# Quantum Fields in the de Sitter Universe
# Mukanov and Winitzki Chap 7 Notes


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Refs:
*Introduction to Quantum Effects in Gravity*, Mukhanov, V., and Winitzki, S. (Cambridge, 2007)
*Student Friendly Quantum Field Theory*, Klauber, R.D., (Sandtrove 2015, 2nd ed, 3rd printing)

**NOTE:** All Section numbers, all section headings, and equation numbers of form (6.X) or (7.X) are with reference to Mukhanov and Winitzki (M&W), Chaps. 6 and 7.

**TYPOS:**

pg. 85, (7.2). $H$ should be squared and have a $\Lambda$ subscript.
pg. 86, 3rd line after (7.4): "that" should be – than –.
pg. 91, 2nd line: $\Lambda$ should be subscript.

Big picture note

M&W Chap. 6 dealt with homogeneous, spatially isotropic, spatially flat (but generally not 4D flat), expanding universe of metric form

$$ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j = a^2(\eta)\eta_{\mu\nu}dx^\mu dx^\nu , \quad \text{(general form)} \qquad \text{(6.1) and (6.2) in M\&W} \qquad (1)$$

where $\eta$ is different from $\eta_{\mu\nu}$. In M&W Chap. 7, we deal with the de Sitter universe, a special case of (1) with the specific form for $a(t)$ of

$$a(t) = a_0 e^{H_\Lambda t} . \quad (H_\Lambda = \text{constant}) \qquad \text{(7.3) in M\&W} \qquad (2)$$

### 7.1 De Sitter universe

It turns out (see below) we get (2) via an equation of state (M&W use $\varepsilon$ for energy density, often $\rho$ in other books)

$$p = -\varepsilon \quad \text{(with cosmological constant } \Lambda = 0) \qquad (3)$$

or equivalently, a positive cosmological constant with no mass-energy or pressure

$$\Lambda > 0 \quad \text{(with } p = \varepsilon = 0 \text{ everywhere).} \qquad (4)$$

M&W choose to use (3) above, but employ subscripts $\Lambda$ in (7.1) to indicate this is effectively the same as using a cosmological constant $\Lambda$ as in (4). To see how this works out, check the Brief Summary of Cosmology wholeness chart at www.quantumfieldtheory.info , second block down relations (A) and (B). Either (3) or (4) yields the same form for (A) and (B).

(3) may seem a little weird, as any mass density, for conserved mass, will decrease as the universe expands (due to greater volume for same mass). However, if the vacuum itself has the property of mass-energy density, one could expect that to stay constant as the universe expands since, presumably, the vacuum itself (and thus its properties) will not change during the expansion. Similar logic for any pressure associated with the vacuum, as, for example, in (3).

Constant energy density

The divergence of the energy-momentum tensor gives us

$$T^{\mu\nu}{}_{;\nu} = 0 \quad \xrightarrow{\text{for } \mu = 0} \quad \dot{\varepsilon} = -3(\varepsilon + p)\frac{\dot{a}}{a} . \qquad (5)$$

Given (3), the energy density $\varepsilon$ is constant in time, and thus so is pressure $p$.

Getting (7.3) [our (2)]

The Friedmann equation (Einstein's field equation $G_{\mu\nu} + \Lambda g_{\mu\nu} = \dfrac{8\pi G}{c^4}T_{\mu\nu}$ for $\mu = 0$, $\nu = 0$ with metric (1)) is

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G\,\varepsilon}{3} + \frac{\Lambda}{3} - k\left(\frac{a_0}{a}\right)^2, \tag{6}$$

where $k$ = 0 for spatially (3D) flat spaces, 1 for positive curvature 3D spaces, and – 1 for negatively curved. M&W treat the $k$ = 0 case, and also assume $\Lambda = 0$ (so the $\varepsilon$ and $p$, as in (3) act as an effective cosmological constant).

With those assumptions, which make solving (6) easier, we get (7.2) (note typos above), with subscripts $\Lambda$ where appropriate to remind us that we are dealing with an equation of state that mimics (has the effect of) a cosmological constant. The solution to (7.2) is (7.3) [our (2)].

General assumptions for de Sitter universe/space

The basic assumptions used by de Sitter, and commonly used to define the de Sitter universe (from which above relations are derived) are:

1. The universe is homogeneous and everywhere isotropic. (So, the metric of (1) is applicable.)
2. Mass-energy density is constant in time and $\varepsilon = -p$ and $\Lambda = 0$. ((3) is applicable.)

(Instead of 2, one could take $\varepsilon = p = 0$ but with a cosmological constant $\Lambda > 0$.)

