Pedagogic Aids to Supersymmetry

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These notes are best used with Aitchison (*Supersymmetry in Particle Physics,* Cambridge 2007), but can aid with other texts, as well.

Table of Contents

1 Gauge Hierarchy Fine Tuning Background

NOTE: This Section 1 is an attempt to explain a somewhat obscure part of Aitchison's Chap. 1, from the bottom paragraph of pg. 6 to the middle of pg. 8. However, IMO, it contributes little to an understanding of SUSY, so the reader can skip it (that part in Aitchison and this present Section 1), if she/he chooses, with little impact on the learning process.

1.1 Photons and Loop Corrections

There is no photon mass term in the QED Lagrangian. From Klauber Vol. 1, second line of (11-31), pg. 293

$$\mathcal{L}_{QED} = \underbrace{-\tfrac{1}{2}\left(\partial^{\nu} A^{\mu}\partial_{\nu} A_{\mu} - \partial^{\nu} A^{\mu}\partial_{\mu} A_{\nu}\right)}_{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}} + \bar{\psi}\left(i\gamma^{\nu}\partial_{\nu} - m\right)\psi + e\bar{\psi}\gamma^{\nu}\psi A_{\nu} \tag{1-1}$$

$$\text{no such term} \quad \rightarrow \quad m_{\gamma}^{2} A^{\mu} A_{\mu}\,. \tag{1-2}$$


free massless photon

Figure 1. Free Photon Feynman Diagram

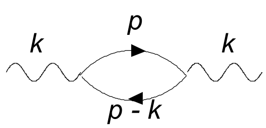
1.1.1 Symmetry of the Photon Terms

In the cited reference, after the cited equation, it is shown that the Lagrangian (1-1) is symmetric, but would not be symmetric if a term like (1-2) existed.

Bottom line: U(1) symmetry (gauge invariance) means the U(1) gauge field (the photon) is massless.

1.1.2 Higher Order Corrections

This is true at tree level. Higher order corrections might be expected to modify this.



might be equivalent to 

where **X** is a mass correction.

Figure 2. One Loop Correction to Free Photon

However, Fig. 2 modifies Fig. 1 because the amplitude for Fig. 2 corrects the amplitude for Fig. 1. But Fig. 1 is not due to terms like (1-2), but to terms like those in the first parentheses after the equal sign in (1-1). There are still no terms like (1-2), even with higher order corrections included.

The divergence from Fig. 2 is absorbed into the running coupling constant $e(p)$, and does not affect the massless nature of the photon.

Bottom line: Symmetry of $\mathcal{L}$ under a U(1) transformation enforces no mass for, and then no mass correction to, the photon.

1.2 Electrons and Loop Corrections

1.2.1 Symmetry and Asymmetry for the Electron Terms

There is a mass term in (1-1), however, for the electron. With this term, the entire $\mathcal{L}$ is symmetric (gauge invariant) under the U(1) transformation. BUT, it is not symmetric under a chiral transformation (in spinor space), i.e., under

$$\psi \rightarrow e^{i\alpha\gamma_5}\psi \quad \text{and} \quad \bar{\psi} = \psi^{\dagger}\gamma^{0} \rightarrow \psi^{\dagger} e^{-i\alpha\gamma_5^{\dagger}}\gamma^{0} = \psi^{\dagger} e^{-i\alpha\gamma_5}\gamma^{0}\,. \tag{1-3}$$

The mass term in (1-1), transforms under (1-3) as

$$m\bar{\psi}\psi \rightarrow m\psi^{\dagger} e^{-i\alpha\gamma^5}\gamma^{0} e^{i\alpha\gamma^5}\psi \xrightarrow[\gamma^5\gamma^{\mu} = -\gamma^{\mu}\gamma^5]{\text{with}} -m\psi^{\dagger}\gamma^{0} e^{-i\alpha\gamma^5} e^{i\alpha\gamma^5}\psi = -m\bar{\psi}\psi\,. \tag{1-4}$$

The mass term is not invariant (it changes sign).

The kinetic electron term in (1-1) is, however, invariant under (1-3).

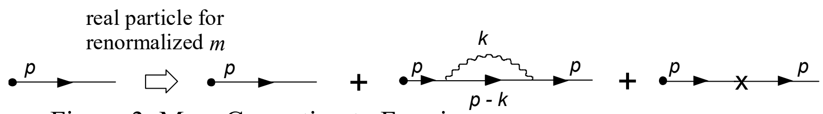
$$\bar{\psi} i\gamma^{\nu}\partial_{\nu}\psi \rightarrow i\psi^{\dagger}e^{-i\alpha\gamma^5}\gamma^0\gamma^{\nu}e^{i\alpha\gamma^5}\partial_{\nu}\psi \xrightarrow[\gamma^5\gamma^{\mu}=-\gamma^{\mu}\gamma^5]{\text{with}} i\psi^{\dagger}\gamma^0\gamma^{\nu}e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\partial_{\nu}\psi = \bar{\psi} i\gamma^{\nu}\partial_{\nu}\psi \quad (1\text{-}5)$$

H.W.: Show that the interaction term in (1-1) is invariant under (1-3).

Bottom line: (1-1) is symmetric under U(1) transformation, but, due to the mass term, is not symmetric under a chiral transformation. However, if the electron were massless, then (1-4) would be zero, and (1-1) would be symmetric under chiral transformation.

1.2.2 Higher Order Corrections

The kinetic (derivative) term inside the second parentheses after the equal sign in (1-1) is expressed graphically in the left side of Fig. 3 for a free electron. When we incorporate higher order terms mathematically, part of the correction comes out as a loop diagram as in the next to last diagram on the right side of Fig. 3, and another part of the correction manifests as a correction to the mass term, as in the last diagram on the right side of Fig. 3.



X represents δm, the mass correction

Figure 3. Mass Correction to Fermion

Detailed calculation shows the mass correction proportional to mass and the log of Λ, which after regularization is taken to infinity.

$$\delta m \propto \alpha m \ln \Lambda \quad (1\text{-}6)$$

So, if mass is initially zero, we get no higher order corrections to the mass. For zero mass term in the Lagrangian, there is zero mass at tree level and at all higher order correction levels. In other words, if the Lagrangian (1-1) is symmetric under a chiral transformation, the electron (or any massless fermion) is massless (at all orders).

Bottom line: Unbroken gauge symmetry keeps gauge vector bosons that are massless at tree level, massless at all higher orders. Unbroken chiral symmetry keeps fermions that are massless at tree level, massless at all levels.

1.3 Impact for SUSY

The symmetries investigated above enforce a "no mass corrections" effect on particles.

We might, therefore, look for some kind of symmetry that groups scalars (like the Higgs) with either massless fermions or massless vector bosons. That could protect the mass of the scalar from ballooning upward due to higher order corrections, which is the gauge hierarchy problem (for the Higgs).

SUSY symmetry is just such a symmetry. It groups the Higgs with fermions (and also the vector bosons with fermions).

2 Gauge Hierarchy and SUSY Cancellations

2.1 Higgs Field in the Standard Model

2.1.1 Review of Higgs Symmetry Breaking

Before symmetry breaking, the free Higgs terms in the Lagrangian[1] are

$$\mathcal{L}_{Higgs} = \partial_\mu \phi \partial^\mu \phi + \left|\mu^2\right| \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 \qquad \text{(high energy)} \qquad (2\text{-}1)$$

For a complex scalar field, the coefficient of $-\phi^\dagger \phi$ represents the mass squared. For the high energy Higgs ϕ, the mass squared at high energy (before breaking) would therefore be

$$m_\phi^2 = -\left|\mu^2\right| \qquad \text{(high energy Higgs "mass" squared; } m_\phi \text{ imaginary).} \qquad (2\text{-}2)$$

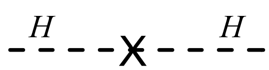
Generally, however, we think of (2-1) as negative of the potential (which shows up in the Lagrangian) for the field ϕ, and all particles as massless. All SM particle masses, after symmetry breaking, depend on μ and λ.

After symmetry breaking, we deal with a low energy Higgs, designated herein by σ, which turns out to be a real (not complex) scalar field. (See Klauber (2021) Chap. 7, summarized in Wholeness Chart 7-8, pg. 240). The mass squared of that field is

$$M_H^2 = 2\left|\mu^2\right| \qquad \text{(low energy Higgs mass squared; } M_H \text{ is real and positive).} \qquad (2\text{-}3)$$

The low energy Higgs mass differs from the high energy "mass" of (2-2) by the factor $i\sqrt{2}$.

In a Feynman diagram, mass is represented by X. For dashed lines representing the Higgs, this looks like Fig. 1, which has external Higgses in and out. The term in the Lagrangian for this (at low energy) is shown in (2-4).



$$\mathcal{L}_{\sigma\, mass} = -\frac{M_H^2}{2}\sigma\sigma = -\left|\mu^2\right|\sigma\sigma \qquad \mathcal{H}_{\sigma\, mass} = \frac{M_H^2}{2}\sigma\sigma = \left|\mu^2\right|\sigma\sigma \qquad (2\text{-}4)$$

Figure 1. Feynman Diagram Representing the Higgs Mass

Note that the mass squared term in the Lagrangian changes sign as energy falls below the symmetry breaking scale. We can determine $|\mu^2|$ (theoretically, or in principle, experimentally) at high energy or at low. In either case, it will be the coefficient of the Higgs field bi-linear term (either $\phi^\dagger\phi$ or $\sigma\sigma$), just with different signs in each.

The expectation value (what we measure, symbolized by an overbar) for $|\mu^2|$ is, where we recognize that σ creates and destroys states with a single low energy Higgs particle,

$$\overline{\left|\mu^2\right|} = \langle\sigma|\left(\left|\mu^2\right|\sigma\sigma\right)|\sigma\rangle = \left|\mu^2\right|\langle\sigma|\sigma|0\rangle = \left|\mu^2\right|\langle\sigma||\sigma\rangle = \left|\mu^2\right|. \qquad (2\text{-}5)$$

2.1.2 Our Goal: Determine Radiative Corrections to the Higgs Mass

We are looking to find modifications to the contemporary Higgs mass M_H, or, due to (2-3), $|\mu|$, caused by radiative corrections. Radiative corrections typically entail integrations over particle loops, where such integrations are carried out in principle from $-\infty$ to $+\infty$ energy levels inside the loop. However, practically, one considers the energy levels to be restricted to values below the Planck scale. In either case, the integration limits are taken as $-\Lambda$ to $+\Lambda$, where Λ can be taken as infinite, or as the Planck energy, or as some other yet to be discovered level at which our physics changes and the $\mathcal{L}$ we are using is no longer valid.

(2-1) plus interaction terms in the high energy (false vacuum) Lagrangian (Klauber (2021) (6-48) pg. 176) result, after symmetry breaking, into terms in the low energy (true vacuum) Lagrangian, (Klauber (2021) pg.251 with Feynman rules on pgs. 290-293). We will be using these Feynman rules in what follows.

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[1] μ^2 is taken as positive (real μ) in Aitshison (2007) with $+\mu^2\phi^\dagger\phi$ appearing in the Lagrangian. In Klauber Vol. 2 μ^2 is taken as negative with $-\mu^2\phi^\dagger\phi$ appearing in $\mathcal{L}$. We hopefully avoid confusion by using the term $+|\mu^2|\phi^\dagger\phi$ in (2-1). Also, Aitchison uses $\lambda/4$, where Klauber uses λ. We opt for the later choice, as we feel it simpler. Appendix A summarizes of the differences in the two approaches.

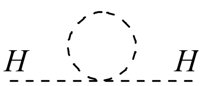
As the reader familiar with QFT should be aware, loop integrations typically result in divergences, to different degrees (quadratic, linear, logarithmic) in the integration limit Λ, as that limit tends to infinity. We will examine corrections to $|\mu|$, and thus to M_H, from each of these, starting in the next section with the quadratic ones, the most severe.

