## Summary of Special Relativity

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## Wholeness Chart 1. Overview of the Development of Special Relativity

|  | Details | Comment |
| :---: | :---: | :---: |
| Frames considered | Observers in inertial frames only. (But, accelerating object can be handled by special relativity.) |  |
| Chronology |  |  |
| Pre-Einstein: Galilean transformation | Speed of light: not same for all (not invariant) <br> Laws of nature: <br> $\mathbf{F}=$ ma valid for all observers (invariant) <br> Maxwell's equations not valid for all (not invariant) |  |
| Problems before 1905 | Michelson-Morley experiment | Implied light speed not obey Galilean transformation |
| Einstein postulates | 1) Speed of light same for all observers (invariant) <br> 2) Laws of nature same for all observers (invariant in form = covariant) | Invariant in form = vectors in eqs change covariantly |
| Result of postulate \#1 | Lorentz transformation (instead of Galilean) $\begin{array}{rlrl} c t^{\prime} & =\frac{1}{\sqrt{1-v^{2} / c^{2}}}\left(c t-\frac{v}{c} x\right) & c t & =\frac{1}{\sqrt{1-v^{2} / c^{2}}}\left(c t^{\prime}+\frac{v}{c} x^{\prime}\right) \\ x^{\prime} & =\frac{1}{\sqrt{1-v^{2} / c^{2}}}(x-v t) & x & =\frac{1}{\sqrt{1-v^{2} / c^{2}}}\left(x^{\prime}+v t^{\prime}\right) \\ y^{\prime} & =y & y & =y^{\prime} \\ z^{\prime} & =z & z & =z^{\prime} \end{array}$ | Resulted in Lorentz contraction, time dilation, simultaneity not the same for all (not invariant), $E=m c^{2}$, and more. <br> Reciprocal: each observer sees other frame with Lorentz contraction, time dilation, etc. |
| Impact on postulate \#2 | Maxwell's equation valid for all (invariant in form = covariant) <br> $\mathbf{F}=$ ma not valid for all (not invariant in form = not covariant) |  |
| So, Einstein changed mechanics | New 4D law of mechanics: $F^{\mu}=m \frac{d u^{\mu}}{d \tau}$ where $u^{\mu}$ is 4 -velocity | $u^{\mu}=\frac{d x^{\mu}}{d \tau} \tau=$ proper time on object (see below) |
| Result of $\uparrow$ in 1905 | All laws of nature same for all observers (invariant in form = covariant) | Only mechanics and e/m known then. |
| Result of $\uparrow$ up to modern day: <br> Postulate \#2 valid | Invariance in form of laws of nature is now a general principle used in all physics. Any law must be covariant under Lorentz transformation. | True for weak and strong force laws. A "must have" for any new proposed theory (SUSY, GUTs, strings, etc.) |
| Minkowski in 1908 | Space and time $=4 \mathrm{D}$ spacetime continuum |  |
| Concepts and Relations |  |  |
| 4D position vector | $\begin{aligned} & x^{\mu}=(c t, x, y, z)=\left(x^{0}, x^{1}, x^{2}, x^{3}\right) \text { contravariant components } \\ & x_{\mu}=(-c t, x, y, z)=\left(x_{0}, x_{1}, x_{2}, x_{3}\right)=\left(-x^{0}, x^{1}, x^{2}, x^{3}\right) \text { covariant components } \end{aligned}$ |  |
| Invariant interval | $(\Delta s)^{2}=-(c \Delta t)^{2}+\left(\Delta x^{1}\right)^{2}+\left(\Delta x^{2}\right)^{2}+\left(\Delta x^{3}\right)^{2}$ same for all observers between same two events. (We need a minus sign for $(c t)^{2}$ to get a Lorentz invariant "length" for the position vector between two events.) | Not seen before Minkowski because assumed + sign for $(c \Delta t)^{2} . \Delta s$ not then invariant. |


