

Sect. 18.9.2 Alternative Method

A reader, Luc Longtin, came up with an alternative method for evaluating the many paths transition amplitude of Sect. 18.9.2, pgs. 504-505. Those lacking familiarity with the convolution integral approach should find this method easier to understand than that used in the text. However, it is a bit longer.

Evaluating the Amplitude Section, pg. 504

To evaluate (18-50), we make repeated use of (18-46). We first perform the integration over the x_1 variable; that is, the rightmost integral in (18-50). We have the following:

$$\begin{aligned} & \dots \int_{-\infty}^{+\infty} e^{\frac{im}{2h(\Delta t)}(x_2-x_1)^2} e^{\frac{im}{2h(\Delta t)}(x_1-x_i)^2} dx_1 = \dots \int_{-\infty}^{+\infty} e^{\frac{im}{2h(\Delta t)}(x_2^2-2x_2x_1+x_1^2)} e^{\frac{im}{2h(\Delta t)}(x_1^2-2x_1x_i+x_i^2)} dx_1 \\ & = \dots \int_{-\infty}^{+\infty} e^{\frac{im}{2h(\Delta t)}(x_2^2+x_i^2)} e^{-\frac{im}{h(\Delta t)}(x_2+x_i)x_1} e^{\frac{im}{h(\Delta t)}(x_1^2)} dx_1 = \dots e^{\frac{im}{2h(\Delta t)}(x_2^2+x_i^2)} \int_{-\infty}^{+\infty} e^{\frac{im}{h(\Delta t)}(x_1^2)} e^{-\frac{im}{h(\Delta t)}(x_2+x_i)x_1} dx_1 \end{aligned}$$

The integral on the RHS is seen to be of form (18-46) with:

$$a = -\frac{im}{h(\Delta t)} \quad \text{and} \quad b = -\frac{im}{h(\Delta t)}(x_2 + x_i)$$

Using the result of (18-46), we can write:

$$\begin{aligned} & = \dots e^{\frac{im}{2h(\Delta t)}(x_2^2+x_i^2)} \sqrt{\frac{\pi h(\Delta t)}{-im}} e^{\left(-\frac{im}{h(\Delta t)}(x_2+x_i)\right)^2 / 4 \left(-\frac{im}{h(\Delta t)}\right)} = \dots \sqrt{\frac{i\pi h(\Delta t)}{m}} e^{\frac{im}{2h(\Delta t)}(x_2^2+x_i^2)} e^{-\frac{im}{4h(\Delta t)}(x_2+x_i)^2} \\ & = \dots \sqrt{\frac{i\pi h(\Delta t)}{m}} e^{\frac{im}{2h(\Delta t)}\left[(x_2^2+x_i^2) - \frac{(x_2+x_i)^2}{2}\right]} = \dots \sqrt{\frac{i\pi h(\Delta t)}{m}} e^{\frac{im}{2h(\Delta t)}\frac{(x_2-x_i)^2}{2}} = \dots \sqrt{\frac{ih(\Delta t)}{m}} \frac{1}{2} e^{\frac{im}{2h(\Delta t)}\frac{(x_2-x_i)^2}{2}} \end{aligned}$$

We now proceed to the second integration in (18-50), over the x_2 variable; that is:

$$\begin{aligned} & = \dots \sqrt{\frac{ih(\Delta t)}{m}} \frac{1}{2} \int_{-\infty}^{+\infty} e^{\frac{im}{2h(\Delta t)}(x_3-x_2)^2} e^{\frac{im}{2h(\Delta t)}\frac{(x_2-x_i)^2}{2}} dx_2 \\ & = \dots \sqrt{\frac{ih(\Delta t)}{m}} \frac{1}{2} \int_{-\infty}^{+\infty} e^{\frac{im}{2h(\Delta t)}(x_3^2-2x_3x_2+x_2^2)} e^{\frac{im}{2h(\Delta t)}\frac{(x_2^2-2x_2x_i+x_i^2)}{2}} dx_2 \\ & = \dots \sqrt{\frac{ih(\Delta t)}{m}} \frac{1}{2} \int_{-\infty}^{+\infty} e^{\frac{im}{2h(\Delta t)}\left(x_3^2+\frac{x_i^2}{2}\right)} e^{\frac{im}{2h(\Delta t)}\left(\frac{3x_2^2}{2}\right)} e^{-\frac{im}{2h(\Delta t)}(2x_3+x_i)x_2} dx_2 \\ & = \dots \sqrt{\frac{ih(\Delta t)}{m}} \frac{1}{2} e^{\frac{im}{2h(\Delta t)}\left(x_3^2+\frac{x_i^2}{2}\right)} \int_{-\infty}^{+\infty} e^{\frac{im}{2h(\Delta t)}\left(\frac{3x_2^2}{2}\right)} e^{-\frac{im}{2h(\Delta t)}(2x_3+x_i)x_2} dx_2 \end{aligned}$$

The integral on the RHS is seen to be of form (18-46) with:

$$a = -\frac{im}{\hbar(\Delta t)} \frac{3}{4} \quad \text{and} \quad b = -\frac{im}{\hbar(\Delta t)} \left(x_3 + \frac{x_i}{2} \right)$$