2.2 Higgs Mass Divergences

2.2.1 Quadratic Divergences

4 Higgs Vertex Loop (Quadratic Contribution)

The $-\lambda\left(\phi^{\dagger}\phi\right)^2$ term in (2-1), part of the high energy Lagrangian, manifests in the low energy Lagrangian as $-\frac{\lambda}{4}(\sigma\sigma)^4$, and represents a four-particle vertex *HHHH* as shown in Feynman rule #15-17 of the above cited reference. Two of the σ fields can result in a propagator loop as illustrated in Fig. 2. Note that we have an external Higgs in and an external Higgs out, as we did in Fig. 1 and (2-5). So, we need to add the amplitude associated with Fig. 2 to that of (2-5) , to find the physical value we would measure for $|\mu^2|$. The vertex factor of the afore cited Feynman rule and the scalar propagator give us (2-6), where λ is known from experiment to be approximately ½,



$$-\left(-i6\lambda\int\frac{i}{k^2-m^2+i\varepsilon}d^4k\right)\sigma\sigma\approx-\left(6\lambda\int_0^{\Lambda}\frac{1}{k^2}2\pi^3k^3dk\right)\sigma\sigma=-6\pi^3\lambda\Lambda^2\sigma\sigma \quad (2\text{-}6)$$

Figure 2. SM Higgs λ Loop

(2-6) contributes a quadratic radiative correction to (2-5). Note that although at low energy, $-\left|\mu^2\right|\sigma\sigma$ in (2-5) and $-\frac{\lambda}{4}(\sigma\sigma)^4$ (which gives rise to (2-6)), have the same signs, at high energy (see (2-1)), the terms have opposite signs. We are concerned with the high energy corrections to $|\mu^2|$, so we will consider the difference between (2-5) and (2-6), rather than the sum. Thus, (2-5) is corrected to become

$$\overline{\left|\mu^2\right|}=\left\langle\sigma\right|\left(\left|\mu^2\right|-6\pi^3\lambda\Lambda^2\right)\sigma\sigma\left|\sigma\right\rangle=\left|\mu^2\right|-6\pi^3\lambda\Lambda^2\,. \quad (2\text{-}7)$$

We can consider the RHS of (2-7) as representing a "physical" μ value, which includes the original μ term plus the radiative correction term, and represents what we would measure at high energy from including both in the Lagrangian. That is,

$$\left|\mu_{phys}^2\right|=\left|\mu^2\right|-6\pi^3\lambda\Lambda^2\,. \quad (2\text{-}8)$$

Essentially, (2-8) replaces $|\mu^2|$ in (2-1), and hence, our Higgs mass diverges quadratically with Λ. This effect cascades into all SM particle masses, as they depend on μ (now μ_{phys}).

This implies, assuming Λ is merely large and not infinite, that not only the Higgs mass, but all SM particle masses should be on the order of Λ, i.e., far higher than the SM mass scale. This is true unless the two terms in (2-8) are of the same order. But since μ_{phys} is known to be on the order of the weak scale (10^2 GeV) and Λ is considered to be on the order of the Plank scale (10^{18} GeV), an extraordinary fine tuning would be required. That is, μ would have to comprise 17 or so digits, yet only differ from Λ in the last digit. This is the hierarchy problem, or at least, the first part of the hierarchy problem. (We need to include further radiative corrections to μ to define it completely.)

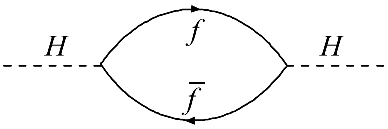
Note two things. First, (2-8) implies, for $\left|\mu^2\right|<6\pi^3\lambda\Lambda^2$, that the actual coefficient of $\phi^{\dagger}\phi$ in (2-1) is negative, turning the Mexican hat diagram for the Higgs potential energy density into a bowl diagram. That means no symmetry breaking, as the Higgs field would then be in a stable configuration. That doesn't work.

Second, if there were new physics at a lower scale than Planck, it would mean our naïve Feynman diagrams and amplitudes should not be integrated to a Plank scale energy, but something lower. If that scale were low enough, then (2-8) would not be so problematic. The $|\mu^2|$ and $\lambda\Lambda^2$ terms could be in the same ballpark, and (2-8) would not give us a hierarchy problem.

Fermion Loops (Quadratic Contribution)

There are other contributions to μ_{phys}, however, that also need to be considered. Any fermion f can form loops with the Higgs like that shown in Fig. 3. See Klauber Vol. 2, pgs 292-293, Feynman rules #15-13 to #15-16, for the particular fermion f. The vertex factor, ignoring some subtleties for neutrinos (whose mass oscillates), is shown there to be, where g_f is the Yukawa coupling and m_f is the fermion mass,

vertex factor for Fig. 3 $\quad \frac{-i}{\sqrt{2}}g_f = \frac{-i}{v}m_f \qquad v = \frac{|\mu_{phys}|}{\sqrt{\lambda}}$ from symmetry breaking theory. (2-9)



$$\left(-\left(\frac{-i}{\sqrt{2}}g_f\right)^2\int Tr\left(\frac{i}{\left(\left(\not{k}_H-\not{p}\right)-m\right)+i\varepsilon}\right)\left(\frac{i}{\left(\not{p}-m\right)+i\varepsilon}\right)d^4p\right)\sigma\sigma$$

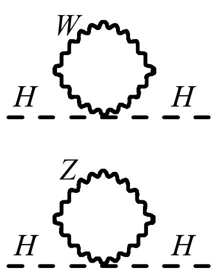
$$\approx\frac{g_f^2}{2}\int_0^{\Lambda}Tr\left(\frac{1}{p^2}\right)2\pi^3p^3dp\sigma\sigma=\pi^3g_f^2\int_0^{\Lambda}Tr\left(I_{4X4}\right)pdp\phi_H\phi_H=2\pi^3g_f^2\Lambda^2\sigma\sigma \qquad (2\text{-}10)$$

Figure 3. SM Fermion Loop

Each possible fermion must be considered, so (2-10) must be summed over f. This result is another quadratic divergence (a series of them, actually) that must be added to (2-7) and (2-8), further modifying our value of μ_{phys}. Before doing so, however, we need to consider yet other quadratic contributions, shown in the following sub-sections.

Gauge Boson 4-Vertex Loops (Quadratic Contributions)

Gauge boson loops also make quadratic contributions. See Klauber Vol. 2, pgs. 292-293, Feynman rules #15-19 and #15-21 for the vertex factors associated with Fig. 4.



For W loop: $$\left(\frac{i}{2}g^2g^{\mu\nu}\int_0^{\Lambda}\frac{-ig_{\mu\nu}+k_\mu k_\nu/m^2}{k^2-m^2+i\varepsilon}d^4k\right)\sigma\sigma$$

$$\xrightarrow[\text{propagator acts like massless}]{\text{high energy range}}\approx\left(\frac{g^2}{2}\int_0^{\Lambda}\frac{4}{k^2}2\pi^3k^3dk\right)\sigma\sigma=2g^2\pi^3\Lambda^2\sigma\sigma \qquad (2\text{-}11)$$

Z loop: $$\left(\frac{i}{2cos^2\theta_W}g^2g^{\mu\nu}\int_0^{\Lambda}\frac{-ig_{\mu\nu}+k_\mu k_\nu/m^2}{k^2-m^2+i\varepsilon}d^4k\right)\sigma\sigma\approx\frac{2g^2\pi^3}{cos^2\theta_W}\Lambda^2\sigma\sigma \qquad (2\text{-}12)$$

Figure 4. SM 4-Boson Vertex Loop

So, we'll need to add contributions from (2-11) and (2-12) to (2-8), along with (2-10), *en route* to determining μ_{phys}, the radiatively corrected μ.

Tadpole Divergences

In a non-renormalized theory, quadratic divergences also arise from amplitudes for what are known as tadpole diagrams. However, in a renormalized theory, which the SM is, tadpole diagram interactions make no contributions and can be ignored. So, we ignore them here.

Nature of Divergences

Note that all of the divergences shown in Figs. 2 to 4 are quadratic, i.e., have energy scale Λ^2. Together these give us a quadratically corrected $|\mu^2|$

from quadratic terms $$\left|\mu_{phys}^2\right|=\left|\mu^2\right|-\overbrace{6\pi^3\lambda\Lambda^2}^{HHHH}+\overbrace{\sum_f 2\pi^3g_f^2\Lambda^2}^{HHf+HH\bar{f}}+\overbrace{2g^2\pi^3\Lambda^2}^{HHWW}+\overbrace{\frac{2g^2\pi^3}{cos^2\theta_W}\Lambda^2}^{HHZZ}$$

$$=\left|\mu^2\right|-\left(6\pi^3\lambda-\sum_f 2\pi^3g_f^2-2g^2\pi^3\left(1+\frac{1}{cos^2\theta_W}\right)\right)\Lambda^2. \qquad (2\text{-}13)$$

We know μ_{phys} from experiment to be about 88 GeV ($\mu_{phys} = M_H / \sqrt{2} \approx 125 / \sqrt{2}\,\text{GeV}$), with order on that of the weak scale ~10^2. And in the SM, we have no intermediate new physics (no symmetry breaking) between the weak and Planck scales, with the latter ~10^{18}. So, the order of magnitudes in (2-13) can be expressed as

$$\underbrace{\left|\mu_{phys}^2\right|}_{\sim 10^4} = \left|\mu^2\right| - \underbrace{\left(\underbrace{6\pi^3\lambda}_{\sim 10^2} - \underbrace{\sum_f 2\pi^3 g_f^2}_{\sim 10^1} - \underbrace{2g^2\pi^3\left(1 + \frac{1}{\cos^2\theta_W}\right)}_{\sim 10^1} \right) \underbrace{\Lambda^2}_{\sim 10^{38}}}_{\text{Call this } X^2 \text{, which has order } \sim 10^{40}} . \tag{2-14}$$

In order to get the μ_{phys} we see in nature, the constant $|\mu^2|$ has to be greater than X^2, but equal it to up the 36th digit. Only then does subtracting the two give a quantity of order 10^4. Put another way, nature had to pick a constant $|\mu|$ that matches the radiative corrections to 18 digits, in order to give a μ_{phys} of order 10^2. This, to say the least, is unnatural.

In other words, we would expect μ_{phys}, and thus the contemporary Higgs mass M_H, to be on the order of the Planck scale, due to the huge radiative corrections. But it is not. This is the biggest part of the gauge hierarchy problem in all its (infamous) glory.

2.2.2 Linear Divergences

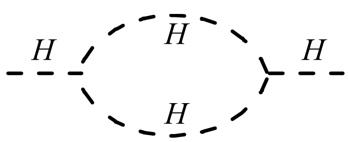
There are no linear divergences for radiative corrections to the Higgs mass – no terms with factors of just Λ.

2.2.3 Logarithmic Divergences

There are logarithmic divergences, as we show in the following.

3 Higgs Vertex Loop (Log Contribution)

Higgs 3 vertex terms lead to additional corrections dependent on the logarithm of energy level. See above cited reference, Feynman rule #15-18, where the vertex factor is $-i6\lambda v$.



$$\left((-i6\lambda v)^2 \int \left(\frac{i}{\left((k_H - k)^2 - m^2\right) + i\varepsilon} \right) \left(\frac{i}{k^2 - m^2 + i\varepsilon} \right) d^4k \right) \sigma\sigma$$

$$\approx \left(36\lambda^2 v^2 \int_0^\Lambda \frac{1}{k^4} 2\pi^3 k^3 dk \right) \sigma\sigma = 72\pi^3 \lambda^2 \frac{\left|\mu_{phys}\right|^2}{\lambda} \ln\Lambda\sigma\sigma \tag{2-15}$$

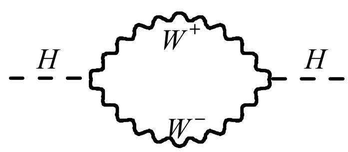
$$= 72\pi^3 \lambda \frac{M_H^2}{4} \ln\Lambda\sigma\sigma = 18\pi^3 \lambda M_H^2 \ln\Lambda\sigma\sigma$$

Figure 5. SM 3 Higgs Vertex Loop

Note this correction varies with the log of Λ, rather than its square, unlike (2-6), (2-10), (2-11) and (2-12).

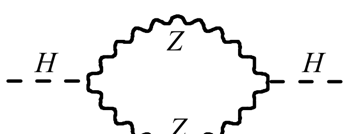
Gauge Bosons with Higgs 3-Vertex Loops (Log Contributions)

The appropriate vertex relations represented in Fig. 6 are shown in the above cited reference as Feynman rules #15-20 and #15-22. The resulting amplitudes are logarithmically divergent.



$$W \text{ loop: } \left(\left(\frac{i}{2} g^2 v \right)^2 \int \left(\frac{-g_{\mu\nu} + k_\mu k_\nu / m^2}{\left((k_H - k)^2 - m^2\right) + i\varepsilon} \right) \left(\frac{-g^{\mu\nu} + k^\mu k^\nu / m^2}{k^2 - m^2 + i\varepsilon} \right) d^4k \right) \sigma\sigma$$

$$\approx -\left(\frac{g^4 v^2}{4} \int_0^\Lambda \frac{4}{k^4} 2\pi^3 k^3 dk \right) \sigma\sigma = -\, 2g^4 \frac{\left|\mu_{phys}^2\right|}{\lambda} \pi^3 \ln\Lambda\sigma\sigma = -\pi^3 g^4 \frac{M_H^2}{\lambda} \ln\Lambda\sigma\sigma \tag{2-16}$$



$$Z\text{: } \left(\left(\frac{i}{2\cos^2\theta_W} g^2 v \right)^2 \int \left(\frac{-g_{\mu\nu} + k_\mu k_\nu / m^2}{\left((k_H - k)^2 - m^2\right) + i\varepsilon} \right) \left(\frac{-g^{\mu\nu} + k^\mu k^\nu / m^2}{k^2 - m^2 + i\varepsilon} \right) d^4k \right) \sigma\sigma$$

$$\approx -\left(\frac{g^4 v^2}{4\cos^4\theta_W} \int_0^\Lambda \frac{4}{k^4} 2\pi^3 k^3 dk \right) \sigma\sigma = -\pi^3 \frac{g^4}{\cos^4\theta_W} \frac{M_H^2}{\lambda} \ln\Lambda\sigma\sigma \tag{2-17}$$

Figure 6. SM 2 Bosons with Higgs Vertex Loop

2.2.4 Impact of Logarithmic Divergences

Log corrections to μ are nowhere near as impactful as quadratic divergences. Indeed, 10^{19} is quite a different animal from $\ln 10^{19} \approx 44$. Nevertheless, the log terms contribute, when one considers the other factors the log factor is multiplied by, on the order of 10^4 to $|\mu_{phys}|$. As we learned in QED, there are ways to tame log corrections (by renormalization in QED, for example). However, there are circumstances, such as GUTs, where they can become problematic.

