Chapter 6 Problem Revisions and One Solution

Original Prob 13 of 1st printing, 1st edition below.

13. Show that for photons $j^\mu = 0$. Do this two ways. i) Assume temporarily that A^μ is complex, so we can write the Lagrangian as $\mathcal{L}_0^{e/m} = -\frac{1}{2}\left(\partial_\nu A_\mu(x)\right)^\dagger\left(\partial^\nu A^\mu(x)\right)$. Use Noether's theorem with the transformation $A^\mu \rightarrow A^\mu e^{-i\alpha}$, to obtain j^μ with $\partial_\mu j^\mu = 0$. Then, show that by taking A^μ as real, we must have $j^\mu = 0$. ii) Note that the Lagrangian with real A^μ, $\mathcal{L}_0^{e/m} = -\frac{1}{2}\left(\partial_\nu A_\mu(x)\right)\left(\partial^\nu A^\mu(x)\right)$ is not symmetric under $A^\mu \rightarrow A^\mu e^{-i\alpha}$. So, there is no conserved current, i.e., $j^\mu = 0$. In either case, there is always no charge, so $Q = 0$ is conserved.

Correction/addition to 1st edition, 1st printing $\rightarrow$ It will be easier algebraically, if we express the Lagrangian by raising, lowering, and exchanging dummy indices as $\mathcal{L}_0^{e/m} = -\frac{1}{2}A^\dagger_{\nu,\mu}A^{\nu,\mu} = -\frac{1}{2}A^{\dagger\nu}{}_{,\mu}A_\nu{}^{,\mu} = -\frac{1}{2}A^{\dagger,\mu}_\nu A^\nu{}_{,\mu}$ with each of last two ways above to express the Lagrangian used where appropriate,

Original Prob 14 of 1st printing, 1st edition below.

14. Use Noether's theorem for scalars and the transformation $x^i \rightarrow x^i + \alpha^i$ to show that three-momentum k_i is conserved. Then, show the same result via commutation of the three-momentum operator of Chap. 3 (which can be found in Wholeness Chart 5-4 at the end of Chap. 5) with the Hamiltonian.

Prob 14, Correction version

14. Show that the total (not density) 3-momentum k^i for free scalars is conserved. Use our knowledge that the conjugate momentum for x^i is k_i, the total (not density) 3-momentum (expressed in covariant components), and it is conserved if L is symmetric (invariant) under the coordinate translation transformation $x^i \rightarrow x'^i = x^i + \alpha^i$, where α^i is a constant 3D vector. Then, show the same result via commutation of the three-momentum operator of Chap. 3 (see Wholeness Chart 5-4, pg. 158) with the Hamiltonian. (Solution is posted on book website. See pg.xvi, opposite pg. 1.)

Ans. (first part):

The Lagrangian density is $\mathcal{L}_0^0 = \phi^\dagger{}_{,\mu}\phi^{,\mu} - \mu^2\phi^\dagger\phi$. We must integrate this over all volume to get the total Lagrangian L. $L = \int \mathcal{L}_0^0 dV$. If k_i is conserved, then of course, so is k^i. So, we need to show L is invariant under $x^i \rightarrow x'^i = x^i + \alpha^i$.

The 1st term in $\mathcal{L}_0^0$, $\phi^\dagger{}_{,\mu}\phi^{,\mu}$

$$\phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right) \qquad \phi^\dagger = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$

$$\phi_{,\mu} = \sum_{\mathbf{k}} \frac{i\,k_\mu}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$

$$\phi^{,\mu} = \sum_{\mathbf{k}} \frac{i\,k^\mu}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right) \qquad \phi^\dagger{}_{,\mu} = \sum_{\mathbf{k}} \frac{i\,k_\mu}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-b(\mathbf{k})e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$

$$\phi^\dagger{}_{,\mu}\,\phi^{,\mu} = \sum_{\mathbf{k}}\sum_{\mathbf{k}''} \frac{-1}{2V}\frac{k_\mu k''^\mu}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}''}}}\Big(b(\mathbf{k})a(\mathbf{k}'')e^{-ik_\mu x^\mu}e^{-ik''_\mu x^\mu} - b(\mathbf{k})b^\dagger(\mathbf{k}'')e^{-ik_\mu x^\mu}e^{ik''_\mu x^\mu}$$
$$- a^\dagger(\mathbf{k})a(\mathbf{k}'')e^{ik_\mu x^\mu}e^{-ik''_\mu x^\mu} + a^\dagger(\mathbf{k})b^\dagger(\mathbf{k}'')e^{ik_\mu x^\mu}e^{ik''_\mu x^\mu}\Big)$$