Using the result of (18-46), we can write:

$$\begin{aligned} &= \dots \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{1}{2}} e^{\frac{im}{2\hbar(\Delta t)} \left(x_3^2 + \frac{x_i^2}{2} \right)} \sqrt{\frac{\pi \hbar(\Delta t)}{-im} \frac{4}{3}} e^{\left[-\frac{im}{\hbar(\Delta t)} \left(x_3 + \frac{x_i}{2} \right) \right]^2 / 4 \left(-\frac{im}{\hbar(\Delta t)} \frac{3}{4} \right)} \\ &= \dots \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{1}{2}} e^{\frac{im}{2\hbar(\Delta t)} \left(x_3^2 + \frac{x_i^2}{2} \right)} \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{2}{3}} e^{-\frac{im}{\hbar(\Delta t)} \frac{1}{3} \left(x_3 + \frac{x_i}{2} \right)^2} \\ &= \dots \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{1}{2}} \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{2}{3}} e^{\frac{im}{2\hbar(\Delta t)} \left(x_3^2 + \frac{x_i^2}{2} - \frac{2}{3} \left(x_3 + \frac{x_i}{2} \right)^2 \right)} \\ &= \dots \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{1}{2}} \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{2}{3}} e^{\frac{im}{2\hbar(\Delta t)} \left(\frac{x_3^2 - 2x_3x_i + x_i^2}{3} \right)} = \dots \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{1}{2}} \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{2}{3}} e^{\frac{im}{2\hbar(\Delta t)} \frac{(x_3 - x_i)^2}{3}} \end{aligned}$$

We see a pattern emerging: for the total of n integrations in (18-50), we will get a product of n square roots and an exponential factor. We have found that exponential factor to be:

$$\text{after one integration, } e^{\frac{im}{2\hbar(\Delta t)} \frac{(x_2 - x_i)^2}{2}}, \text{ after two integrations, } e^{\frac{im}{2\hbar(\Delta t)} \frac{(x_3 - x_i)^2}{3}}.$$

$$\text{We thus expect the exponential factor after } n \text{ integrations to be } e^{\frac{im}{2\hbar(\Delta t)} \frac{(x_{n+1} - x_i)^2}{n+1}} = e^{\frac{im}{2\hbar T} (x_f - x_i)^2}.$$

In the last equality, we have used the fact that $x_f = x_{n+1}$ and $T = (n+1)\Delta t$.

As to the product of square roots, we might guess the following result after three integrations:

$$\dots \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{1}{2}} \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{2}{3}} \sqrt{\frac{i\hbar(\Delta t)}{m} \frac{3}{4}} = \dots \left(\frac{i\hbar(\Delta t)}{m} \right)^{3/2} \sqrt{\frac{1}{4}} \quad (\text{our guess; validated in problem})$$

To determine the actual result for the square root factors, we must perform the third integration, over the x_3 variable. This is left as an exercise for you to do, and confirm our guess.

Having these results, we easily extrapolate to write the end result for the n integrations of (18-50). We thus have:

$$U(x_i, x_f; T) \approx C \left(\frac{i\hbar(\Delta t)}{m} \right)^{n/2} \sqrt{\frac{1}{n+1}} e^{\frac{im}{2\hbar T} (x_f - x_i)^2} = C \left(\frac{i\hbar(\Delta t)}{m} \right)^{n/2} \sqrt{\frac{\Delta t}{T}} e^{\frac{im}{2\hbar T} (x_f - x_i)^2}$$

$$U(x_i, x_f; T) \approx C \left(\frac{i\hbar(\Delta t)}{m} \right)^{n/2} \sqrt{\frac{i\hbar(\Delta t)}{m}} \sqrt{\frac{m}{i\hbar T}} e^{\frac{im}{2\hbar T} (x_f - x_i)^2}$$

$$U(x_i, x_f; T) \approx C \left(\frac{i\hbar(\Delta t)}{m} \right)^{(n+1)/2} \sqrt{\frac{m}{i\hbar T}} e^{\frac{im}{2\hbar T}(x_f - x_i)^2} = C \left(\frac{i2\pi\hbar(\Delta t)}{m} \right)^{(n+1)/2} \sqrt{\frac{m}{i2\pi\hbar T}} e^{\frac{im}{2\hbar T}(x_f - x_i)^2},$$

where $n + 1 = N$ of Chap. 18, (18-59). n = number of integrations.

From this result, we see that the dependence on x_i , x_f and T , is *wholly* contained in the last two factors on the RHS, namely the square root and the exponential factor. The constant C can be determined by comparison with (18-47).

$$U(x_i, x_f; T) = \sqrt{\frac{m}{i2\pi\hbar T}} e^{\frac{im}{2\hbar T}(x_f - x_i)^2} \quad \text{from Schroedinger approach} \quad (18-47)$$

to get

$$C = \left(\frac{m}{i\hbar(\Delta t)} \right)^{(n+1)/2}.$$

Thus, we arrive at (18-61),

$$\left| U(x_i, x_f; T) \right|^2 = \frac{m}{2\pi\hbar T} \quad (18-61)$$