In general, the primary concern is with quadratic divergences, but we need also need to keep an eye on the logarithmic ones, whose impact is, relatively speaking, far less, but nevertheless can be difficult to reconcile.

2.2.5 Physical μ with Quadratic plus Logarithmic Corrections

For all corrections, quadratic plus logarithmic, up to second order, we need to add (2-15), (2-16), and (2-17) to (2-13)

$$\left|\mu_{phys}^2\right| = \left|\mu^2\right| - \left(6\pi^3\lambda - \sum_f 2\pi^3 g_f^2 - 2g^2\pi^3\left(1+\frac{1}{cos^2\theta_W}\right)\right)\Lambda^2 + M_H^2\pi^3\left(18\lambda - \frac{g^4}{\lambda}\left(1+\frac{1}{cos^4\theta_W}\right)\right) ln\,\Lambda. \quad (2\text{-}18)$$

Note that the first log term is positive, and as one sees after plugging in numbers for the various constants, is an order of magnitude greater than the final two log terms combined. So, the log terms add to $|\mu|^2$, whereas the quadratic ones subtract from it. All in all, a mess. And all together, the gauge hierarchy problem.

2.2.6 Other Considerations

2.2.7 Corrections to Higher Order

We have, of course, only dealt with second order corrections up to here. For a complete analysis, one would have to include higher order corrections, as well. Any solution to the hierarchy problem would need to handle corrections at all orders.

2.2.8 Low vs High Energy Analysis

We have worked with the low energy (after symmetry breaking) Feynman rules, with associated low energy propagators. These propagators have mass terms in them, but above the electroweak symmetry breaking scale, particles have no mass. In the loop integration analyses herein, however, we ignored the mass terms because they became insignificant over higher energies, which comprised, for all practical purposes, virtually the entire integration range. Additionally, contributions from portions of the integral over high energy dwarf, by an enormous amount, the contributions from lower energy.

Further, we used propagator expressions for low energy weak boson gauge fields W and Z, rather than for the high energy $SU(2)$ and $U(1)$ vector gauge fields W^1, W^2, W^3, and B. However, since the former fields are but linear combinations of the latter, the result either way is the same, i.e., (2-18).

2.3 Naïve Power Counting vs Proper Regularization

We have used the cutoff regularization method (see Klauber (2013) Chap. 15, summarized in Wholeness Chart 15-2, pg. 397) to evaluate the divergent integrals. But as noted in the cited reference, the cut-off method doesn't work, primarily because it is not Lorentz nor gauge invariant. (See the cited reference for details.) In fact, for interactions like that of Fig. 3, the true divergence turns out to be logarithmic, rather than quadratic, which the naïve power counting of the cutoff method gives us.

This brings up the non-trivial concern that the quadratic divergences cited above, and the concomitant position that the Higgs mass should be of Planck scale energy, may not, in the final and correct analysis, be valid. This position is to be found throughout the literature, but one familiar with photon self-energy in QED will have serious difficulty buying into it.

Nevertheless, for the time being, we proceed in the following with the traditional rendition of gauge hierarchy.

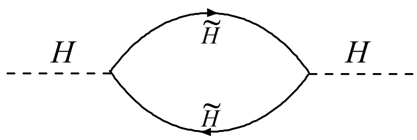
2.4 Supersymmetry and Gauge Hierarchy

2.4.1 Supersymmetry Contributions to Higgs Mass Corrections

As the reader should be aware, supersymmetry (SUSY) adds a SUSY boson (sfermion) for every SM fermion and a fermion (bosino) for every SM boson. Couplings (constants and interactions) for the SUSY spartners mirror those of the SM partners. Masses of spartners equal those of their SM partners above some presumed SUSY breaking scale.

2.4.2 Higgsino Loop

Consider $\tilde{H}$ as a Higgsino (fermionic), where $M_{\tilde{H}} = M_H$ (see appendix for final relation)



$$\left(-\left(\frac{-i}{\sqrt{2}}g_{H\tilde{H}\tilde{H}}\right)^2\int_0^\Lambda Tr\left(\frac{1}{\left(\left(\not k-\not p\right)-m\right)+i\varepsilon}\right)\left(\frac{1}{\left(\not p-m\right)+i\varepsilon}\right)d^4p\right)\phi^\dagger\phi$$
$$\approx\left(\frac{g^2_{H\tilde{H}\tilde{H}}}{2}\int_0^\Lambda Tr\left(\frac{1}{p^2}\right)2\pi^3p^3dp\right)\phi^\dagger\phi = 2\pi^3 g^2_{H\tilde{H}\tilde{H}}\Lambda^2\phi^\dagger\phi \tag{2-19}$$

Figure 7. SUSY Higgsino Loop

2.4.3 Sfermion Loops

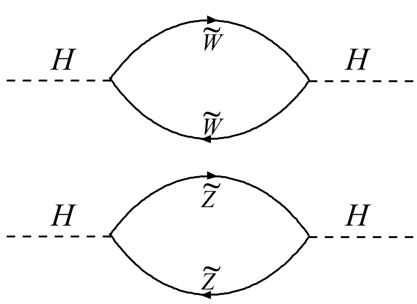
There are other contributions to μ_{phys}, however, such as the fermion loops of Fig. 3, which manifest as the sum over f term in (2-23). These might be cancelled by other SUSY amplitudes/diagrams, such as that shown in Fig. 8, where $\tilde{f}$ is bosonic, a SUSY partner of a standard model fermion. We get a similar result as in (2-6) but with a different constant $\lambda_{HH\tilde{f}\tilde{f}}$ in place of λ.

$$-\left(-i6\lambda_{HH\tilde{f}\tilde{f}}\int\frac{i}{k^2-m^2+i\varepsilon}d^4k\right)\sigma\sigma \approx -\left(6\lambda_{HH\tilde{f}\tilde{f}}\int_0^\Lambda\frac{1}{k^2}2\pi^3k^3dk\right)\sigma\sigma$$
$$= -6\pi^3\lambda_{HH\tilde{f}\tilde{f}}\Lambda^2\sigma\sigma \tag{2-20}$$

Figure 8. SUSY Sfermion Loop

2.4.4 Gaugino Loops

In SUSY one might have loops of winos and zinos, as shown in Fig. 9, with concomitant contributions to the Higgs mass as in (2-21) and (2-22).



$$\left(-\left(\frac{-i}{\sqrt{2}}g_{H\tilde{W}\tilde{W}}\right)^2\int_0^\Lambda Tr\left(\frac{1}{\left(\left(\not k-\not p\right)-m\right)+i\varepsilon}\right)\left(\frac{1}{\left(\not p-m\right)+i\varepsilon}\right)d^4p\right)\phi^\dagger\phi \tag{2-21}$$
$$\approx 2\pi^3 g^2_{H\tilde{W}\tilde{W}}\Lambda^2\phi^\dagger\phi$$

$$\approx 2\pi^3 g^2_{H\tilde{Z}\tilde{Z}}\Lambda^2\phi^\dagger\phi \tag{2-22}$$

Figure 9. Weak Gaugino Loops

2.4.5 Adding SUSY Terms to Mass Correction

The $|\mu^2|$ factor of the $\phi^\dagger\phi$ term in (2-1) is thus, in SUSY, modified by adding (2-19), (2-20), (2-21), and (2-22) to (2-18).

$$\left|\mu_{phys}^2\right| = \left|\mu^2\right| - \left(\overbrace{3\lambda}^{HHHH} \overbrace{-\,g_{H\tilde{H}\tilde{H}}^2}^{H\tilde{H}\tilde{H}+H\tilde{H}\tilde{H}} \overbrace{-\sum_f g_f^2}^{Hff+Hff} + \overbrace{\sum_{\tilde{f}} 3\lambda_{HH\tilde{f}\tilde{f}}}^{HH\tilde{f}\tilde{f}} - g^2 \left(\overbrace{1}^{HHWW} + \overbrace{\frac{1}{cos^2\theta_W}}^{HHZZ} \right) \overbrace{+g_{H\tilde{W}\tilde{W}}^2}^{H\tilde{W}\tilde{W}+H\tilde{W}\tilde{W}} \overbrace{+g_{H\tilde{Z}\tilde{Z}}^2}^{H\tilde{Z}\tilde{Z}+H\tilde{Z}\tilde{Z}} \right) 2\pi^3\Lambda^2 \leftarrow \text{quadratic terms}$$

$$+ \left(\underbrace{18\lambda}_{HHH+HHH} - \frac{g^4}{\lambda} \left(\underbrace{1}_{HWW+HWW} + \underbrace{\frac{1}{cos^4\theta_W}}_{HZZ+HZZ} \right) \right) M_H^2 \pi^3 \ln\Lambda . \quad \leftarrow \text{log terms}$$

. (2-23)

If $3\lambda = g_{H\tilde{H}\tilde{H}}^2$, we would get cancellation of the first two divergent terms in the top row of (2-23). That is, Fig. 7 would cancel Fig. 2. If $3\lambda_{HH\tilde{f}\tilde{f}} = g_f^2$, the next two divergent terms would cancel. That is, Fig. 8 would cancel Fig. 3. If $g = g_{H\tilde{W}\tilde{W}}$ and $g = cos\,\theta_W\, g_{H\tilde{Z}\tilde{Z}}$, then the last terms in (2-23) would also cancel. In SUSY, one finds, after considerable analysis, that this can indeed be true. The signs in (2-19) to (2-22) are critical for this cancellation.

We still have log terms to worry about, but one can see that cancellation of terms can occur in SUSY that could eliminate, or at least ameliorate, the gauge hierarchy problem.

2.5 Problems?

In the standard model, there are no vertices like that of Fig. 8 for SM particles, i.e., no *HHff* vertices. So, one can't expect vertices in SUSY like $HH\tilde{f}\tilde{f}$. See Klauber, Vol. 2, pgs 292-293, 251, and 176, for a summary of Higgs vertices in the SM.

And as mentioned in Sect. 2.3, divergences, such as the fermion loop in boson propagation are naively quadratic, and that is assumed herein, but when fully evaluated, are only logarithmically divergent.

2.6 Appendix A. Comparison of Symmetry Breaking in Aitchison and Klauber

See Wholeness Chart 1 on the next page.

Wholeness Chart 2-1. Symmetry Breaking in Aitchison vs Klauber
ϕ is high energy Higgs doublet (complex). σ is low energy singlet (real)

Entity	Typical Scalar	Higgs Scalar in Klauber	Higgs Scalar in Aitchison	Difference/Comment
$\phi^\dagger\phi$ Part of Potential	$\mathcal{V}_{mass} = m^2\phi^\dagger\phi$	$\mathcal{V}_{mass+\lambda} = \mu^2\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2 = -\lvert\mu^2\rvert\phi^\dagger\phi + \lambda\left(\phi^\dagger\phi\right)^2$	$\mathcal{V}_{mass+\lambda} = -\mu^2\phi^\dagger\phi + \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2$	$\mu_{Kl}^2 < 0 \quad \mu_{Atich}^2 > 0$ $\lambda_{Kl} = \frac{\lambda_{Aitch}}{4}$
$\phi^\dagger\phi$ Part of Lagrangian	$\mathcal{L}_{mass} = -m^2\phi^\dagger\phi$	$\mathcal{L}_{mass+\lambda} = -\mu^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2 = \lvert\mu^2\rvert\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2$	$\mathcal{L}_{mass+\lambda} = \mu^2\phi^\dagger\phi - \frac{\lambda}{4}\left(\phi^\dagger\phi\right)^2$	ϕ is complex $m = \mu$ (nat. units)
Mass, high energy	$m^2 > 0$ m real	μ = high energy mass $\mu^2 < 0 \quad \mu$ imaginary	$i\mu$ = high energy mass $\mu^2 > 0 \quad \mu$ real	
Mass, low energy	$M^2 > 0$ M real > 0	M_H = low energy mass $M_H = \sqrt{-2\mu^2} = \sqrt{2}\lvert\mu\rvert$	M_H = low energy mass $M_H = \sqrt{2\mu^2} = \sqrt{2}\mu = \sqrt{2}\lvert\mu\rvert$	unitary gauge
$\mathcal{J}$ Mass term, low energy	$\mathcal{L}_{mass} = -\frac{M^2}{2}\phi\phi$	$\mathcal{L}_{\sigma\, mass} = -\frac{M_H^2}{2}\sigma\sigma_H = -\lvert\mu^2\rvert\sigma\sigma$	$\mathcal{L}_{\sigma\, mass} = -\frac{M_H^2}{2}\sigma\sigma = -\lvert\mu^2\rvert\sigma\sigma$	σ is real
ϕ_{min}	N/A	$\phi_{min} = v = \sqrt{\frac{-\mu^2}{\lambda_{Kl}}} = \frac{\lvert\mu\rvert}{\sqrt{\lambda_{Kl}}}$	$\phi_{min} = \frac{v}{\sqrt{2}} = \sqrt{\frac{2\mu^2}{\lambda_{Atich}}} = \frac{\sqrt{2}\lvert\mu\rvert}{\sqrt{\lambda_{Atich}}}$	
v	N/A	$v = \frac{\lvert\mu\rvert}{\sqrt{\lambda_{Kl}}}$	$v = \frac{2\lvert\mu\rvert}{\sqrt{\lambda_{Atich}}} = \frac{\lvert\mu\rvert}{\sqrt{\lambda_{Atich}/4}} = \frac{\lvert\mu\rvert}{\sqrt{\lambda_{Kl}}}$	v is same for both; ϕ_{min} defined differently
M_H in v, λ	N/A	$M_H^2 = 2\lvert\mu^2\rvert = 2v^2\lambda_{Kl}$	$M_H^2 = 2\lvert\mu^2\rvert = v^2\frac{\lambda_{Aitch}}{2} = 2v^2\lambda_{Kl}$	See (1.3) [3] Aitchison
Bottom line:	Because Aitchison uses $\lambda/4$ instead of λ, he needs to define ϕ_{min} differently to get the same v value. In summary, the two approaches define the μ^2 term in the Lagrangian with opposite signs and λ different by a factor of 4. The same $\lvert\mu\rvert$ and v are obtained in both, but one needs to exchange Aitchison's $\lambda/4$ for Klauber's λ to get all parameters like particle masses to agree in the two approaches. ϕ_{min} is not used to determine low energy parameters, as they are expressed in terms of μ, λ, and v, so one does not need to work with ϕ_{min}. But, ϕ_{min} differs in the two approaches by a factor of $\sqrt{2}$.			

