)), and because (19) was used only to glean an important result in the simplest way and is not so reflective of what is going on with Riemann surfaces. Thus, we employ the mapping (18) for four punctures. and the values for those punctures as shown in Fig. 11. That is, we employ the analytic map

$$w = -2\alpha' p_1^+ \ln z - 2\alpha' p_2^+ \ln(z - \lambda) + 2\alpha' p_3^+ \ln(z - 1) \quad z_4 = \infty. \quad (25)$$

Recall from an earlier section of these notes that for the choice of gauge family employed (which includes the light-cone gauge as one case), the parameter τ increases monotonically with time t of any observer. So, one will see authors like Zwiebach cavalierly refer to τ values, such as T in Fig. 11, as "time", even though τ is only a parameter.

To show that (25) actually does map the points as described, we calculate the w position of points in z -space in Table 2. To use that table, we remind readers of some key relations.

$$x = e^{\ln x} \rightarrow \ln 0 = -\infty \quad \ln 1 = 0 \quad \ln \infty = \infty \quad \ln(-1) = i\pi \quad \ln(-x) = \ln x + i\pi \quad \ln(-\infty) = \infty + i\pi \quad (26)$$

Table 2. Finding Points in w -space from Points on Real Axis in z -space Using (25)

Point	z Value	$w = w(z)$	w Value
P_1	$z_1 = 0$	$w_1 = -2\alpha' p_1^+ \ln 0 - 2\alpha' p_2^+ \ln(-\lambda) + 2\alpha' p_3^+ \ln(-1)$	$w_1 = \infty + i2\pi\alpha' p_3^+ = \infty$
P_2	$z_2 = \lambda$	$w_2 = -2\alpha' p_1^+ \ln \lambda - 2\alpha' p_2^+ \ln 0 + 2\alpha' p_3^+ \ln(\lambda-1)$	$w_2 = \infty + i2\pi\alpha' p_3^+ = \infty$
P_3	$z_3 = 1$	$w_3 = -2\alpha' p_1^+ \ln 1 - 2\alpha' p_2^+ \ln(1-\lambda) + 2\alpha' p_3^+ \ln 0$	$w_3 = -\infty$
P_4	$z_4 = \infty$	$w_4 = -2\alpha' p_1^+ \ln \infty - 2\alpha' p_2^+ \ln(\infty-\lambda) + 2\alpha' p_3^+ \ln(\infty-1)$	$w_4 = -\infty$
For unknown (at this point) x_A and x_B in z-space			
Q_A	$0 < x_A < \lambda$	$w_A = -2\alpha' p_1^+ \ln x_A - 2\alpha' p_2^+ \ln(x_A - \lambda) + 2\alpha' p_3^+ \ln(x_A - 1)$	$w_A = T_A + i2\pi\alpha' p_1^+$
Q_B	$1 < x_B < \infty$	$w_B = -2\alpha' p_1^+ \ln x_B - 2\alpha' p_2^+ \ln(x_B - \lambda) + 2\alpha' p_3^+ \ln(x_B - 1)$	$w_B = T_B + i2\pi\alpha' p_4^+$

5.3 Finding λ for Given T

At this point, we have three unknowns, the locations in z -space for P_2 , Q_A , and Q_B , i.e., λ , x_A , and x_B . We know the locations of all six points in w -space. (We assume $\text{Re}\{w_A\} = T_A$ and $\text{Re}\{w_B\} = T_B$ are known, as well as all p_i^+ .)

So, we need three equations in these three unknowns. Once we have these, we will be particularly interested in $T = T_A - T_B$. Because of that, we will take our first of the three equations to be the difference between the real parts of w_A and w_B , as shown in the last two rows, third column of Table 2.

The First Equation

Note from (26), that

$$\Re\{\ln C\} = \ln|C| \quad \text{for } C \text{ a positive or negative real number.} \quad (27)$$

Thus, from Table 1,

$$\begin{aligned} T_A &= \Re\{w_A\} = -2\alpha' p_1^+ \ln x_A - 2\alpha' p_2^+ \ln|x_A - \lambda| + 2\alpha' p_3^+ \ln|x_A - 1| \\ T_B &= \Re\{w_B\} = -2\alpha' p_1^+ \ln x_B - 2\alpha' p_2^+ \ln|x_B - \lambda| + 2\alpha' p_3^+ \ln|x_B - 1|. \end{aligned} \quad (28)$$