| Proper time $\tau$ on an object | Time $\tau$ passing on standard clock at rest with respect to object. $\tau=\frac{t}{\gamma}=\sqrt{1-(v / c)^{2}} t \quad \tau=t^{\prime}$ at $x^{\prime}=y^{\prime}=z^{\prime}=$ constant | Found from invariant interval between 2 events on object world line. |
| :---: | :---: | :---: |
| Proper length $L_{o}$ of an object | Length measured with meter sticks at rest with respect to object. $L_{0}=\gamma L=\frac{1}{\sqrt{1-(v / c)^{2}}} L \quad \Delta t^{\prime}=0 \text { and } \Delta t=0$ | Found from invariant interval between 2 events at ends of object at same time in each frame. |
| 4-vector | $w^{\mu}=\left(w^{0}, w^{1}, w^{2}, w^{3}\right)$ contravariant components <br> $w_{\mu}=\left(w_{0}, w_{1}, w_{2}, w_{3}\right)=\left(-w^{0}, w^{1}, w^{2}, w^{3}\right)$ covariant components |  |
| Magnitude of a 4-vector | $\begin{aligned} (w)^{2}=\sum_{\mu=0}^{3} w^{\mu} w_{\mu}=w^{\mu} w_{\mu} & =w^{0} w_{0}+w^{1} w_{1}+w^{2} w_{2}+w^{3} w_{3} \\ & =-w^{0} w^{0}+w^{1} w^{1}+w^{2} w^{2}+w^{3} w^{3} \end{aligned}$ | Einstein convention after $2^{\text {nd }}$ equal sign. |
| Covariance of a 4-vector | To qualify as a legitimate four-vector, its magnitude must be Lorentz invariant. Components can vary, but not magnitude. |  |
| Minkowski metric and 4-vector length | $\begin{array}{r} \eta_{\mu \nu}=\left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad(w)^{2}=w^{\mu} w_{\mu}=\left[\begin{array}{llll} w^{0} & w^{1} & w^{2} & w^{3} \end{array}\right]\left[\begin{array}{c} -w^{0} \\ w^{1} \\ w^{2} \\ w^{3} \end{array}\right] \\ =\left[\begin{array}{llll} w^{0} & w^{1} & w^{2} & w^{3} \end{array}\right]\left[\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]\left[\begin{array}{c} w^{0} \\ w^{1} \\ w^{2} \\ w^{3} \end{array}\right]=\eta_{\mu \nu} w^{\mu} w^{\nu} \\ \left.(s)^{2}=\eta_{\mu \nu} x^{\mu} x^{\nu} \quad \text { (assumes initial } s_{0}=0, \text { so } \Delta s=s-s_{0}=s\right) \end{array}$ | Compare to $(w)^{2}$ above <br> Compare to $(\Delta s)^{2}$ above |
| 4-velocity | $u^{\mu}=\frac{d x^{\mu}}{d \tau}=\left[\begin{array}{c}u^{0} \\ u^{1} \\ u^{2} \\ u^{3}\end{array}\right]=\gamma\left[\begin{array}{c}c \\ v^{1} \\ v^{2} \\ v^{3}\end{array}\right]=\frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^{2}}}\left[\begin{array}{c}c \\ v^{1} \\ v^{2} \\ v^{3}\end{array}\right] \quad v^{i}=$ Newtonian velocity $=\frac{d x^{i}}{d t}$ | Always tangent to particle world line |
| 4-velocity squared | $(u)^{2}=u^{\mu} u_{\mu}=-c^{2} \quad$ Massive particles. | Invariant. Same for any particle and any observer. |
| 4-momentum | Massive particles $p^{\mu}=m u^{\mu} . \quad \downarrow$ Valid for all particles $p^{\mu}=\left[\begin{array}{c} p^{0} \\ p^{1} \\ p^{2} \\ p^{3} \end{array}\right]=\left[\begin{array}{c} E / c \\ p^{1} \\ p^{2} \\ p^{3} \end{array}\right] \quad E=\frac{m c^{2}}{\sqrt{1-\left(\frac{v}{c}\right)^{2}}}=m c^{2} \underbrace{+\frac{1}{2} m v^{2}+\ldots}_{K E_{\text {rel }}} \quad p^{i}=\frac{m v^{i}}{\sqrt{1-\left(\frac{v}{c}\right)^{2}}}$ | $E=$ relativistic energy; $p^{i}=$ relativistic 3-momentum |
| 4-momentum squared | $\begin{aligned} & (p)^{2}=p^{\mu} p_{\mu}=-m^{2} c^{2} \quad \text { Massive and massless particles. } \\ & p^{\mu} p_{\mu}=-\frac{E^{2}}{c^{2}}+p^{i} p_{i}=-\frac{E^{\prime 2}}{c^{2}}+p^{i} p_{i}^{\prime}=p^{\prime \mu} p_{\mu}^{\prime}=-m^{2} c^{2} \end{aligned}$ | Invariant. Same for any observer, any velocity. Different for different mass |
| 4D unit basis vectors | $\boldsymbol{e}_{\mu}=\boldsymbol{e}_{0}, \boldsymbol{e}_{1}, \boldsymbol{e}_{2}$, and $\boldsymbol{e}_{3}$. | Like i, j, k in 3D |
| 4-vectors | $\mathbf{A}=A^{0} \boldsymbol{e}_{0}+A^{1} \boldsymbol{e}_{1}+A^{2} \boldsymbol{e}_{2}+A^{3} \boldsymbol{e}_{3}=A^{\mu} \boldsymbol{e}_{\mu} \quad$ Same $\mathbf{A}$, diff frames: $A^{\mu} \boldsymbol{e}_{\mu}=A^{\prime} \boldsymbol{e}^{\prime}{ }_{\mu}$ |  |
| Invariance vs conservation | Invariance $=$ no change for different coordinate systems (observers) Conservation $=$ no change over time | $\Delta s$ invariant, not conserved $E$ conserved, not invariant |