In QFT, a real scalar mass squared term differs from a complex scalar by a factor of 2.

2.7 Appendix B

For RHS of (2-19),

$$\frac{1}{(\not k - m)^2} = \frac{(\not k + m)^2}{(\not k - m)^2(\not k + m)^2} = \frac{\not k^2 + 2\not k m + m^2}{((\not k - m)(\not k + m))^2} = \frac{\not k^2 + 2\not k m + m^2}{\left(\not k^2 - m^2\right)^2} = \frac{k^2 + 2\not k m + m^2}{\left(k^2 - m^2\right)^2}. \tag{2-24}$$

The denominator is just a number, so only consider trace of numerator. Trace of a single gamma matrix is zero, so middle term in numerator drops out.

$$\int_0^\Lambda Tr\left(\frac{k^2 + 2\not k m + m^2}{\left(k^2 - m^2\right)^2}\right)d^4k \sim \int_0^\Lambda Tr\left(\frac{k^2 + m^2}{\left(k^2 - m^2\right)^2}\right)k^3dk \xrightarrow[\text{energy}]{\text{high}} \sim \int_0^\Lambda Tr\left(\frac{1}{k^2}\right)k^3dk \sim \Lambda^2. \tag{2-25}$$

2.8 References

Aitchison, I. (2007). Supersymmetry in Particle Physics: An Elementary Introduction, Cambridge

Klauber, R. D. (2013). Student Friendly Quantum Field Theory Volume 1: Basic Principles and QED, Sandtrove

Klauber, R. D. (2021). Student Friendly Quantum Field Theory Volume 2: The Standard Model, Sandtrove

3 Infinitesimal Lorentz and Rotation Vector Transformations

3.1 Boost

$$ct' = \frac{1}{\sqrt{1-v^2/c^2}}\left(ct - \frac{v}{c}x\right) \qquad x' = \frac{1}{\sqrt{1-v^2/c^2}}\left(x - \frac{v}{c}ct\right) \qquad y' = y \qquad z' = z \tag{3-1}$$

$$\begin{pmatrix} E'/c \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{-v^1/c}{\sqrt{1-v^2/c^2}} & & \\ \frac{-v^1/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \xrightarrow{v \ll c} \begin{pmatrix} E'/c \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{bmatrix} 1 & -v^1/c & & \\ -v^1/c & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \tag{3-2}$$

For $v/c \ll 1$

$$E' = E - v^1 p^1 \qquad p'^1 = p^1 - \frac{v^1}{c}\frac{E}{c} = p^1 - \frac{v^1}{c}\frac{mc^2}{c} = p^1 - mv^1 \qquad \frac{v^1}{c} \ll 1 \tag{3-3}$$

More generally,

$$E' = E - \mathbf{v}\cdot\mathbf{p} \qquad \mathbf{p}' = \mathbf{p} - m\mathbf{v} \qquad \frac{v}{c} \ll 1 \tag{3-4}$$

For notation, sometimes we find $\boldsymbol{\eta} = (v^1, v^2, v^3)$ used. So, for low velocity,

$$E' = E - \boldsymbol{\eta}\cdot\mathbf{p} \qquad \mathbf{p}' = \mathbf{p} - m\boldsymbol{\eta} \qquad \frac{v}{c} \ll 1 \tag{3-5}$$

For natural units, where $c = 1$, and $E = m$,

$$E' = E - \boldsymbol{\eta}\cdot\mathbf{p} \qquad \mathbf{p}' = \mathbf{p} - \boldsymbol{\eta}E \qquad v \ll 1 \qquad \text{Aitchison (2.16) [20]} \tag{3-6}$$

3.2 Rotation

See Klauber, Vol. 2, (2-13), pg. 13 for the general rotation matrix, where the angles are successive ccw transformations about the axis labeled by the subscript on θ. (To be precise, θ_3 occurs first, θ_2, second, and θ_1 last.)

$$\begin{pmatrix} E'/c \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{bmatrix} 1 & & & \\ & \cos\theta_2\cos\theta_3 & -\cos\theta_2\sin\theta_3 & \sin\theta_2 \\ & \cos\theta_1\sin\theta_3 + \sin\theta_1\sin\theta_2\cos\theta_3 & \cos\theta_1\cos\theta_3 - \sin\theta_1\sin\theta_2\sin\theta_3 & -\sin\theta_1\cos\theta_2 \\ & \sin\theta_1\sin\theta_3 - \cos\theta_1\sin\theta_2\cos\theta_3 & \sin\theta_1\cos\theta_3 + \cos\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_1\cos\theta_2 \end{bmatrix} \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix} \tag{3-7}$$

For infinitesimal rotations, i.e., $\theta_i \to 0$, we have (Klauber, Vol. 2, (2-14), pg. 14)

$$\begin{pmatrix} E'/c \\ p'^1 \\ p'^2 \\ p'^3 \end{pmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & -\theta_3 & \theta_2 \\ & \theta_3 & 1 & -\theta_1 \\ & -\theta_2 & \theta_1 & 1 \end{bmatrix} \begin{pmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{pmatrix}. \tag{3-8}$$

$$E' = E \qquad p'^1 = -\theta_3 p^2 + \theta_2 p^3 \qquad p'^2 = \theta_3 p^1 - \theta_1 p^3 \qquad p'^3 = -\theta_2 p^1 + \theta_1 p^2. \tag{3-9}$$

The θ_i are really contravariant, so we should represent them by θ^i. Doing that we can write (3-9) as

$$p'^i = \varepsilon^{ijk}\theta^j p^k \quad \rightarrow \quad \mathbf{p}' = \boldsymbol{\theta}\times\mathbf{p} \qquad \theta^i \ll 1. \tag{3-10}$$

If we define a vector $\boldsymbol{\varepsilon} = (-\theta_1, -\theta_2, -\theta_3)$, then (3-9) and (3-10) become

$$E' = E \qquad \mathbf{p}' = -\boldsymbol{\varepsilon}\times\mathbf{p} \qquad \varepsilon^i \ll 1 \qquad \text{Aitchison (2.15) [20].} \tag{3-11}$$

4 Spinor Transforms in QFT

Page numbers and equation numbers with dashes in them below reference those in Klauber, *Student Friendly QFT Volume 1, Basic Principles and QED,* and *Volume 2, The Standard Model*. Be aware that notation is different from Aitchison, as explained earlier in this document. At the end of this Sect. 4, we will convert the result to Aitchison notation for use with that text.

4.1 Background

4.1.1 Weyl Representation Relations

In the Weyl rep, from Vol. 2, pg. 138,

$$ {}_W\psi = \begin{bmatrix} {}_W\psi^L \\ {}_W\psi^R \end{bmatrix} = \begin{bmatrix} {}_W\psi_1 \\ {}_W\psi_2 \\ {}_W\psi_3 \\ {}_W\psi_4 \end{bmatrix}, \qquad (4\text{-}1) $$

where the Weyl spinor is composed of a two-component left chiral (L superscript) spinor (also called a *L* Weyl spinor) and a two-component right chiral (R superscript) spinor (also called a *R* Weyl spinor).

Weyl rep gamma matrices, from vol. 2, pgs. 136 and 160, are

$$ {}_W\gamma^0 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} \quad {}_W\gamma^1 = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} \quad {}_W\gamma^2 = \begin{bmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{bmatrix} \quad {}_W\gamma^3 = \begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix} $$

$$ {}_W\gamma^5 = \begin{bmatrix} -I & \\ & I \end{bmatrix} = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \qquad \text{(5-10) and (5-88)} $$

4.1.2 Lorentz Transformation Group for Spinors

A general Lorentz transformation of spinors, including 3D rotations and boosts, is effected by the operator *D,* as shown in Vol. 2, pg. 153, LHS of (5-64) (and Vol. 1, pg. 171, (6-21))

$$ \psi' = D\psi \ , \qquad (4\text{-}2) $$

where, in Vol. 2, pg. 153, RHS of (5-64) (or Vol. 1 pg. 171, eq (6-22)), we stated without proof,

$$ D = e^{-i\left(L^k\Theta^k + M^k Q^k\right)} \qquad \text{Klauber Vol 2 (5-64)} $$

$$ L^k = -\tfrac{i}{2}\varepsilon_{ij}{}^k\gamma^i\gamma^j \, , \quad \Theta^k = \underbrace{\left(\theta^1,\theta^2,\theta^3\right)}_{\substack{\text{rotation}\\ \text{angles}}}, \quad M^k = -\tfrac{i}{2}\gamma^0\gamma^k \, , \quad Q^k = \underbrace{\underbrace{\left(v^1,v^2,v^3\right)}_{\text{boost velocity comps}}}_{\text{for } v \ll 1} \ . \qquad \text{Klauber Vol 2 (5-65)} $$

Note that (5-65) is for low velocities (and this is a correction listed in the corrections lists). For boost velocity a substantial fraction of the speed of light, Q^k takes a more complicated form. See the appendix herein.

4.2 Proofs in References

Relations (5-64) and (5-65) of Vol. 2 above are proven in the references of the footnote on the cited page of Vol. 1 (and also in S. Coleman, Quantum Field Theory, Lectures of Sidney Coleman, World Scientific 2019, pgs. 369-377), but they done first for the LC spinor field in its own SU(2) space, then for the RC spinor fields in its separate SU(2) space. As can be seen by comparing the following with those derivations, (5-64) and (5-65) above express the same results in 4D spinor space instead of two separate 2D spaces.

4.3 3D Rotations

Dropping the *w* subscripts on the gamma matrices, but remembering we are working in the Weyl rep, we have, for a rotation only, without boost, from (5-64).

$$ D = e^{-iL^k\Theta^k} \ , \qquad (4\text{-}3) $$

where, from the LHS of Vol. 2 (5-65),

$$ L^1 = -\tfrac{i}{2}\left(\gamma^2\gamma^3 - \gamma^3\gamma^2\right) \qquad L^2 = -\tfrac{i}{2}\left(\gamma^3\gamma^1 - \gamma^1\gamma^3\right) \qquad L^3 = -\tfrac{i}{2}\left(\gamma^1\gamma^2 - \gamma^1\gamma^2\right) . \qquad (4\text{-}4) $$

From Vol. 1, pg. 414, where the commutation relations for gamma matrices are shown, we have

$$
\begin{gathered}
L^3 = -\tfrac{i}{2}\left(\gamma^1\gamma^2 - \gamma^1\gamma^2\right) = -\begin{bmatrix} \sigma_3 & \\ & \sigma_3 \end{bmatrix} = -\sigma_3 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \\
L^1 = -\begin{bmatrix} \sigma_1 & \\ & \sigma_1 \end{bmatrix} = -\sigma_1 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \qquad L^2 = -\begin{bmatrix} \sigma_2 & \\ & \sigma_2 \end{bmatrix} = -\sigma_2 \begin{bmatrix} 1 & \\ & 1 \end{bmatrix},
\end{gathered} \tag{4-5}
$$

or, generally,

$$
L^k = -\sigma_k \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}. \tag{4-6}
$$

So, (4-3) becomes

$$
D = e^{-iL^k\Theta^k} = e^{i\sigma_k \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\theta^k} \tag{4-7}
$$

For small rotations

$$
D\psi = e^{i\sigma_k \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\theta^k} \begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} \approx \left(I + i\sigma_k \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}\theta^k \right) \begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} \tag{4-8}
$$

and the effect of a rotation on the R Weyl spinor is the same as that on the L Weyl spinor.