We have to integrate each term in $\mathcal{L}$ over all volume to find L. When we do this to the first term $\phi^\dagger{}_{,\mu}\,\phi^{,\mu}$ above, the first sub-term on the RHS inside the parentheses above will only survive if $k_i = -k''_i$. The same is true of the last sub-term. The 2nd and 3rd sub-terms will only survive if $k_i = k''_i$. So, therefore,

$$\underbrace{\int \phi^\dagger{}_{,\mu}\,\phi^{,\mu}dV}_{\text{original term in } L} = \sum_{\mathbf{k}} \frac{-k_\mu k^\mu}{2V\omega_{\mathbf{k}}}\left(b(\mathbf{k})a(-\mathbf{k}) - b(\mathbf{k})b^\dagger(\mathbf{k}) - a^\dagger(\mathbf{k})a(\mathbf{k}) + a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})\right) \qquad \text{(A)}$$

Now, let's see what we get when we transform the spatial coordinates via $x^i \rightarrow x'^i = x^i + \alpha^i$.

$$\phi^{\dagger}{}_{,\mu}\phi^{,\mu} \xrightarrow{x^i \rightarrow x^i = x'^i - \alpha^i} = \phi'^{\dagger}{}_{,\mu}\phi'^{,\mu} = \sum_{\mathbf{k}}\sum_{\mathbf{k}''} \frac{-1}{2V}\frac{k_\mu k''^\mu}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}''}}}\Big(b(\mathbf{k})a(\mathbf{k}'')e^{-ik_\mu x'^\mu}e^{ik_i\alpha^i}e^{-ik''_\mu x'^\mu}e^{ik''_i\alpha^i}$$
$$- b(\mathbf{k})b^{\dagger}(\mathbf{k}'')e^{-ik_\mu x'^\mu}e^{ik_i\alpha^i}e^{ik''_\mu x'^\mu}e^{-ik''_i\alpha^i} - a^{\dagger}(\mathbf{k})a(\mathbf{k}'')e^{ik_\mu x'^\mu}e^{-ik_i\alpha^i}e^{-ik''_\mu x'^\mu}e^{ik''_i\alpha^i} + a^{\dagger}(\mathbf{k})b^{\dagger}(\mathbf{k}'')e^{ik_\mu x'^\mu}e^{-ik_i\alpha^i}e^{ik''_\mu x'^\mu}e^{-ik_i\alpha^i}\Big)$$

Once again, the first and last sub-terms above, when integrated over all space, can only be non-zero if $k_i = -k''_i$, and in those cases $e^{ik_i\alpha^i}e^{ik''_i\alpha^i} = 1$. The 2nd and 3rd sub-terms will only survive if $k_i = k''_i$. In that case, $e^{ik_i\alpha^i}e^{-ik''_i\alpha^i} = 1$. When we do this, we get

$$\underbrace{\int \phi'^{\dagger}{}_{,\mu}\phi'^{,\mu}dV}_{\text{transformed term in } L} = \sum_{\mathbf{k}} \frac{-k_\mu k^\mu}{2V\omega_{\mathbf{k}}}\Big(b(\mathbf{k})a(-\mathbf{k}) - b(\mathbf{k})b^{\dagger}(\mathbf{k}) - a^{\dagger}(\mathbf{k})a(\mathbf{k}) + a^{\dagger}(\mathbf{k})b^{\dagger}(-\mathbf{k})\Big) \quad . \qquad \text{(B)}$$

Since (A) and (B) are the same, the first term in *L* is symmetric under the transformation.

<u>The 2nd term</u> in $\mathcal{L}_0^0$, $-\mu^2\phi^{\dagger}\phi$

The second term in *L* follows in almost identical fashion (and is simpler, since no derivatives exist in it) to the first.