| Spacetime Diagrams | See figures below. |  |  |
| :---: | :---: | :---: | :---: |
| Concept/Entity | Timelike Interval (AB, Fig. 1) | Spacelike Interval (AC) | Lightlike Interval (AD) |
| Region on spacetime diagram | Inside light cone | Outside light cone | On surface of light cone |
| Space vs time components | $c \Delta t>\Delta x$ | $c \Delta t<\Delta x$ | $c \Delta t=\Delta x$ |
| $(\Delta s)^{2}=-(c \Delta t)^{2}+(\Delta x)^{2}$ | negative | positive | zero (null) |
| $\Delta s$ | imaginary | real | zero |
| Travel from first event to second? | Yes | No | Only light can. |
| Find proper time $\tau$ from $(\Delta s)^{2}=(c \tau)^{2}$ ? | Yes, for a particle traveling from first to second event | No. Particle would have to travel faster than light. | $\tau=0$ for a photon. |
| Can first event affect (cause) the second event? | Yes. At or below light speed, a signal from $1^{\text {st }}$ event can reach $2^{\text {nd }}$ event. | No. A signal would have to travel faster than light. | Yes, but only an electromagnetic signal. All others too slow. |
| Can the two events be simultaneous for some observer? | No. Inside light cone can never have the $x^{\prime}$ axis (the axis where all events occur at the same time) | Yes. Can have $x^{\prime}$ axis extending from first event to second. | No. Observer would have to travel at light speed between events to see them simultaneous. |
| Is the time order of events the same for all observers? | Yes | No. <br> (See Fig. 2 below.) | Yes |
| $\Delta s$ invariant? | Yes | Yes | Yes |
| Is the primed time axis timelike, spacelike, or null? | Yes, timelike. | No. | No. |
| Is the primed space axis timelike, spacelike, or null? | No. | Yes, spacelike. | No. |
| Is the light path for all observers timelike, spacelike, or null? | No. | No. | Yes, null. |



Figure 11-1 Kinds of Intervals Figure 11-2. Order of Spacelike Separated Events Different for Different Observers
(Not for timelike such as AB. Cannot rotate space axis through both events, so never simultaneous nor reversed in order.)

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## Wholeness Chart 2. Electromagnetism: Classical 3D + 1 vs Relativistic 4D Spacetime

Equation numbers are for Griffiths, Introduction to Electromagnetism, $4^{\text {th }}$ ed.