As noted, this effect is typically derived in texts separately on each of the two *SU*(2) spinors ψ^L and ψ^R, each in its own *SU*(2) space. Here, we have shown those relations in the full 4D spinor space of QFT.

4.4 Lorentz Boosts

From Vol. 2 (5-64) with boost, but no rotation,

$$
D = e^{-\frac{1}{2}\gamma^0\gamma^k\Omega^k}\,. \tag{4-9}
$$

where

$$
\begin{gathered}
\tfrac{1}{2}\gamma^0\gamma^1 = \tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} & -1 & & \\ -1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} -\sigma_1 & \\ & \sigma_1 \end{bmatrix} \\
\tfrac{1}{2}\gamma^0\gamma^2 = \tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & & -i \\ & & i & \\ & i & & \\ -i & & & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} & i & & \\ -i & & & \\ & & & -i \\ & & i & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} -\sigma_2 & \\ & \sigma_2 \end{bmatrix} \\
\tfrac{1}{2}\gamma^0\gamma^3 = \tfrac{1}{2}\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix}\begin{bmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = \tfrac{1}{2}\begin{bmatrix} -\sigma_3 & \\ & \sigma_3 \end{bmatrix}.
\end{gathered} \tag{4-10}
$$

Or, generally,

$$
\tfrac{1}{2}\gamma^0\gamma^k = \tfrac{1}{2}\begin{bmatrix} -\sigma_k & \\ & \sigma_k \end{bmatrix} = -\tfrac{1}{2}\sigma_k \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}. \tag{4-11}
$$

For a small boost, from Vol. 2 (5-65), (4-9), and (4-11),

$$
D = e^{-\frac{1}{2}\gamma^0\gamma^k v^k} = e^{\frac{1}{2}\sigma_k \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} v^k} \approx I + \tfrac{1}{2}\sigma_k \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} v^k \qquad v^k \ll 1\,. \tag{4-12}
$$

so, from (4-1), (4-2), and (4-12),

$$
D\psi = e^{-\frac{1}{2}\gamma^0\gamma^k v^k} \begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} \approx \left(I + \tfrac{1}{2}\sigma_k \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} v^k \right) \begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} \qquad v^k \ll 1\,, \tag{4-13}
$$

and the effect of a boost on the R Weyl spinor is the opposite of that on the L Weyl spinor.

As is the case for rotation, this effect is typically derived in texts separately on each of the two *SU*(2) spinors ψ^L and ψ^R, each in its own *SU*(2) space. Here, we have shown those relations in the full 4D spinor space of QFT.

4.5 Visualizing the Transformations

In this section, we show a heuristic visualization of a particle undergoing first a rotation and then, a boost. But, we should first note one thing about spins and momentum of RC (right chiral) and LC (left chiral) fermions.

4.5.1 Spin and Momentum of RC and LC Particles

From Vol. 2, pg. 139, (5-20), we know that the spin operator in the Weyl rep has form

$$\Sigma_3 = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}. \tag{4-14}$$

So, the action of (4-14) on (4-1) is the same on the top two components of ψ, as on the bottom two components.

$$\Sigma_3 \psi = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix}. \tag{4-15}$$

At speeds approaching the speed of light, chirality is the same as helicity. Consider, in that case, RC and LC particles to each have spin aligned with momentum **p**. From (4-15), they will have the same spin. But, at the speed of light, RC and LC particles are also RH (right hand helicity) and LH (left hand helicity) particles, so if they have the same spin, they must have opposite direction momentum.

We can generalize to any speed. That is, RC and LC particles with spin in the same direction have momentum in the opposite directions. We show this in Fig. 4-1 of the following section.

4.5.2 3D Rotation Visualization

In the top part of Fig. 4-1, we show RC (right chiral) and LC (left chiral) fermions at almost the speed of light, where 1) the spin virtually aligns with the momentum (see Vol. 1, pgs. 95-96) and 2) helicity and chirality are essentially the same, i.e., the RC particle has RH (right hand helicity) and the LC particle has LH (left hand helicity).

In the lower part, LHS of the figure, we show RC and LC particles at speed much less than light, so spin does not align with momentum. The chirality of the particles remains the same, but the helicity changes (and the particles are no longer in helicity eigenstates).

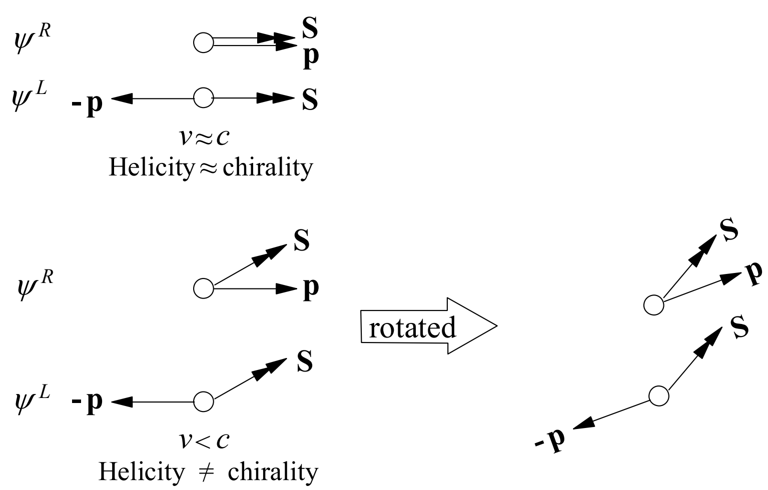


Figure 4-1. Heuristic Visualization of Fermion Rotation Transformation

In the lower part, RHS of the figure, we have rotated our reference frame. Note that both the RC and LC particles are affected in the same way, as we noted at the end of Sect. 4.3. The relationship between the spin and the momentum stays the same for both particles.

4.5.3 Boost Visualization

In Fig. 4-2, we show the same particles on the LHS as in the lower part LHS of Fig. 4-1. But note that when we boost both particles in the same direction, the momenta changes in opposite ways (one gets greater, the other lesser). Further, the spins change in opposite ways, as well. For an increase in momentum, spin gets closer to aligning with the direction of momentum. For a decrease in momentum, spin gets further from such alignment.

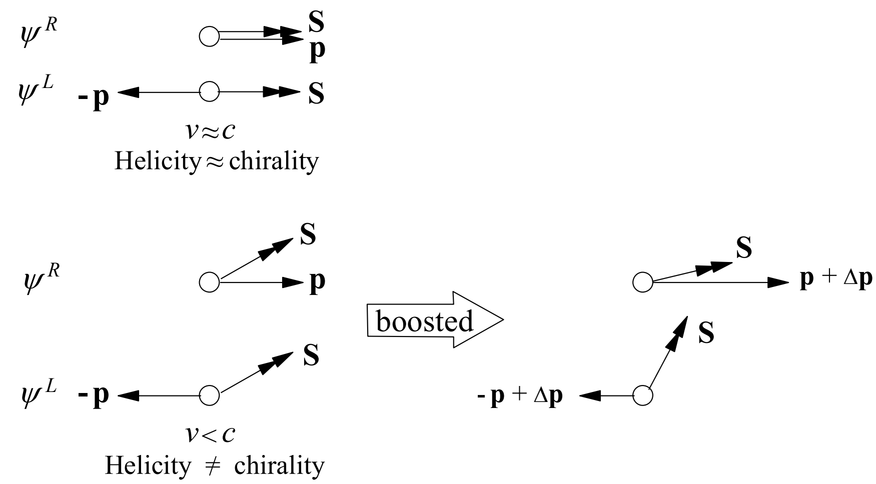


Figure 4-2. Heuristic Visualization of Fermion Boost Transformation

So, a boost transformation of an RC particle has the opposite effect of a boost on an LC particle, as we noted in Sect. 4.4.

4.6 Note on Derivation

Note that we have not derived the transformation of Vol. 2 (5-64). That is done in the references of the footnote of Vol. 1, pg. 171, and the derivations are long and complex. What we have done here is justify (5-64) to some degree in order to gain some intuitive level of comfort with the relation.

4.7 Summary and Notation Comparison

We can combine the small rotation transformation of (4-8) with the small Lorentz transformation of (4-13), and express the result in terms of the LC and RC Weyl fermions separately, as

$$\begin{bmatrix} \psi'^L \\ \psi'^R \end{bmatrix} \approx \begin{bmatrix} \left(I + i\sigma_k\theta^k + \frac{1}{2}\sigma_k v^k\right)\psi^L \\ \left(I + i\sigma_k\theta^k\psi^R - \frac{1}{2}\sigma_k v^k\right)\psi^R \end{bmatrix} \qquad v^k \ll 1 \tag{4-16}$$

For Aitchison's notation (where he takes the LC field in the bottom position of the column vector and the RC field in the top position), we have

$$\psi^L \xrightarrow{\text{Aitchinson}} \chi \qquad \psi^R \xrightarrow{\text{Aitchinson}} \psi$$
$$\theta^k \xrightarrow{\text{Aitchinson}} \frac{\varepsilon}{2} \qquad v^k \xrightarrow{\text{Aitchinson}} \eta \qquad (\text{bold} = \text{3-vector}) \tag{4-17}$$
$$\begin{bmatrix} \psi^L \\ \psi^R \end{bmatrix} \xrightarrow{\text{Aitchinson}} \begin{bmatrix} \psi^R \\ \psi^L \end{bmatrix} = \begin{bmatrix} \psi \\ \chi \end{bmatrix}$$

so, for Aitchison notation,

LC Weyl field $$\chi'_a = \Big(1 + \underbrace{i\boldsymbol{\varepsilon}\cdot\frac{\boldsymbol{\sigma}}{2}}_{\text{rotation}} + \underbrace{\boldsymbol{\eta}\cdot\frac{\boldsymbol{\sigma}}{2}}_{\text{boost}}\Big)_a^{\ b} \chi_b = \left[V^{*-1}\right]_a^{\ b} \chi_b$$ infinitesimal, a,b = 1,2 (4-18)

simpler form $\rightarrow \chi' = V^{\dagger -1}\chi$

RC Weyl field $$\psi'^{\dot a} = \Big(1 + \underbrace{i\boldsymbol{\varepsilon}\cdot\frac{\boldsymbol{\sigma}}{2}}_{\text{rotation}} - \underbrace{\boldsymbol{\eta}\cdot\frac{\boldsymbol{\sigma}}{2}}_{\text{boost}}\Big)^{\dot a}_{\ \dot b} \psi^{\dot b} = V^{\dot a}_{\ \dot b}\psi^{\dot b}$$ infinitesimal, a,b = 1,2 (4-19)

simpler form $\rightarrow \psi' = V\psi$

4.8 Appendix

The equivalent of (5-65) for v^k a substantial fraction of the speed of light is more complicated. We start by assuming our coordinate system x^3 is aligned with the direction of the boost **v**, and defining a parameter ϕ via

$$cosh\,\phi = \frac{1}{\sqrt{1-v^2}} \qquad sinh\,\phi = \frac{v}{\sqrt{1-v^2}} \qquad tanh\,\phi = v\,. \tag{4-20}$$

Then, Q^k, which we state without proof (see earlier cited references), is

$$Q^k = \left(0,0,\phi\right), \tag{4-21}$$

and (5-64) is

$$D = e^{-i\left(L^k\Theta^k + M^kQ^k\right)} = e^{-i\left(L^k\Theta^k + M^3Q^3\right)} = e^{-i\left(L^k\Theta^k + M^3\phi\right)}. \tag{4-22}$$

This can be generalized to

$$D = e^{-i\left(L^k\Theta^k + M^k\phi^k\right)}, \tag{4-23}$$

where there are three ϕ^k for the general case of three components v^k.

In the limit of small v, from (4-20),

$$sinh\,\phi = \frac{v}{\sqrt{1-v^2}} \quad \xrightarrow{v \ll 1} \quad sinh\,\phi \approx \phi \approx v\,, \tag{4-24}$$

And (4-22) becomes

$$D = e^{-i\left(L^k\Theta^k + M^3 v\right)} = e^{-i\left(L^k\Theta^k\right)} e^{-\frac{1}{2}\gamma^0\gamma^3 v^3}\,, \tag{4-25}$$

which we can generalize to (5-64) with (5-65) and (4-12) as

$$D = e^{-i\left(L^k\Theta^k\right)} e^{-\frac{1}{2}\gamma^0\gamma^k v^k} \qquad v^k \ll 1\,. \tag{4-26}$$

5 Spinor Space Notation, Metrics, and Transformations

5.1 **Wholeness Chart 5-1. Metrics and Invariants in Different Spaces**

Note: $\sigma_2 = \begin{bmatrix} & -i \\ i & \end{bmatrix}$ Also: using QFT spacetime metric form, not usual special relativity theory (SRT) form.