From Fig. 11, we can glean the signs of the quantities inside absolute value lines of (28), and

$$\begin{aligned} T_A &= -2\alpha' p_1^+ \ln x_A - 2\alpha' p_2^+ \ln(\lambda - x_A) + 2\alpha' p_3^+ \ln(1 - x_A) \\ T_B &= -2\alpha' p_1^+ \ln x_B - 2\alpha' p_2^+ \ln(x_B - \lambda) + 2\alpha' p_3^+ \ln(x_B - 1) \end{aligned} \quad (29)$$

$$T = T_A - T_B = -2\alpha' p_1^+ \ln \frac{x_A}{x_B} - 2\alpha' p_2^+ \ln \frac{(\lambda - x_A)}{(x_B - \lambda)} + 2\alpha' p_3^+ \ln \frac{(1 - x_A)}{(x_B - 1)}.$$

$$T = 2\alpha' p_1^+ \ln \frac{x_B}{x_A} + 2\alpha' p_2^+ \ln \frac{(x_B - \lambda)}{(\lambda - x_A)} + 2\alpha' p_3^+ \ln \frac{(1 - x_A)}{(x_B - 1)} \quad \text{1st equation in } \lambda, x_A, \text{ and } x_B. \quad (30)$$

The Second Equation

The second and third equations are arrived at by noting that at the points Q_A and Q_B in w -space, the orientation of the boundary of the degenerate polygon changes, i.e., the real part of w changes, while the imaginary part is constant. At the same points in z -space, z is real and continues increasing. Thus, we can consider the derivative of w with respect to z as zero.

For Q_A , from (25), we have

$$\left. \frac{\partial w}{\partial x} \right|_{x=x_A} = \left. \frac{\partial w}{\partial z} \right|_{z=x_A} = 0 = -2\alpha' p_1^+ \frac{1}{x_A} - 2\alpha' p_2^+ \frac{1}{x_A - \lambda} + 2\alpha' p_3^+ \frac{1}{x_A - 1} \quad \text{2nd equation in } \lambda \text{ and } x_A. \quad (31)$$

Third Equation

For Q_B , again using (25), we have

$$\left. \frac{\partial w}{\partial x} \right|_{x_B} = \left. \frac{\partial w}{\partial z} \right|_{z=x_B} = 0 = -2\alpha' p_1^+ \frac{1}{x_B} - 2\alpha' p_2^+ \frac{1}{x_B - \lambda} + 2\alpha' p_3^+ \frac{1}{x_B - 1} \quad 3^{\text{rd}} \text{ equation in } \lambda \text{ and } x_B. \quad (32)$$

The Result

So, we have three equations (30), (31), and (32) in the three unknowns λ , x_A , and x_B , where we assume T and all p_i^+ are known. Note one does not need to know the individual values of T_A and T_B to solve these equations, just the difference between them T .

In finding amplitudes, one typically takes the incoming and outgoing 4-momenta as given, since the external particles are external to the interaction region. However, the characteristics of the interaction region itself and the virtual particles/strings therein are not measurable. Such is the case with the time T between interaction points. Hence, one must integrate over all possible T to get the amplitude for given external particles of given 4-momenta.

Since the three unknowns, including λ , are all pinned down by the three equations, for given T , we can find λ as a function of T . We can also find x_A and x_B as functions of T , if we were interested in those. Essentially, in principle, we can find

$$\lambda = \lambda(T) \text{ for given } p_i^+ \quad \text{more generally, } \lambda = \lambda(T, p_i^+), \quad (33)$$

so, integration in w -space over T is equivalent to integration in z -space over λ .

In practice, solving these equations is not trivial, but one can more easily investigate the behavior of the various parameters in certain extrema, as we discuss in the next sub-section.