| Entity | $3 \mathrm{D}+1$ | 4D | Comment |
| :---: | :---: | :---: | :---: |
| E/m field tensor | N/A | $F^{\mu \nu}=\left[\begin{array}{cccc}0 & E^{1} / c & E^{2} / c & E^{3} / c \\ -E^{1} / c & 0 & B^{3} & -B^{2} \\ -E^{2} / c & -B^{3} & 0 & B^{1} \\ -E^{3} / c & B^{2} & -B^{1} & 0\end{array}\right] \quad$ (12.119) | $\begin{gathered} \text { Antisymmetric. } \\ F^{\mu v}=F^{\mu \nu}\left(t, x^{i}\right)= \\ F^{\mu \nu}\left(x^{\beta}\right) \end{gathered}$ |
| $\mathrm{E} / \mathrm{m}$ dual field tensor | N/A | $G^{\mu \nu}=\left[\begin{array}{cccc}0 & B^{1} & B^{2} & B^{3} \\ -B^{1} & 0 & -E^{3} / c & E^{2} / c^{2} \\ -B^{2} & E^{3} / c^{2} & 0 & -E^{1} / c \\ -B^{3} & -E^{2} / c^{2} & E^{1} / c & 0\end{array}\right]$ (12.120) | Antisymmetric, $\begin{gathered} G^{\mu \nu}=G^{\mu \nu}\left(t, x^{i}\right)= \\ G^{\mu \nu}\left(x^{\beta}\right) \end{gathered}$ |
| Electric field | 3-vector $\mathbf{E}$ or $\boldsymbol{E}^{i}$ | $c F^{01}, c F^{02}, c F^{03}$ | Components of the |
| Magnetic field | 3 -vector B or $\boldsymbol{B}^{\boldsymbol{i}}$ | $F^{23}, F^{31}, F^{12}$ | tensor |
| Charge | $Q$ | $Q$ | Invariant |
| Proper charge density | $\rho=\rho_{0}=\frac{Q}{V_{0}}=\frac{Q}{V}(V \rightarrow 0) \quad$ page 565 | $\rho_{0}=\frac{Q}{V_{0}}(V \rightarrow 0) \quad(12.121),(12.122)$ | Invariant, $\rho=\rho\left(t, x^{i}\right)=\rho\left(x^{\mu}\right)$ |
| Charge density | $\rho=\frac{Q}{V}$ same as above | $\begin{equation*} \rho=\frac{Q}{V}=\frac{Q}{V_{0} \sqrt{1-v^{2} / c^{2}}}=\frac{\rho_{0}}{\sqrt{1-v^{2} / c^{2}}} \tag{12.122} \end{equation*}$ | Not 4D invariant, since $V$ not |
| Current density | $\mathbf{J}=\rho \mathbf{v} \quad$ page 565 | $\begin{equation*} \mathbf{J}=\rho \mathbf{v}=\frac{\rho_{0} \mathbf{v}}{\sqrt{1-v^{2} / c^{2}}}=\rho_{0} u^{i} \tag{12.122} \end{equation*}$ | Griffiths uses $\mathbf{u}$ for our $\mathbf{v}, \eta$ for our $u$ |
| 4-current | N/A | $\begin{gathered} J^{\mu}=\left[J^{0}, J^{1}, J^{2}, J^{3}\right]=\left[\rho c, \rho v^{1}, \rho v^{2}, \rho v^{3}\right](12.124) \\ =\rho \rho u^{\mu} \quad J_{\mu}=\left[-\rho c, \rho v^{1}, \rho v^{2}, \rho v^{3}\right]=\rho o u_{\mu} \end{gathered}$ | Not 4D invariant, $J^{\mu}=J^{\mu}\left(t, x^{i}\right)=J^{\mu}\left(x^{\mu}\right)$ |
| Continuity equation (charge) | $\nabla \cdot \mathbf{J}+\frac{\partial \rho}{\partial t}=0 \quad$ below (12.124) | $\frac{\partial J^{\mu}}{\partial x^{\mu}}=0 \quad$ (12.126) | $x^{0}=c t$ |
| Maxwell's equations | $\begin{align*} \nabla \cdot \mathbf{E} & =\frac{1}{\varepsilon_{0}} \rho \\ \nabla \cdot \mathbf{B} & =0 \\ \nabla \times \mathbf{E} & =-\frac{\partial \mathbf{B}}{\partial t}  \tag{7.40}\\ \nabla \times \mathbf{B} & =\mu_{0} \mathbf{J}+\mu_{0} \varepsilon_{0} \frac{\partial \mathbf{E}}{\partial t} \end{align*}$ | $\frac{\partial F^{\mu \nu}}{\partial x^{\nu}}=\mu_{0} J^{\mu} \quad \frac{\partial G^{\mu \nu}}{\partial x^{\nu}}=0 \quad$ (12.127) |  |
| Lorentz force law | $\mathbf{F}=q(\mathbf{E}+(\mathbf{v} \times \mathbf{B})) \quad$ below (12.129) | $K^{\mu}=q u_{v} F^{\mu \nu} \quad$ (12.128) | $\begin{aligned} & \text { Griffith's } K^{\mu} \\ & \text { is our } F^{\mu} \end{aligned}$ |
| Scalar \& vector potential | $\Phi$ and $\mathbf{A}$ | $A^{\mu}=\left[\Phi / c, A^{i}\right]^{\mathrm{T}}=\left[A^{0}, A^{1}, A^{2}, A^{3}\right]$ (12.132) | $\begin{aligned} & \text { Griffith's } V \\ & \text { is our } \Phi \\ & A^{\mu}=A^{\mu}\left(x^{\nu}\right) \end{aligned}$ |