Math Entity	**3D Space, Cartesian**	**4D Spacetime, Minkowski, QFT**	**2D Spinor Space, Weyl Rep, LC**	**Comment**
Typical vector notation	x^i and x_i $\quad i,j = 1,2,3$	x^μ and x_μ $\quad \mu,\nu = 0,1,2,3$	χ_a and χ^a $\quad a,b = 1,2$	Not doing RC spinors yet.
Which is true vector?	x^i	x^μ	χ_a	LC covariant by convention
Which is calculation aid?	x_i	x_μ	χ^a	
General metric symbol	$g_{\mu\nu}$	as at left	as at left	
Specific metric symbol	$\delta_{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$ $\delta^{ij} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$	$\eta_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$ $\eta^{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}$	$\varepsilon^{ab} = i\sigma_2 = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}$ $\varepsilon_{ab} = -i\sigma_2 = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}$	Each metric has form it does because it leads to invariant inner products. (See below.) $g_{\mu\beta}\, g^{\beta\nu} = \delta_\mu{}^\nu$
Raising and lowering	$x_i = \delta_{ij} x^j \quad x^i = \delta^{ij} x_j$ $x^i = x_i$	$x_\mu = \eta_{\mu\nu} x^\nu \quad x^\mu = \eta^{\mu\nu} x_\nu$ $x_0 = x^0 \quad x_i = -x^i$	$\chi^a = \varepsilon^{ab}\chi_b \quad \chi_a = \varepsilon_{ab}\chi^b$ $\chi^1 = \chi_2 \quad \chi^2 = -\chi_1$	
Vector length squared	$x^i x_i = \delta_{ij} x^i x^j$	$x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu$	$\chi_a \chi^a = \varepsilon^{ab}\chi_a \chi_b$	
Inner product, 2 vectors	$\underset{\sim}{x}^i x_i = \delta_{ij} \underset{\sim}{x}^i x^j$ $= \underset{\sim}{x}^1 x^1 + \underset{\sim}{x}^2 x^2 + \underset{\sim}{x}^3 x^3$ $= \underset{\sim}{x}_1 x_1 + \underset{\sim}{x}_2 x_2 + \underset{\sim}{x}_3 x_3$	$\underset{\sim}{x}^\mu x_\mu = \eta_{\mu\nu} \underset{\sim}{x}^\mu x^\nu$ $= \underset{\sim}{x}^0 x_0 + \underset{\sim}{x}^1 x_1 + \underset{\sim}{x}^2 x_2 + \underset{\sim}{x}^3 x_3$ $= \underset{\sim}{x}^0 x^0 - \underset{\sim}{x}^1 x^1 - \underset{\sim}{x}^2 x^2 - \underset{\sim}{x}^3 x^3$	$\underset{\sim}{\chi}^a \chi_a = \varepsilon_{ab} \underset{\sim}{\chi}^a \chi^b$ $= \underset{\sim}{\chi}^1 \chi_1 + \underset{\sim}{\chi}^2 \chi_2$ $= \underset{\sim}{\chi}_2 \chi_1 - \underset{\sim}{\chi}_1 \chi_2$	One can consider the negative of any inner product as invariant, as well.
Invariance	Above inner product invariant	Above inner product invariant	Above inner product invariant	Lorentz, rotation, and translation invariance, here
Where did above invariance come from?	Above is obvious.	We proved above in SRT course.	We haven't proven above yet.	

5.2 Two Component Spinor Notation: Weyl Rep

$$\sigma^1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} & -i \\ i & \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \quad \sigma^0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \qquad \sigma^{\mu\dagger} = \sigma^{\mu} \tag{5-1}$$

$$\gamma^i = \begin{bmatrix} & -\sigma^i \\ \sigma^i & \end{bmatrix} \quad \gamma^0 = \begin{bmatrix} & I \\ I & \end{bmatrix} \quad \left(\gamma^{\mu} = \begin{bmatrix} & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & \end{bmatrix}\right) \qquad \gamma_5 = \begin{bmatrix} I & \\ & -I \end{bmatrix} \tag{5-2}$$

Weyl rep form is Aitchison's, not Klauber (or Swartz or Peskin & Schroeder)

$$\text{Aitchison} \quad \rightarrow \Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix} = \begin{pmatrix} \Psi^R \\ \Psi^L \end{pmatrix} = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \quad \psi = \text{RC} \quad \chi = \text{LC} \qquad \text{Klauber, Schwartz, P\&S} \rightarrow \Psi = \begin{pmatrix} \Psi^L \\ \Psi^R \end{pmatrix} \tag{5-3}$$

Wholeness Chart 5-2. Comparing LC and RC Fields

Entity	**LC χ**	**RC Ψ**	**Aitchison**
Default indices	lower $\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$	upper $\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$	convention
Index notation	not dotted, $\xi_a = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is LC	dotted, $\xi^{\dot a} = \begin{pmatrix} \xi^{\dot 1} \\ \xi^{\dot 2} \end{pmatrix}$ is RC	convention
Raised and lowered	$\chi^a = \varepsilon^{ab}\chi_b \quad \chi_a = \varepsilon_{ab}\chi^b$ Both LC, not dotted	$\psi_{\dot a} = \varepsilon_{\dot a \dot b}\psi^{\dot b} \quad \psi^{\dot a} = \varepsilon^{\dot a \dot b}\psi_{\dot b}$ Both RC, dotted	(2.61) [26] (2.71) [28]
Metric	$\varepsilon^{ab} = i\sigma_2^{ab} = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} = \varepsilon^{\dot a \dot b}$	$\varepsilon_{\dot a \dot b} = \left(-i\sigma_2\right)_{ab} = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} = \varepsilon_{ab}$	(2.63) [26] (2.72) [28]
χ, ψ notation	$\chi^a = \begin{bmatrix} \chi^1 \\ \chi^2 \end{bmatrix} = \varepsilon^{ab}\chi_b = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \end{bmatrix} = \begin{bmatrix} \chi_2 \\ -\chi_1 \end{bmatrix}$	$\psi_{\dot a} = \begin{bmatrix} \psi_{\dot 1} \\ \psi_{\dot 2} \end{bmatrix} = \varepsilon_{\dot a \dot b}\psi^{\dot b} = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} \psi^{\dot 1} \\ \psi^{\dot 2} \end{bmatrix} = \begin{bmatrix} -\psi^{\dot 2} \\ \psi^{\dot 1} \end{bmatrix}$	(2.57) [25] (2.67) [27]
General notation	$\xi^a = \begin{bmatrix} \xi^1 \\ \xi^2 \end{bmatrix} = \varepsilon^{ab}\xi_b = \begin{bmatrix} \xi_2 \\ -\xi_1 \end{bmatrix}$	$\xi_{\dot a} = \begin{bmatrix} \xi_{\dot 1} \\ \xi_{\dot 2} \end{bmatrix} = \varepsilon_{\dot a \dot b}\xi^{\dot b} = \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}\begin{bmatrix} \xi^{\dot 1} \\ \xi^{\dot 2} \end{bmatrix} = \begin{bmatrix} -\xi^{\dot 2} \\ \xi^{\dot 1} \end{bmatrix}$	
Infinitesimal transformations	$\chi'_a = \Big(1 + \underbrace{i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2}}_{\text{rotation}} + \underbrace{\boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}}_{\text{boost}}\Big)_a^{\ b}\chi_b = \left[V^{*-1}\right]_a^{\ b}\chi_b$ simpler form $\rightarrow \chi' = V^{\dagger -1}\chi$	$\psi'^{\dot a} = \Big(1 + \underbrace{i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2}}_{\text{rotation}} - \underbrace{\boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}}_{\text{boost}}\Big)^{\dot a}_{\ \dot b}\psi^{\dot b} = V^{\dot a}_{\ \dot b}\psi^{\dot b}$ simpler form $\rightarrow \psi' = V\psi$	(2.24), (2.27), & (2.28) [21]
for raised/lowered	$\chi'^a = \left(1 - i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}\right)^a_{\ b}\chi^b = \left[V^*\right]^a_{\ b}\chi^b$	$\psi'_{\dot a} = \left(1 - i\boldsymbol{\varepsilon}\bullet\frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\eta}\bullet\frac{\boldsymbol{\sigma}}{2}\right)_{\dot a}^{\ \dot b}\psi_{\dot b} = \left[V^{-1}\right]_{\dot a}^{\ \dot b}\psi_{\dot b}$	Transform= above inverse transpose. Proof later.
Invariants	$\bar\Psi\Psi = \Psi^\dagger\gamma^0\Psi = \psi^\dagger\chi + \chi^\dagger\psi$	each of $\psi^\dagger\chi$ and $\chi^\dagger\psi$ independently invariant	(2.31) [22] (2.32) [22]
Inner products of spinor fields	$\underset{\sim}{\chi}^a\chi_a = \left(\underset{\sim}{\chi}^1 \ \ \underset{\sim}{\chi}^2\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \underset{\sim}{\chi}^1\chi_1 + \underset{\sim}{\chi}^2\chi_2$ $= \left(\underset{\sim}{\chi}_2 \ \ -\underset{\sim}{\chi}_1\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \underset{\sim}{\chi}_2\chi_1 - \underset{\sim}{\chi}_1\chi_2$ positive is top left to bottom right indices	$\underset{\sim}{\psi}_{\dot a}\psi^{\dot a} = \left(\underset{\sim}{\psi}_{\dot 1} \ \ \underset{\sim}{\psi}_{\dot 2}\right)\begin{pmatrix} \psi^{\dot 1} \\ \psi^{\dot 2} \end{pmatrix} = \underset{\sim}{\psi}_{\dot 1}\psi^{\dot 1} + \underset{\sim}{\psi}_{\dot 2}\psi^{\dot 2}$ $= \left(-\underset{\sim}{\psi}^{\dot 2} \ \ \underset{\sim}{\psi}^{\dot 1}\right)\begin{pmatrix} \psi^{\dot 1} \\ \psi^{\dot 2} \end{pmatrix} = -\underset{\sim}{\psi}^{\dot 2}\psi^{\dot 1} + \underset{\sim}{\psi}^{\dot 1}\psi^{\dot 2}$ positive is bottom left to top right indices	Proof of invariance on next pages. (2.60) [26] (2.70) [28]
Covariants	$\bar\Psi\gamma^\mu\Psi = \psi^\dagger\sigma^\mu\chi + \chi^\dagger\bar\sigma^\mu\psi$ $\sigma^\mu = \left(1, \sigma_i\right) \quad \bar\sigma^\mu = \left(1, -\sigma_i\right)$	each of $\psi^\dagger\sigma^\mu\chi$ and $\chi^\dagger\bar\sigma^\mu\psi$ independently covariant $\gamma^\mu = \begin{bmatrix} & \bar\sigma^\mu \\ \sigma^\mu & \end{bmatrix}$	(2.33) [22] (2.34) & (2.35) [22]
New notation		$\psi^{\dot 1} = \bar\psi^{\dot 1} \quad \psi^{\dot 2} = \bar\psi^{\dot 2} \rightarrow \underset{\sim}{\bar\psi}\bullet\bar\psi = \underset{\sim}{\psi}_{\dot a}\bar\psi^{\dot a}$ Bar $\neq$ bar of QFT Dirac $\bar\Psi$	(2.76) [28]

5.3 Invariance of Weyl Spinor Inner Product Proof

5.3.1 Background

Important and Useful Relation

A very key relation, which is proven below is this.

$$\boxed{\sigma_2\chi^* \text{ transforms like } \psi \quad \rightarrow \quad [\sigma_2]^{ab}\,\chi_b^* \text{ transforms like } \psi^a.} \tag{5-4}$$

Proof of Above Relation (5-4)

The following paragraphs are copied from Aitchison, pg. 23. For the (infinitesimal) Lorentz and rotation transformations for χ and ψ, referenced as (2.39) and (2.24) below, see Wholeness Chart 5-2 herein. Those transformation relations were derived in Sect. 4 herein, titled Spinor Transforms in QFT.

Aitchison, pg. 23 copy:

complex conjugate of χ, denoted by χ^*, transforms under Lorentz transformations. We have

$$\chi' = (1 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}/2 + \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2)\chi. \tag{2.39}$$

Taking the complex conjugate gives

$$\chi^{*\prime} = (1 - \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}^*/2 + \boldsymbol{\eta}\cdot\boldsymbol{\sigma}^*/2)\chi^*. \tag{2.40}$$

Now observe that $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$, $\sigma_3^* = \sigma_3$, and that $\sigma_2\sigma_3 = -\sigma_3\sigma_2$ and $\sigma_1\sigma_2 = -\sigma_2\sigma_1$. It follows that

$$\begin{aligned}
\sigma_2\chi^{*\prime} &= \sigma_2(1 - \mathrm{i}\boldsymbol{\epsilon}\cdot(\sigma_1, -\sigma_2, \sigma_3)/2 + \boldsymbol{\eta}\cdot(\sigma_1, -\sigma_2, \sigma_3)/2)\chi^* && (2.41)\\
&= (1 + \mathrm{i}\boldsymbol{\epsilon}\cdot\boldsymbol{\sigma}/2 - \boldsymbol{\eta}\cdot\boldsymbol{\sigma}/2)\sigma_2\chi^* && (2.42)\\
&= V\sigma_2\chi^*, && (2.43)
\end{aligned}$$

referring to (2.24) for the definition of V, which is precisely the matrix by which ψ transforms.