Re-arranging Two of the Equations

Zwiebach rearranges the 2nd and 3rd equations (31) and (32) to gain insight into the amplitude. To do this, first solve (32) for all p_1^+ and substitute that into (31).

$$(32) \rightarrow p_1^+ = x_B \left(-p_2^+ \frac{1}{x_B - \lambda} + p_3^+ \frac{1}{x_B - 1} \right) \quad (34)$$

$$(31) \rightarrow p_1^+ = x_A \left(-p_2^+ \frac{1}{x_A - \lambda} + p_3^+ \frac{1}{x_A - 1} \right) = x_B \left(-p_2^+ \frac{1}{x_B - \lambda} + p_3^+ \frac{1}{x_B - 1} \right) \quad (35)$$

$$\begin{aligned} -\frac{x_A}{x_B} p_2^+ \frac{1}{x_A - \lambda} + \frac{x_A}{x_B} p_3^+ \frac{1}{x_A - 1} &= -p_2^+ \frac{1}{x_B - \lambda} + p_3^+ \frac{1}{x_B - 1} \\ p_2^+ \frac{1}{x_B - \lambda} - \frac{x_A}{x_B} p_2^+ \frac{1}{x_A - \lambda} &= p_3^+ \frac{1}{x_B - 1} - \frac{x_A}{x_B} p_3^+ \frac{1}{x_A - 1} \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{p_2^+}{p_3^+} \left(\frac{1}{x_B - \lambda} - \frac{x_A}{x_B} \frac{1}{x_A - \lambda} \right) &= \frac{1}{x_B - 1} - \frac{x_A}{x_B} \frac{1}{x_A - 1} \\ \frac{p_2^+}{p_3^+} \left(\frac{x_B(x_A - \lambda) - x_A(x_B - \lambda)}{(x_B - \lambda)x_B(x_A - \lambda)} \right) &= \frac{x_B(x_A - 1) - x_A(x_B - 1)}{(x_B - 1)x_B(x_A - 1)} \end{aligned} \quad (37)$$

$$\frac{p_2^+}{p_3^+} \left(\frac{\lambda(x_A - x_B)}{(x_B - \lambda)(x_A - \lambda)} \right) = \frac{x_A - x_B}{(x_B - 1)(x_A - 1)}$$

$$\frac{p_2^+}{p_3^+} = \frac{(\lambda - x_A)(x_B - \lambda)}{(1 - x_A)(x_B - 1)\lambda} \quad \text{Zwiebach (25.87) [620] 2nd line (our } x_A, x_B = \text{his } x_1, x_2) \quad (38)$$

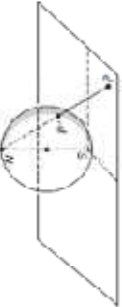
In similar fashion, we show without proof,

$$\frac{p_2^+}{p_1^+} = \frac{(\lambda - x_A)(x_B - \lambda)}{(1 - \lambda)x_A x_B} \quad \text{Zwiebach (25.87) [620] 1st line.} \quad (39)$$

Our 1st equation (30) is Zwiebach's (25.86) [619]. On pages 620-621, he uses (30), (38), and (39) to show that as $T \rightarrow +\infty$, $\lambda \rightarrow 0$, and as $T \rightarrow -\infty$, $\lambda \rightarrow 1$. As T varies continuously from $-\infty$ to $+\infty$, λ varies continuously from 0 to 1. Each different value of T in w -space leads to a different value of λ , and thus, an inequivalent Riemann surface in z -space. Hence, as T is a modulus in w -space, λ is a modulus in z -space.

Summary of this Section 5: See Wholeness Chart 1 below.