| $\mathrm{E} / \mathrm{m}$ field tensor via 4-potential | N/A | $F^{\mu \nu}=\frac{\partial A^{\mu}}{\partial x_{\nu}}-\frac{\partial A^{\nu}}{\partial x_{\mu}}$ | (12.133) | Antisymmetric. $F^{\mu v}=F^{\mu v}\left(x^{\beta}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| Maxwell's equations via 4potential | $\begin{gathered} \frac{\partial}{c \partial t}(\nabla \cdot \mathbf{A})-\nabla^{2} \Phi=c \mu_{0} \rho \\ -\nabla\left(\frac{\partial \Phi}{c \partial t}+(\nabla \cdot \mathbf{A})\right)+\frac{\partial^{2} \mathbf{A}}{c^{2} \partial t^{2}}-\nabla^{2} \mathbf{A}=\mu_{0} \mathbf{J} \end{gathered}$ | $\frac{\partial}{\partial x_{\mu}}\left(\frac{\partial A^{\nu}}{\partial x^{\nu}}\right)-\frac{\partial}{\partial x_{\nu}}\left(\frac{\partial A^{\mu}}{\partial x^{\nu}}\right)=\mu_{0} J^{\mu}$ | (12.134) | For 3D + 1, different units, see Jackson 1975 pg 220 (6.32), (6.33) |
| Lorenz gauge | $\nabla \cdot \mathbf{A}=-\frac{1}{c} \frac{\partial \Phi}{\partial t} \quad$ pg. 570 | $\frac{\partial A^{\nu}}{\partial x^{\nu}}=0$ | (12.136) | $\Phi=V / c$ |
| Maxwell's equations in Lorenz gauge | $\begin{aligned} -\frac{\partial^{2} \Phi}{c^{2} \partial t^{2}}-\nabla^{2} \Phi & =c \mu_{0} \rho \\ \frac{\partial^{2} \mathbf{A}}{c^{2} \partial t^{2}}-\nabla^{2} \mathbf{A} & =\mu_{0} \mathbf{J} \end{aligned}$ | $-\frac{\partial}{\partial x_{v}} \frac{\partial}{\partial x^{\nu}} A^{\mu}=\mu_{0} J^{\mu}$ | (12.137) | For 3D + 1, different units, see Jackson 1975 pg 220 (6.37), (6.38) |

NOTE: For no source charges or 3-currents (i.e., $J^{\mu}=0$ ) last equation above ((12.137) in Griffiths) is just the wave equation

$$
\frac{\partial}{\partial x_{v}} \frac{\partial}{\partial x^{\nu}} A^{\mu}=0 \xrightarrow[\text { in } x^{1} \text { direction }]{\text { for } \mu=2} \frac{\partial}{\partial x_{0}} \frac{\partial}{\partial x^{0}} A^{2}+\frac{\partial}{\partial x_{1}} \frac{\partial}{\partial x^{1}} A^{2} \underbrace{+\frac{\partial}{\partial x_{2}} \frac{\partial}{\partial x^{2}} A^{2}+\frac{\partial}{\partial x_{3}} \frac{\partial}{\partial x^{3}} A^{2}}_{=0 \text { since } A^{2} \text { only depends on } x^{1}, t}=0
$$

$\xrightarrow{\text { take } x^{1} \text { as } x}-\frac{\partial^{2} A^{2}}{c^{2} \partial t^{2}}+\frac{\partial^{2} A^{2}}{\partial x^{2}}=0 \quad \rightarrow \quad \frac{\partial^{2} A^{2}}{\partial t^{2}}=c^{2} \frac{\partial^{2} A^{2}}{\partial x^{2}} \quad c=$ wave speed