We have therefore established the important result that

$$\sigma_2\chi^* \text{ transforms like a } \psi. \tag{2.44}$$

We will use this result for one step of the proof carried out in the next section.

5.3.2 Proof of Invariance of Weyl Spinor Inner Product

We proceed in numbered steps.

Step 1

We know from QFT that $\overline{\Psi}\Psi$ is a Lorentz, and rotation, invariant scalar.

$$\begin{aligned}
\overline{\Psi}\Psi &= \begin{pmatrix}\psi\\ \chi\end{pmatrix}^{\dagger}\gamma^0\begin{pmatrix}\psi\\ \chi\end{pmatrix} = \begin{pmatrix}\psi^\dagger & \chi^\dagger\end{pmatrix}\begin{bmatrix} & I\\ I & \end{bmatrix}\begin{pmatrix}\psi\\ \chi\end{pmatrix} = \psi^\dagger\chi + \chi^\dagger\psi\\
&= \begin{pmatrix}\psi^{1*} & \psi^{2*}\end{pmatrix}\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}\chi_1^* & \chi_2^*\end{pmatrix}\begin{pmatrix}\psi^1\\ \psi^2\end{pmatrix} = \psi^{a\dagger}\chi_a + \chi_a^\dagger\psi^a = \text{invariant}
\end{aligned} \tag{5-5}$$

Step 2

It turns out that each of the terms in (5-5), $\psi^\dagger\chi$ and $\chi^\dagger\psi$ are invariant on their own, as we prove below.

For the first term, the factors transform (see Wholeness Chart 5-2) as

$$\chi' = V^{\dagger -1}\chi \quad \psi' = V\psi \;\rightarrow\; \psi'^{\dagger} = \psi^{\dagger}V^{\dagger}. \tag{5-6}$$

So, $\psi^{\dagger}\chi$ is invariant via

$$\psi'^{\dagger}\chi' = \left(\psi^{\dagger}V^{\dagger}\right)\left(V^{\dagger -1}\chi\right) = \psi^{\dagger}\left(V^{\dagger}V^{\dagger -1}\right)\chi = \psi^{\dagger}\chi \;. \tag{5-7}$$

The proof of invariance of the $\chi^{\dagger}\psi$ term is straightforward and left as an exercise for the reader.

Step 3

We now combine the invariance result (5-7) with (5-4), but first note that if (5-4) is true, then, also,

$$i\sigma_2\chi^* = \varepsilon^{ab}\chi_b^* = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix}\chi_1^* \\ \chi_2^*\end{bmatrix} \text{ transforms like } \psi \; (= \psi^a). \tag{5-8}$$

And then,

$$-i\sigma_2^*\chi \text{ transforms like } \psi,^* \tag{5-9}$$

and, taking the transpose of (5-9),

$$\chi^T\left(-i\sigma_2^{\dagger}\right) = \chi^T\left(-i\sigma_2\right) \text{ transforms like } \psi^{\dagger}. \tag{5-10}$$

Step 4

So, for any $\psi^{\dagger}$ we might have, there exists a $\underset{\sim}{\chi}$ for which $\underset{\sim}{\chi}^T\left(-i\sigma_2\right)$ is equal to that $\psi^{\dagger}$. So, we can substitute (5-10) for $\psi^{\dagger}$ in (5-4). We'd actually like to use the symbol $\underset{\sim}{\chi}$ instead of χ in (5-10), because $\psi^{\dagger}$ in (5-4) could be unrelated to χ in (5-4). $\underset{\sim}{\chi}$ represents any LC 2-spinor. This gives us

$$\psi^{\dagger}\chi = \underset{\sim}{\chi}^T\left(-i\sigma_2\right)\chi = -\underset{\sim}{\chi}_a\varepsilon^{ab}\chi_b = -\begin{pmatrix}\underset{\sim}{\chi}_1 & \underset{\sim}{\chi}_2\end{pmatrix}\begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = -\underset{\sim}{\chi}_1\chi_2 + \underset{\sim}{\chi}_2\chi_1 = \underset{\sim}{\chi}_2\chi_1 - \underset{\sim}{\chi}_1\chi_2 \text{ is invariant.} \tag{5-11}$$

Step 5

From Wholeness Chart X-X

$$\chi^a = \begin{bmatrix}\chi^1 \\ \chi^2\end{bmatrix} = \varepsilon^{ab}\chi_b = \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}\begin{bmatrix}\chi_1 \\ \chi_2\end{bmatrix} = \begin{bmatrix}\chi_2 \\ -\chi_1\end{bmatrix} \;\rightarrow\; \chi^1 = \chi_2 \quad \chi^2 = -\chi_1 \tag{5-12}$$

So,

$$\underset{\sim}{\chi}^a\chi_a = \begin{pmatrix}\underset{\sim}{\chi}^1 & \underset{\sim}{\chi}^2\end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = \begin{pmatrix}\underset{\sim}{\chi}_2 & -\underset{\sim}{\chi}_1\end{pmatrix}\begin{pmatrix}\chi_1 \\ \chi_2\end{pmatrix} = \underset{\sim}{\chi}_2\chi_1 - \underset{\sim}{\chi}_1\chi_2. \tag{5-13}$$

Comparing (5-13) to (5-11), we see that the inner product of two LC spinors is invariant and equals the last part of (5-13), as shown in Wholeness Chart 5-2.

For the inner product of a spinor with itself, the length squared in spinor space, (5-13) becomes

$$\chi^a\chi_a = \chi_2\chi_1 - \chi_1\chi_2. \tag{5-14}$$

If χ were a number, (5-14) would be zero, but in QFT, χ is a field which anti-commutes with itself, so (5-14) is non-zero.

First Note

Consider

$$\underset{\sim}{\chi}_a\chi^a = \begin{pmatrix}\underset{\sim}{\chi}_1 & \underset{\sim}{\chi}_2\end{pmatrix}\begin{pmatrix}\chi^1 \\ \chi^2\end{pmatrix} = \begin{pmatrix}\underset{\sim}{\chi}_1 & \underset{\sim}{\chi}_2\end{pmatrix}\begin{pmatrix}\chi_2 \\ -\chi_1\end{pmatrix} = \underset{\sim}{\chi}_1\chi_2 - \underset{\sim}{\chi}_2\chi_1, \tag{5-15}$$

which is the negative of (5-13), though we would naively expect it to be equal to (5-13). This is an idiosyncrasy of the spinor space metric. Conventionally, (5-13) is considered the positive inner product of two LC spinors – index order from top right to bottom left.

<u>Second Note</u>

We know from (5-13) that $\underset{\sim}{\chi}^a \chi_a$ is invariant, and we know from Aitchison's (2.39) above, as well as Wholeness Chart 5-2, what the infinitesimal transformation is for an LC spinor like χ_a. So, we can deduce the transformation for the contravariant form of a LC spinor $\underset{\sim}{\chi}^a$.

$$\underset{\sim}{\chi}'^a \chi'_a = \left[S^a{}_b \underset{\sim}{\chi}^b \right] \left[V^{\dagger -1} \right]_a^{\ c} \chi_c = S^a{}_b \left[V^{\dagger -1} \right]_a^{\ c} \underset{\sim}{\chi}^b \chi_c = S^a{}_b \left(1 + i\boldsymbol{\varepsilon} \bullet \frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\eta} \bullet \frac{\boldsymbol{\sigma}}{2} \right)_a^{\ c} \underset{\sim}{\chi}^b \chi_c \tag{5-16}$$

For

$$S^a{}_b = \left(1 - i\boldsymbol{\varepsilon} \bullet \frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta} \bullet \frac{\boldsymbol{\sigma}}{2} \right)^a{}_b , \tag{5-17}$$

(5-16) becomes

$$\begin{aligned} \underset{\sim}{\chi}'^a \chi'_a &= \left(1 - i\boldsymbol{\varepsilon} \bullet \frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta} \bullet \frac{\boldsymbol{\sigma}}{2} \right)^a{}_b \left(1 + i\boldsymbol{\varepsilon} \bullet \frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\eta} \bullet \frac{\boldsymbol{\sigma}}{2} \right)_a^{\ c} \underset{\sim}{\chi}^b \chi_c \\ &= \left(1 - i\boldsymbol{\varepsilon} \bullet \frac{\boldsymbol{\sigma}}{2} + i\boldsymbol{\varepsilon} \bullet \frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta} \bullet \frac{\boldsymbol{\sigma}}{2} + \boldsymbol{\eta} \bullet \frac{\boldsymbol{\sigma}}{2} + \textit{higher order in } \boldsymbol{\varepsilon} \textit{ and } \boldsymbol{\sigma} \right)_b^{\ c} \underset{\sim}{\chi}^b \chi_c . \end{aligned} \tag{5-18}$$

Since $\boldsymbol{\varepsilon}$ and $\boldsymbol{\eta}$ are infinitesimal, the higher order terms are meaningless, so given (5-17),

$$\underset{\sim}{\chi}'^a \chi'_a = \delta^c_b \underset{\sim}{\chi}^b \chi_c = \underset{\sim}{\chi}^b \chi_b = \underset{\sim}{\chi}^a \chi_a , \tag{5-19}$$

i.e., the inner product is invariant, as we know it must be. Therefore, (5-17) is the correct transformation for the contravariant form of the LC spinor χ.

$$\underset{\sim}{\chi}'^a = \left(1 - i\boldsymbol{\varepsilon} \bullet \frac{\boldsymbol{\sigma}}{2} - \boldsymbol{\eta} \bullet \frac{\boldsymbol{\sigma}}{2} \right)^a{}_b \underset{\sim}{\chi}^b = \left[V^* \right]^a_b \underset{\sim}{\chi}^b , \tag{5-20}$$

as we show in Wholeness Chart 5-2.

6 Overview of How SUSY is Deduced

6.1 Steps to Deduce SUSY

The steps one uses to deduce supersymmetry parallel those used for the standard model U(1), SU(2), and SU(3). The following discussion of the steps involved in all these cases can be followed more easily by tracking them in Wholeness Chart 6-2 herein. The steps are

1. Propose a suitable Lagrangian density $\mathcal{L}$.
2. Find an internal symmetry of $\mathcal{L}$, typically represented by matrices operating on multiplets (column vectors of fields). Usually easiest to handle if the symmetry transformations are infinitesimal, i.e., each matrix operating on a column vector is multiplied by an arbitrary real parameter, symbolized here by ε_i .
3. Note the commutation relations for the matrices that operate on the field multiplets. The matrices are generators of the Lie Algebra associated with the group transformation acting on the fields.
4. Use Noether's theorem to find the conserved 4-currents $j_i{}^{\mu}$.
5. Find the conserved charges Q_i by integrating $j_i{}^0$ over all space.
6. Determine the commutation relations for the Q_i, which are the generators of the algebra for the group transformations acting on the states.
7. Examine what effects each Q_i has when operating on particular states. (Sometimes they result in an eigenvalue [charge = quantum number] for a state. Sometimes they raise or lower a state, i.e., change it from a state with particular charge quantum number(s) to a state with different charge quantum number (s).)

6.2 Simple SUSY Summary

When the early researchers did all of the above, they found, for the simplest form of SUSY, the following.

There are two charges, labeled Q_1 and Q_2. Q_1 changes an RC fermion (ψ herein in the Weyl rep) into a scalar (ϕ herein). $Q_1{}^\dagger$ turns that scalar back into the original RC fermion. Q_2 changes an LC fermion (χ herein in the Wey rep) into a scalar. $Q_2{}^\dagger$ turns that scalar back into the original LC fermion.

Since the chirality of a LC fermion is –½, we say Q_2 "raises" that fermion to a chirality zero scalar. Since the chirality of a RC fermion is +½, Q_1 is said to lower that fermion to a chirality zero scalar. This is summarized in Wholeness Chart 6–1.

Wholeness Chart 6–1. **The Effects of the Two SUSY Charges on States**

Chirality	Particle	Q_1	$Q_1{}^\dagger$	Q_2	$Q_1{}^\dagger$
+ ½	RC fermion ψ				
		↓	↑		
0	scalar ϕ				
				↑	↓
– ½	LC fermion χ				

Note the chiral multiplets (or super multiplets) for the fields.

$$\text{LC chiral (or super) multiplet} = \begin{bmatrix}\phi \\ \chi\end{bmatrix} \qquad \text{RC chiral (or super) multiplet} = \begin{bmatrix}\psi \\ \phi\end{bmatrix} \tag{5-21}$$

Similar effects, though a bit more complicated, arise for spin 1 (gauge) fields/states and fermion fields/states.