Summary of Interacting Strings Intro

Conformal Mapping Relations between Riemann Spaces	
Mapping Types	
Plane-to-plane	$w = \lambda \alpha' p_1^+ \ln(\tau - \tau_1) + \lambda \alpha' p_2^+ \ln(\tau - \tau_2) + \lambda \alpha' p_n^+ \ln(\tau - \tau_n)$ punctures at τ_i not included
Sphere-to-sphere	$w = \frac{a\tau + b}{c\tau + d}$ Same as $\lambda \alpha' p_1^+ \ln(\tau - \tau_1) + \lambda \alpha' p_2^+ \ln(\tau - \tau_2) + \lambda \alpha' p_n^+ \ln(\tau - \tau_n)$ convention, choose $a\bar{d} - b\bar{c} > 0$; by convention, choose $a\bar{d} - b\bar{c} = 1$
Plane-to-sphere	 <p>(Can use z plane coordinates to label points on sphere. Then, can refer to sphere points with plane diag. au.)</p>
Note:	Spaces mapped to are equivalent to original space. One can draw certain conclusions regarding mapping more easily from the sphere to sphere relation, then via the sphere-to-plane one-to-one correspondence, can adapt those conclusions to Riemann surfaces (planes).
Conclusions	
Planes	Puncture on real z axis mapped to real w axis. 3 punctures z_i can be mapped to any 3 w_j coordinates, but if on boundary of the space, such as the real axis, must be in same order. Only two possible orders for 3 points on real axis. If one more puncture z_4 , the order of points in w -space, must be preserved. Moduli space \mathcal{M}_4 has coordinates of 4th puncture as a modulus, typically labeled λ in z space. Each λ value represents a different z Riemann space. 4-string interactions: Conventionally, punctures taken as $P_1 @ z_1 = 0$ (specified as mapped to $w_1 = +\infty$) $P_3 @ z_3 = 1$ (specified as mapped to $w_3 = -i\pi$) $P_4 @ z_4 = \infty$ (specified as mapped to $w_4 = -\infty$) $P_2 @ z_2 = \lambda$ (specified as mapped to $w_2 = +i\pi$) where λ is between z_1 and z_3
Spheres	Choosing to place all punctures on the real axis simplifies analysis. Use sphere to sphere map along with sphere to plane map to draw conclusion at left. Moduli space \mathcal{M}_3 (subscript 3 means 3 punctures) has two discrete points (two orderings). In general, λ is a complex number (two real numbers) $\text{Dim } \mathcal{M}_4 = 2$ If first 3 punctures on real line, 4th is also. Then, $\text{Dim } \mathcal{M}_3 = 1$. Modulus λ , $0 < \lambda < 1$, because P_2 must lie between P_1 and P_3 . λ is fixed by mapping 2 additional points in w -space, as follows Choose 2 points in w -space Q_A & Q_B , with real coordinates labeled T_A & T_B , with $T = T_A - T_B$. (Zwiebach calls these T_1 & T_2 , but that can be confused with P_1 & P_2 .) Use relation in top row to find 3 equations in 3 z -space unknowns, λ , $\bar{\lambda}$, and λ , where T is known. Solving these, can, in principle, find $\lambda = \lambda(T)$ for given w_i . Integration over T ($-\infty$ to $+\infty$) in w -space yields the same amplitude as integration over λ (0 to 1) in z -space. Can be easier in z -space. $\mathcal{M}_{3,3}$ has no members, no moduli (no ordering or sphere surface) $\mathcal{M}_{3,4}$ has complex λ , 2 real numbers $\text{Dim } \mathcal{M}_{3,4} = 2$

6 Self-energy Interactions for Strings

We now examine what is known in QFT as a self-energy diagram, one of which, for QFT, is displayed in the upper left (Feynman) diagram in Fig. 11. Below that diagram is the equivalent interaction in string theory, where the analysis and symbolism follow that of prior sections. ΔT is the time (the τ parameter, actually) during which the virtual particles/strings exist. Note that the incoming 4-momentum (in light-cone coordinates) is considered given, and since momentum is conserved, only one of the p_{1+} or p_{2+} momenta is unknown. Of the two, we choose p_{1+} to be the unknown, so it and ΔT are the variables. If we were to work with this expression of the interaction, to get the amplitude, we would integrate over all possible ΔT and all possible p_{1+} . These two variables are known as the moduli for this example.

We will be carrying out a sequence of several analytic mappings for this interaction, which are diagrammed in Fig. 11. The period during which the virtual particles/strings exist is represented by a slit, as shown in the top row left diagram. Thus, we can treat that entire diagram like we treated other interactions using degenerate polygons, though we realize the slit actually has zero thickness.