$$-\mu^2\phi^{\dagger}\phi = -\sum_{\mathbf{k}}\sum_{\mathbf{k}''}\frac{\mu^2}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}''}}}\Big(b(\mathbf{k})a(\mathbf{k}'')e^{-ik_\mu x^\mu}e^{-ik''_\mu x^\mu} + b(\mathbf{k})b^{\dagger}(\mathbf{k}'')e^{-ik_\mu x^\mu}e^{ik''_\mu x^\mu}$$
$$+ a^{\dagger}(\mathbf{k})a(\mathbf{k}'')e^{ik_\mu x^\mu}e^{-ik''_\mu x^\mu} + a^{\dagger}(\mathbf{k})b^{\dagger}(\mathbf{k}'')e^{ik_\mu x^\mu}e^{ik''_\mu x^\mu}\Big)$$

$$\underbrace{-\int \mu^2\phi^{\dagger}\phi\, dV}_{\text{original term}} = -\sum_{\mathbf{k}}\frac{\mu^2}{2V\omega_{\mathbf{k}}}\Big(b(\mathbf{k})a(-\mathbf{k}) + b(\mathbf{k})b^{\dagger}(\mathbf{k}) + a^{\dagger}(\mathbf{k})a(\mathbf{k}) + a^{\dagger}(\mathbf{k})b^{\dagger}(-\mathbf{k})\Big) \qquad \text{(C)}$$

When we transform the spatial coordinates via $x^i \rightarrow x'^i = x^i + \alpha^i$, we get

$$-\mu^2\phi^{\dagger}\phi \xrightarrow{x^i \rightarrow x^i = x'^i - \alpha^i} = -\mu^2\phi'^{\dagger}\phi' = -\sum_{\mathbf{k}}\sum_{\mathbf{k}''}\frac{\mu^2}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}''}}}\Big(b(\mathbf{k})a(\mathbf{k}'')e^{-ik_\mu x'^\mu}e^{ik_i\alpha^i}e^{-ik''_\mu x'^\mu}e^{ik''_i\alpha^i}$$
$$+ b(\mathbf{k})b^{\dagger}(\mathbf{k}'')e^{-ik_\mu x'^\mu}e^{ik_i\alpha^i}e^{ik''_\mu x'^\mu}e^{-ik''_i\alpha^i} + a^{\dagger}(\mathbf{k})a(\mathbf{k}'')e^{ik_\mu x'^\mu}e^{-ik_i\alpha^i}e^{-ik''_\mu x'^\mu}e^{ik''_i\alpha^i} + a^{\dagger}(\mathbf{k})b^{\dagger}(\mathbf{k}'')e^{ik_\mu x'^\mu}e^{-ik_i\alpha^i}e^{ik''_\mu x'^\mu}e^{-ik_i\alpha^i}\Big)$$

When we integrate the above over space, the same sub-terms will drop out in the same way as did to get (B). Thus, we end up with

$$\underbrace{-\int \mu^2\phi^{\dagger}\phi\, dV}_{\text{transformed term}} = -\sum_{\mathbf{k}}\frac{\mu^2}{2V\omega_{\mathbf{k}}}\Big(b(\mathbf{k})a(-\mathbf{k}) + b(\mathbf{k})b^{\dagger}(\mathbf{k}) + a^{\dagger}(\mathbf{k})a(\mathbf{k}) + a^{\dagger}(\mathbf{k})b^{\dagger}(-\mathbf{k})\Big). \qquad \text{(D)}$$

Since (C) and (D) are the same, the second term in *L* is also symmetric under the transformation, and thus *L* is symmetric under it.

From macro variational mechanics, we know that if *L* is symmetric in some coordinate, then the conjugate momentum of that coordinate is conserved. k_i, the particle(s) 3-momentum is the conjugate momentum of x^i. Thus, k_i, is conserved.

Ans. (second part):

$$H = \sum_{\mathbf{k}}\omega_{\mathbf{k}}\big(N_a(\mathbf{k}) + N_b(\mathbf{k})\big) \qquad \mathbf{P} = \sum_{\mathbf{k}}\mathbf{k}\big(N_a(\mathbf{k}) + N_b(\mathbf{k})\big) \;\rightarrow\; [H,\mathbf{P}] = 0 \quad \begin{pmatrix}\text{because all number}\\ \text{operators commute}\end{pmatrix}$$

Thus **P** is conserved for the free Hamiltonian.

Chapter 6 Problem Revisions and One Solution

Original Prob 15 of 1st printing, 1st edition below.

15. Use Noether's theorem for scalars and the transformation $x^0 \rightarrow x^0 + \alpha$ to show that energy $\omega_{\mathbf{k}}$ is conserved. Is it immediately obvious that you will get the same results from commutation of the energy operator with the Hamiltonian? (Tricky wording here?)

Prob 15, Correction version

15. Use the transformation $x^0 \rightarrow x'^0 = x^0 + \alpha$ for free scalars to show that energy $\omega_{\mathbf{k}}$ is conserved. Note that the conjugate momentum for time is energy. Is it immediately obvious that you will get the same results from commutation of the energy operator with the Hamiltonian? (Tricky wording here?)