Wholeness Chart 6-2. Operators on Fields vs States: Standard Model and SUSY

	QFT 4D Spin	**SU(2) Isospin**	**SUSY**	**Comment**
Field	$\psi = \sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_\mathbf{p}}} \left(c_r(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + d_r^\dagger(\mathbf{p}) v_r(\mathbf{p}) e^{ipx} \right)$ $u_1 = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 \\ 0 \\ \frac{p^3}{E+m} \\ \frac{p^1+ip^2}{E+m}\end{pmatrix}$ etc. u_2, v_1, v_2	Isospin doublet $\Psi = \begin{pmatrix}\psi_u\\ \psi_d\end{pmatrix}$ or $\begin{pmatrix}\psi_{\nu_e}\\ \psi_e\end{pmatrix}$ generally $\begin{pmatrix}u\\ d\end{pmatrix}$ or $\Psi_a = \begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}$ u and d each like ψ at left and solves Dirac equation on its own; each is a four component (LC) spinor (indices not shown)	LC field, Weyl rep = χ $\Psi = \begin{bmatrix}\psi^{RC}\\ \psi^{LC}\end{bmatrix} = \begin{bmatrix}\psi\\ \chi\end{bmatrix}$ Ψ like ψ at far left or u or d at left. It could be an electron (or other fermion). Ψ has 4 components in spinor space. χ (and ψ) has 2 components	Column vector
Operator on fields	$\Sigma_i = \frac{\hbar}{2}\begin{bmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{bmatrix} \rightarrow \Sigma_1 = \frac{\hbar}{2}\begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$ $\Sigma_3 = \frac{\hbar}{2}\begin{bmatrix}1&&&\\ &-1&&\\ &&1&\\ &&&-1\end{bmatrix}$ etc. for Σ_2	$\frac{1}{2}\tau_i = \frac{1}{2}\sigma_i \quad i=1,2,3$ $SU(2)$ algebra generators for fields (See commutators below.) $\Psi' = \left(1 - i\varepsilon_i \tau_i/2\right)\Psi$ $\delta_{\varepsilon_i}\Psi = \begin{pmatrix}\delta_{\varepsilon_i}\psi_u\\ \delta_{\varepsilon_i}\psi_d\end{pmatrix} = -i\varepsilon_i \frac{\tau_i}{2}\begin{pmatrix}\psi_u\\ \psi_d\end{pmatrix}$	LC fermion & scalar doublet field $\begin{bmatrix}\phi\\ \chi\end{bmatrix}$ (chiral multiplet) $\delta_\xi \begin{bmatrix}\phi\\ \chi\end{bmatrix} = \begin{bmatrix} & \xi\bullet \\ -i\sigma^\mu \bar{\xi}\partial_\mu & \end{bmatrix}\begin{bmatrix}\phi\\ \chi\end{bmatrix}$ $\delta_\xi\phi = \xi\bullet\chi \quad \delta_\xi\chi = -i\sigma^\mu\bar{\xi}\partial_\mu\phi$	Matrix operator
Operator on states	${}_{\text{QFT}}\Sigma_i = \int_V \psi^\dagger \Sigma_i \psi\, d^3x$ $\rightarrow {}_{\text{QFT}}\Sigma_3 = \int_V \psi^\dagger\Sigma_3\psi\,d^3x$ $= \sum_{r,\mathbf{p}} \frac{m}{E_\mathbf{p}} \begin{pmatrix} u_r^\dagger(\mathbf{p})\Sigma_3 u_r(\mathbf{p}) N_r(\mathbf{p}) \\ + v_r^\dagger(\mathbf{p})\Sigma_3 v_r(\mathbf{p})\bar{N}_r(\mathbf{p})\end{pmatrix}$	$T_i = \int_V j_i^0\,d^3x = \int_V \Psi^\dagger \frac{\tau_i}{2}\Psi\,d^3x$ $\rightarrow T_3 = \frac{1}{2}\int_V \left(u^\dagger\ d^\dagger\right)\begin{bmatrix}1&\\ &-1\end{bmatrix}\begin{pmatrix}u\\ d\end{pmatrix}d^3x$ $T_1 = \frac12\int_V \left(u^\dagger\ d^\dagger\right)\begin{bmatrix}&1\\ 1&\end{bmatrix}\begin{pmatrix}u\\ d\end{pmatrix}d^3x$, etc.	$Q_a = \int_V J_a^0 d^3x$ where $a=1,2$ $= \int_V \left(\sigma^\nu\chi(x)\right)_a \partial_\nu\phi^\dagger(x)\,d^3x$ Q_2 raises χ to ϕ. $Q_2^\dagger$ reverse. Q_1 lowers ψ to ϕ. $Q_1^\dagger$ reverse.	From $\mathcal{L}$ symmetry → Noether conserved currents j^μ. Not a matrix.
State	for given r, $\mathbf{p}'$ $\left\|\psi_{r,\mathbf{p}'}\right\rangle$	for given d = LC electron, $\left\|e^L_{r'\mathbf{p}'}\right\rangle$	for χ = LC electron $\left\|e^L_{r'\mathbf{p}'}\right\rangle$	Not a column vector
Operation on example state	For spin up state, ${}_{\text{QFT}}\Sigma_3\left\|\psi_{\uparrow,\mathbf{p}'}\right\rangle$ $= \frac12 \frac{m}{E_{\mathbf{p}'}} \underbrace{u_1^\dagger(\mathbf{p}')u_1(\mathbf{p}')}_{E_{\mathbf{p}'}/m} \left\|\psi_{\uparrow,\mathbf{p}'}\right\rangle$ $=\frac12\left\|\psi_{\uparrow,\mathbf{p}'}\right\rangle$. spin = $+\frac12$	For LC electron, $T_3\left\|e^L_{r'\mathbf{p}'}\right\rangle = \frac12\left(N_{\nu r}(\mathbf{p}) - N_{er}(\mathbf{p})\right)\left\|e^L_{r'\mathbf{p}'}\right\rangle$ $= -\frac12\left\|e^L_{r'\mathbf{p}'}\right\rangle$ Weak isospin charge = $-\frac12$	$Q_2\left\|\chi\right\rangle = A\left\|\phi\right\rangle$ A = a constant For LC electron, $Q_2\left\|e^L_{r'\mathbf{p}'}\right\rangle = A\underbrace{\left\|\tilde{e}^L_{r'\mathbf{p}'}\right\rangle}_{\text{selectron}}$	
Operation on states, in general	${}_{\text{QFT}}\Sigma_i\left\|\text{multipart}\right\rangle$ has no columns or matrices involved. Operator is just number, creation, destruction opers. For single particle up state, ${}_{\text{QFT}}\Sigma_3$ eigenvalue = ½, down state –½. ${}_{\text{QFT}}\Sigma_1$, ${}_{\text{QFT}}\Sigma_2$ raise & lower state spin.	$T_i\left\|e_{r,\mathbf{p}'}\right\rangle$ have no columns or matrices involved. Operator is just number, creation, destruction operators. For single electron state, T_3 eigenval = –½, neutrino = +½. T_1 & T_2 will exchange electron state with neutrino state.	$Q_a\left\|\text{state}\right\rangle$ have no columns or matrices involved. Just change fermion states to scalar states.	Terminology: creation = raising; destruction = lowering
Commutators	$\left[\Sigma_i,\Sigma_j\right] = i2\varepsilon_{ijk}\Sigma_k$ field operators $\left[{}_{\text{QFT}}\Sigma_i, {}_{\text{QFT}}\Sigma_j\right] = i2\varepsilon_{ijk}\,{}_{\text{QFT}}\Sigma_k$ state ops Note no matrices in state operators.	$\left[\sigma_i,\sigma_j\right]=i2\varepsilon_{ijk}\sigma_k$ field opers $\left[T_i,T_j\right] = i2\varepsilon_{ijk}T_k$ state operators Note no matrices in state operators.	Field opers not usually treated State operators: $\left[Q_a, Q_b^\dagger\right]_+ = \left(\sigma^\mu\right)_{ab}P_\mu$ $\left[Q_a,P_\mu\right] = \left[Q_a^\dagger,P_\mu\right]=0$ No matrices in state operators	State com relations due to raising & lowering operators.
Nomenclature	${}_{\text{QFT}}\Sigma$ from diag Σ_3 is spin "charge". Some authors call all ${}_{\text{QFT}}\Sigma_i$ "charge".	T_i are generators of state algebra. T_3 from diag τ_3 is isospin charge. Some authors call all T_i "charge".	Q_a are called SUSY charges, or SUSY generators	

6.3 Clarification on Treating a Field Multiplet as a Vector or an Operator

In Aitchison (4.3) to (4.9), pg. 51, he considers an *SU*(2) doublet of fields labeled *q*, where the *u* and *d* components represent (left chiral) fermion fields such as up and down quark fields, electron neutrino and electron fields, charm and strange quark fields, etc.

He then varies *q* in *SU*(2) space. In (4.3) to (4.4), he treats varies *q* as if it were an entity in a vector space, i.e., the transformation is for a (2-component) vector. But, in (4.5) to (4.8), he treats *q* as if it were an operator that operates on a vector space, i.e., the transformation is for an operator, not a vector. That is, where equation numbers are those in Aitchison, the τ_i are the Pauli matrices, *q* transforms as a vector, the (unitary) transformation is infinitesimal (ε_i are infinitesimal real, arbitrary parameters), and herein we use the symbol $\hat{U}$ to represent that transformation,

$$q = \begin{bmatrix} u \\ d \end{bmatrix} \qquad \text{Aitchison (4.2)}$$

$$q' = q + \delta_\varepsilon q = \underbrace{\left(1 - i\boldsymbol{\varepsilon}\boldsymbol{\cdot}\boldsymbol{\tau}/2\right)}_{\hat{U}} q = \left(1 - i\varepsilon_1 \frac{1}{2}\begin{bmatrix} & 1 \\ 1 & \end{bmatrix} - i\varepsilon_2 \frac{1}{2}\begin{bmatrix} & -i \\ i & \end{bmatrix} - i\varepsilon_3 \frac{1}{2}\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}\right)\begin{bmatrix} u \\ d \end{bmatrix} = \hat{U}q \,. \qquad \text{Aitchison (4.3)}$$

$$\delta_\varepsilon q = -i\boldsymbol{\varepsilon}\boldsymbol{\cdot}\frac{\boldsymbol{\tau}}{2} q \qquad \text{Aitchison (4.4)}$$

Normally, we think of the τ_i as the generators of infinitesimal transformations in the 2D vector space.

Aitchison follows this by transforming *q* as a matrix, or tensor, operator, where the (unitary) transformation is represented by *U*, and the T_i are generators of infinitesimal *SU*(2) transformations.

$$q' = UqU^\dagger \qquad \text{Aitchison (4.5)}$$

$$\begin{aligned} q' &= \left(1 - i\boldsymbol{\varepsilon}\boldsymbol{\cdot}\boldsymbol{T}\right) q \left(1 - i\boldsymbol{\varepsilon}\boldsymbol{\cdot}\boldsymbol{T}\right) = q + i\boldsymbol{\varepsilon}\boldsymbol{\cdot}\boldsymbol{T}q - i\boldsymbol{\varepsilon}\boldsymbol{\cdot}q\boldsymbol{T} + \textit{higher order terms} \\ &= q + i\boldsymbol{\varepsilon}\boldsymbol{\cdot}\left[\boldsymbol{T}, q\right] \end{aligned}$$

$$\delta_\varepsilon q = -i\boldsymbol{\varepsilon}\boldsymbol{\cdot}\left[\boldsymbol{T}, q\right] \qquad \text{part of Aitchison (4.9)}$$

Aitchison (4.4) is deduced considering *q* as a vector, but his (4.9) is deduced considering *q* as an operator (typically operating on a vector). One might ask how can one do such a thing. *q* must be either a vector or an operator, no?

The answer to this question lies in the overall structure of QFT. It is rarely pointed out in texts, but there are really two vector spaces involved.

One such vector space is the vector space of fields, represented in QFT, as, for examples, *u* as an up quark field, *d* as a down quark field, or ν_e as an electron neutrino field. These fields create and destroy states. In *SU*(2) space, for electroweak interactions, the left chiral (LC) parts of these fields are configured into doublets, i.e., 2 component vectors, such as in Aitchison (4.2). (The *u* and *d* therein are actually LC, as the RC *u* and *d* fields are *SU*(2) singlets (think "scalars").

A different (but related via QFT) vector space is the vector space of states, represented in QFT, as, for examples, $|u\rangle$ as an up quark, $|d\rangle$ as a down quark, or $|\nu_e\rangle$ as an electron neutrino. In this context, the *u* up quark field is an operator that creates and destroys up quark states (particles). Similarly, *d* and ν_e and other fields create and destroy other types of particle states.

Wholeness Chart 6-2 lays out the differences between these two vector spaces.

<u>Bottom line:</u> In the transformation of Aitchison (4.3) and (4.4), one considers *q* to be a vector in the *SU(*2) space of fields. In the transformation of Aitchison (4.5) and (4.9), one considers each component of *q* to be an operator that operates on the vector space of particle states. Whereas τ_i in Aitchison (4.4) are matrices, the T_i in (4.9) are not. The T_i are composed of creation, destruction, and number operators, and those result in commutation relations between the T_i, which are considered to define the Lie Algebra of transformations in state space. The particle states are not column vectors *per se*, but simply comprise one or more particles.