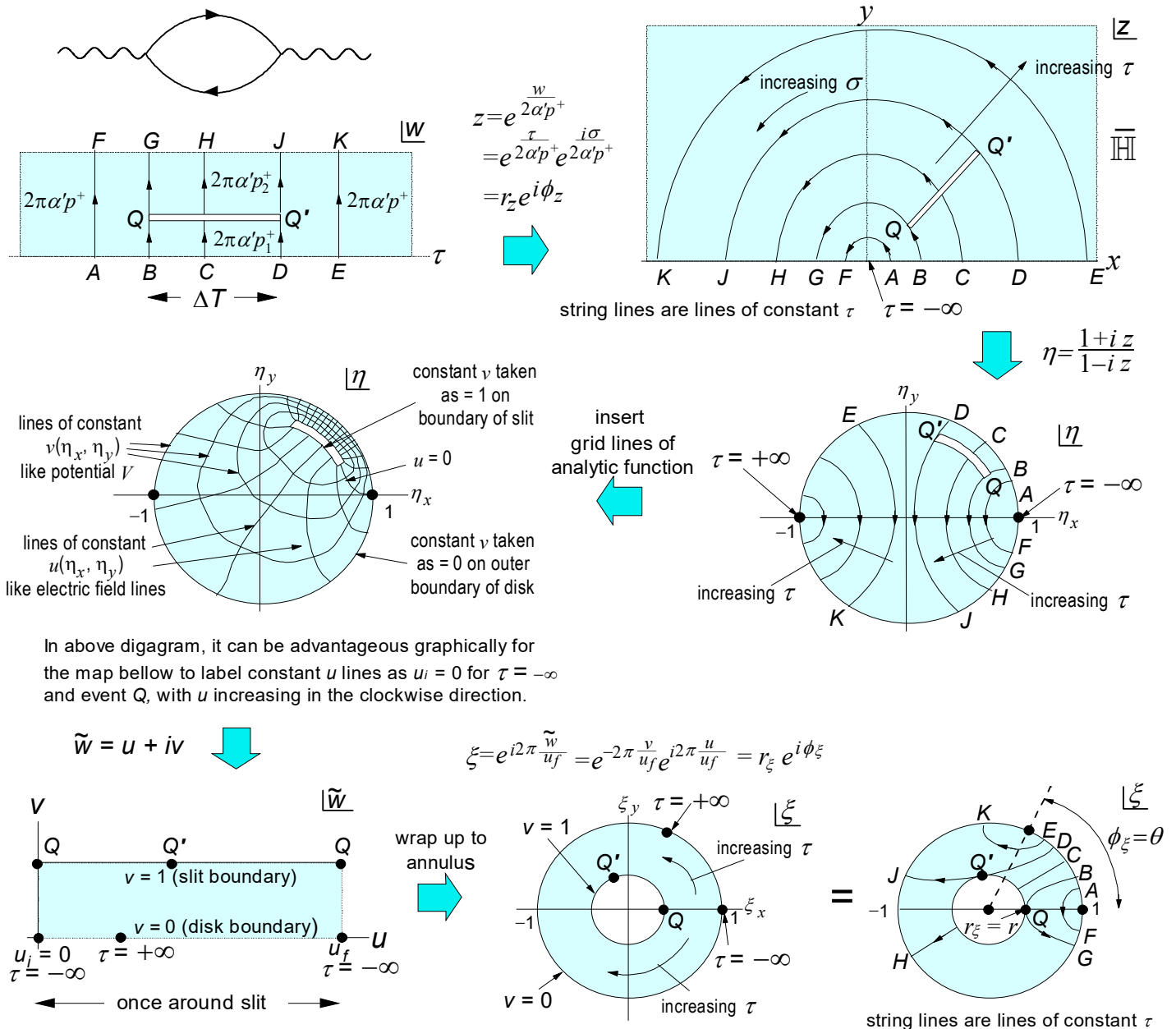


Figure 10. Sequence of Analytic Mappings from Self Energy String Diagram to Canonical Annulus

6.1 The First Mapping: From w -space to z -space

The first mapping is the one we have been dealing with in prior sections. It takes us from the top row left diagram in w of Fig. 11 to the top row right diagram in z and is shown mathematically both in that row and in (7). The diagrams parallel Fig. 7, except there is now a slit that is carried along with the mapping.

Note some things:

1. The string with $\tau=0$ actually lies between the DJ and EK strings and is not shown. It has been taken there (which may seem a bit unusual or unexpected) in order for subsequent mapping diagrams to look like those in Zwiebach. Hence, the entire interaction over ΔT occurs for negative τ . However, ΔT is still positive.
2. The angular distance ϕ_z around any string in z -space is proportional to σ . That is why the slit between Q and Q' aligns longitudinally with the origin in z -space. They both have the same sigma values, so they both have the same ϕ_z values.
3. If ΔT is longer, then the slit in z -space is longer. If p_1^+ is greater (and thus, p_2^+ lesser), the slit in z -space moves counterclockwise, since ϕ_z is proportional to p_1^+ .

6.2 The Second Mapping: From z -space to η -space

The next mapping, shown in the middle row right side, is that of (9), re-expressed below as (40) and takes us from the z complex variable space to the η complex variable space.

$$\eta(z) = \frac{1+iz}{1-iz} \quad (40)$$

In Table 3, we deduce mapped η values from associated z values for some particular points.

Table 3. Finding Points in η -space from Points in z -space

<u>Point in z-space</u> <u>x and y</u>	<u>Point in z-space</u> <u>r_z and ϕ_z</u>	<u>Point in w-space</u> <u>τ and σ</u>	<u>Point in η-space</u> $\eta = \frac{1+iz}{1-iz} = \frac{1-y+ix}{1+y-ix} =$ $\frac{1+ir_z e^{i\phi_z}}{1-ir_z e^{i\phi_z}} = \frac{1+ie^{\frac{\tau}{2\alpha'p^+}} e^{i\frac{\sigma}{2\alpha'p^+}}}{1-ie^{\frac{\tau}{2\alpha'p^+}} e^{i\frac{\sigma}{2\alpha'p^+}}}$	<u>η_x and η_y</u>
$x = y = 0$	$r_z = 0$ any ϕ_z	$\tau = -\infty$ any σ	$\eta = 1$	$\eta_x = 1$ $\eta_y = 0$
$x = +\infty$ $y = 0$	$r_z = \infty$ $\phi_z = 0$	$\tau = +\infty$ $\sigma = 0$	$\eta = -1$	$\eta_x = -1$ $\eta_y = 0$
$x = 0$ $y = +\infty$	$r_z = \infty$ $\phi_z = \pi/2$	$\tau = +\infty$ $\sigma = \pi/2$	$\eta = -1$	$\eta_x = -1$ $\eta_y = 0$
$x = -\infty$ $y = 0$	$r_z = \infty$ $\phi_z = \pi$	$\tau = +\infty$ $\sigma = \pi$	$\eta = -1$	$\eta_x = -1$ $\eta_y = 0$
$x =$ between D & E $y = 0$	$r_z = r_{DE}$ $\phi_z = 0$	$\tau = 0$ $\sigma = 0$	$\eta = \frac{1+i}{1-i} = \frac{e^{i\pi/4}}{e^{-i\pi/4}} = e^{i\pi/2} = i$	$\eta_x = 0$ $\eta_y = 1$
$x =$ between J & K $y = 0$	$r_z = r_{JK} = r_{DE}$ $\phi_z = \pi$	$\tau = 0$ $\sigma = 2\pi\alpha'p^+$	$\eta = \frac{1+ie^{i\pi}}{1-ie^{i\pi}} = \frac{1-i}{1+i} = \frac{1}{i} = -i$	$\eta_x = 0$ $\eta_y = -1$
$x = 0$ $y =$ between DJ & EK strings	$r_z = r_{JK} = r_{DE}$ $\phi_z = \pi/2$	$\tau = 0$ $\sigma = \pi\alpha'p^+$	$\eta = \frac{1+ie^{i\pi/2}}{1-ie^{i\pi/2}} = \frac{1+i^2}{1-i^2} = 0$	$\eta_x = 0$ $\eta_y = 0$
$x = 0$ any y	$r_z =$ any $\phi_z = \pi/2$	any τ $\sigma = \pi\alpha'p^+$	$\eta = \frac{1+ir_z e^{i\pi/2}}{1-ir_z e^{i\pi/2}} = \frac{1-r_z}{1+r_z}$	on real axis $\eta_y = 0$

Note some things:

1. The entire length of the incoming string at $\tau = -\infty$ is located at a single point in η -space, $\eta = 1$. The entire length of the outgoing string at $\tau = +\infty$ is located at a single point in η -space, $\eta = -1$. Time passes moving counterclockwise (positive ϕ_η) in the upper half plane; clockwise (negative ϕ_η) in the lower half.
2. If ΔT is longer, and thus, the slit in z -space is longer, then the slit in η -space is also longer. If p_1^+ is greater (p_2^+ lesser), and thus, the slit in z -space moves counterclockwise (ϕ_z is greater), then the slit in η -space moves inward (further from the starting points B, C, and D), i.e., r_η gets smaller.

6.3 The Third Mapping from η -space to \tilde{w} -space

For the next mapping to what we label \tilde{w} -space, note if you are following along with Zwiebach, that he uses the notation w -space. Using w can be confusing since that was used (by both us and Zwiebach) for the very first string diagram (upper left in Fig. 11), but this space we now encounter is a quite different thing.

The mapping we now employ is akin to taking potential and field lines in electrostatics, as shown in the middle row lefthand diagram of Fig. 11. Think of the middle row righthand diagram as representing the cross section of a capacitor, which extends out of the page. There is a charge distributed along the inner part of the slit, and an opposite charge distributed around the outer circumference. These charges result in potential lines and field line perpendicular to those potential lines, as shown in the middle row lefthand diagram.

We take v to represent potential lines and u to represent field lines. If you have not previously analyzed an electrostatic potential problem using complex variables, simply note that in such cases, for 2D, we can represent the potential and field lines as the real and imaginary parts of a complex variable. Here, call that complex variable \tilde{w} , so our 3rd mapping can be described by

$$\tilde{w} = u + iv \quad u = u(\eta_x, \eta_y) \quad v = v(\eta_x, \eta_y). \quad (41)$$

The relations $u = u(\eta_x, \eta_y)$ and $v = v(\eta_x, \eta_y)$ can be complicated to deduce and vary with the parameters ΔT and p_1^+ . We do not derive them here, but merely note below how they depend in general on η -space, as well as on ΔT and p_1^+ .

By convention, we take $u = 0$ on the u line intersecting $\eta = 1$ ($\tau = -\infty$), and consider u increasing as we travel counterclockwise around the slit (which is counterclockwise around the disk circumference, as well). We label the final u after going once around the slit as u_f . So, for one trip around the slit, we have $\Delta u = u_f$.

Also, by convention, we take the potential v on the boundary of the slit to equal 1, and on the outer boundary of the disk equal to zero.

Things to note:

1. The potential and field lines where the slit is close to the disk circumference (above the slit) are packed tightly, whereas on the other side of (below) the slit they are widely spaced. So, the Δu above the slit has a greater value than the Δu below the slit, even though ΔT is the same for both above and below the slit.
2. If we increase ΔT , the slit gets longer, so the Δu above the slit would be even greater than it was compared to the Δu below the slit. The net effect is an increase in u_f . Thus, u_f is a function of ΔT and increases, or decreases, with it.

$$u_f = u_f(\Delta T) \quad \Delta T \uparrow, u_f \uparrow \quad \Delta T \downarrow, u_f \downarrow \quad (42)$$

This can, alternatively, be thought of via electrostatics, where the boundary of the slit has, for the same charge per unit length, a greater total charge if it is larger. Thus, we would get more field lines emanating from it.

3. If we increase p_1^+ , as noted above, the slit moves inward toward the disk center. This spaces out the u (field) lines above the slit, and tightens them up below it. Hence, Δu above the slit is less, and Δu below the slit is more. Similarly, Δu around the disk circumference from $\tau = -\infty$ ($\eta = 1$) to $\tau = +\infty$ ($\eta = -1$) in the top half of the plane is reduced when p_1^+ increases. We label this Δu as u_{top} .

$$u_{top} = u_{top}(p_1^+) \quad p_1^+ \uparrow, u_{top} \downarrow \quad p_1^+ \downarrow, u_{top} \uparrow. \quad (43)$$

We then plot the u and v values around the slit and around the η disk circumference in the bottom row lefthand diagram. That space is \tilde{w} -space and completes our map from η -space to \tilde{w} -space.

Note in the \tilde{w} -space diagram that time increases on the u axis ($v = 0$, η disk circumference) until we get to $\tau = +\infty$, then it decreases to $\tau = -\infty$ at u_f . Along the $v = 1$ line (slit boundary), time increases from Q to Q' , then decreases from Q' to Q , although u increases continually along that line (and the u axis, as well).

6.4 The Final Mapping: from \tilde{w} -space to ξ -space.

We now carry out the final mapping, which is simply a rolling up of the rectangle in the bottom row lefthand diagram into a canonical annulus. This results in the final two diagrams of the bottom row, which are the same thing, but we simply need two such diagrams to label all the quantities we want to label in a legible way.

Mathematically, the rolling up is expressed as the mapping from \tilde{w} -space to ξ -space of

$$\xi = e^{i2\pi \frac{\tilde{w}}{u_f}} = e^{-2\pi \frac{v}{u_f}} e^{i2\pi \frac{u}{u_f}} = r_\xi e^{i\phi_\xi} \quad r_\xi = e^{-2\pi \frac{v}{u_f}} \quad \phi_\xi = 2\pi \frac{u}{u_f}. \quad (44)$$

Note that for $v = 0$, $r_\xi = 1$, which is what we see in the middle diagram of the bottom row in Fig. 11.

We use the symbol r for the inner radius of the annulus in ξ -space, i.e.,

$$\text{at } v=1 \text{ (boundary of slit), } r_\xi = r = e^{-\frac{2\pi}{u_f}}, \quad (45)$$

and the angular distance from $\tau = -\infty$ to $\tau = +\infty$ in the increasing u direction as θ , i.e.,

$$\text{at } \tau = +\infty, \quad \phi_\xi = \theta. \quad (46)$$

Note the following:

1. The inner radius r of (45) increases in ξ -space if u_f increases. In the last section, we learned that u_f increases when ΔT increases and decreases when ΔT decreases. See (42). Thus,

$$r = r(u_f) = r(\Delta T) \quad \Delta T \uparrow, r \uparrow \quad \Delta T \downarrow, r \downarrow. \quad (47)$$

For $\Delta T \rightarrow \infty$, $r \rightarrow 1$, and the annulus in the bottom row righthand diagram becomes very thin.

2. From (44), we see that ϕ_ξ depends on u_f , and thus, on ΔT , but it also depends on u . The quantity θ of (46) has the u value u_{top} as defined in the prior sub-section. So,

$$\theta = \phi_\xi \Big|_{\tau=+\infty} = 2\pi \frac{u_{top}}{u_f}. \quad (48)$$

As we noted in the prior section, u_{top} decreases as the slit moves away from the disk circumference in η space, and the slit moves away for an increase in p_1^+ . See (43). So,

$$\theta = \theta(u_{top}, u_f) = \theta(p_1^+, \Delta T). \quad (49)$$

3. To find the precise functional relationship between the r, θ parameters of ξ -space and $\Delta T, p_1^+$ of w -space, we would need to find the precise functional dependencies of u and v in \tilde{w} -space on η_x and η_y in η -space, i.e.,

$$u = u(\eta_x, \eta_y) \quad v = v(\eta_x, \eta_y). \quad (50)$$

We do not do that here.

6.5 Summary

To find the amplitude for the interaction of Fig. 11, we need to integrate over all values of ΔT and p_1^+ , the independent variables, which are the moduli in w -space.

Integration over all independent variables is invariant under any analytic mapping. The physics is the same. We just change the parameters. So, the amplitude, the square of whose absolute value is the probability of the interaction, is the same when calculated from any of the diagrams in Fig. 11.

Thus, whereas ΔT and p_1^+ are the moduli for the first string diagram of Fig. 11, r and θ are the moduli for the last diagram. To get the amplitude we integrate over all r and all θ . Note this is not the same as integrating over r_ξ and ϕ_ξ in ξ -space for given r and θ . We must effectively add (integrate) every possible ξ -space diagram, each having different r and θ .

7 Appendix: A Natural Log Relation

$$\ln z = \ln(r_z e^{i\phi_z}) = \ln(r_z) + \ln(e^{i\phi_z}) = \ln(r_z) + i\phi_z \quad (51)$$

$$\begin{aligned} \phi_z = 0 & \quad \rightarrow \quad \ln z = \ln(r_z e^0) = \ln(r_z) \\ \phi_z = \pi & \quad \rightarrow \quad \ln z = \ln(r_z e^{i\pi}) = \ln(-r_z) = \ln(r_z) + i\pi \\ \phi_z = \pi/2 & \quad \rightarrow \quad \ln z = \ln(r_z e^{i\pi/2}) = \ln(ir_z) = \ln(r_z) + i\frac{\pi}{2} \end{aligned} \quad (52)$$

From the first row in (52), in z -space ($\overline{\mathbb{H}}$), we have $z = x = +\infty$ ($r_z = \infty$ and $\phi_z = 0$), and thus, $\ln z = \ln(r_z) = \infty$.

From the second row in (52), we have $z = x = -\infty$ ($r_z = \infty$ and $\phi_z = \pi$), and thus, $\ln z = \ln(r_z) + i\pi = \infty$.

From the third row in (52), we have $z = iy = +\infty i$ ($r_z = \infty$ and $\phi_z = \pi/2$), and thus, $\ln z = \ln(r_z) + i\pi/2 = \infty$.

Due to the minus signs in (10), we will get $w = \tau = -\infty$ for P_1 in w -space, but a semi-circle of infinite radius with ends at $-\infty$ and $+\infty$ in z -space.