Point 1 leads to (6). Point 2 in (6) yields (7.3) [our (2)].

As an aside, an anti-de Sitter universe/space has $\Lambda < 0$ instead of number 2 above.[1]

Getting (7.4)

Using (7.3) [our (2)], where for future convenience we take $a_0 = \frac{1}{H_\Lambda}$, in (6.1) [our (1)] yields (7.4).

Almost halfway down pg. 86 in M&W, we find the conformal time defined as

$$\eta = -\int_t^\infty \frac{dt}{a(t)}, \tag{7}$$

which one can compare with pg. 64 near the bottom where conformal time is defined as

$$\eta = \int^t \frac{dt}{a(t)}. \tag{8}$$

One can get (7) from (8) if the lower integral range value is $+\infty$ in (8), where a reversal in upper and lower integral range values will change the sign of the integral.

The question is why isn't that lower value in (8) zero or $-\infty$? They are more intuitive choices.

The answer appears to be that we can take any lower limit we want to give us a form for $\eta$ that is convenient. No matter what that limit, we would in any case have

$$d\eta^2 = \frac{dt^2}{\left(a(t)\right)^2} \quad \rightarrow \quad dt^2 = \left(a(t)\right)^2 d\eta^2 = \left(a(\eta)\right)^2 d\eta^2 \tag{9}$$

to use in the metric (6.1) of pg. 64 to get (6.2). So for any metric of form (6.1), we can use (8) with any lower limit on the integral.

So, on pg. 86, we take a lower limit in (8) of $+\infty$ because it will turn out to give us a form of $\eta$ that is most convenient for future purposes. Then, reversing the limits leads to the sign change of (7).

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[1] As a further aside, you may have heard that deSitter space has positive curvature, whereas in our case we take $k$ = 0, which means flat space, so how can this be? The answer is that the value of $k$ reflects the 3D spatial curvature. When one talks of deSitter space as curved, it means the 4D spacetime is curved. The 3D space (with same time value everywhere) can be flat, as it is in our case.
Anti-deSitter space (with $\Lambda < 0$ instead of $> 0$) turns out to have negative 4D curvature.

Thus, from (7) and (7.3) [our (2)], where, as noted above, we have taken $a_0 = \frac{1}{H_\Lambda}$ ,

$$a(t) = \frac{e^{H_\Lambda t}}{H_\Lambda} \quad \rightarrow \quad \eta = -\int_t^\infty \frac{dt'}{a(t')} = -H_\Lambda \int_t^\infty e^{-H_\Lambda t'} dt' = -H_\Lambda \left. \frac{e^{-H_\Lambda t'}}{(-H_\Lambda)} \right|_t^\infty = e^{-\infty t} - e^{-H_\Lambda t} = -e^{-H_\Lambda t} \tag{10}$$

Getting (7.5)

Using $\eta = -e^{-H_\Lambda t}$ of (10) and the RHS of (9) in (7.4), we get (7.5).

$$ds^2 = \frac{1}{H_\Lambda^2 \eta^2}\left(d\eta^2 - dr^2 - r^2\left(d\theta^2 + sin^2\theta d\phi\right)\right) \qquad \text{M\&W (7.5)} \tag{11}$$

Note from (10), the limits

$$\underbrace{-\infty}_{t=-\infty} < \eta < \underbrace{0}_{t=+\infty} . \tag{12}$$

Transforming to more convenient coordinates for analysis

M&W then use (7.6) to transform to coordinates that probably seem quite weird, but facilitate analysis. As with other transformations used throughout physics, after getting our answers, we can transform back to coordinates we can use to compare our results to physical measurements.

Getting (7.7)

We need to use the transformation (7.6) in (7.5) to get (7.7) in terms of new coordinates $\tilde{\eta}$ and $\chi$. To do that, we need to find the values for quantities shown below.

$$\eta = \frac{sin\tilde{\eta}}{cos\tilde{\eta} + cos\chi} \qquad r = \frac{sin\chi}{cos\tilde{\eta} + cos\chi} \qquad \text{M\&W (7.6)} \tag{13}$$

$$\begin{aligned} d\eta &= \frac{cos\tilde{\eta}\, d\tilde{\eta}}{cos\tilde{\eta} + cos\chi} - \frac{sin\tilde{\eta}(-sin\tilde{\eta})d\tilde{\eta}}{(cos\tilde{\eta} + cos\chi)^2} - \frac{sin\tilde{\eta}(-sin\chi)d\chi}{(cos\tilde{\eta} + cos\chi)^2} \\ &= \frac{cos\tilde{\eta}\, d\tilde{\eta}(cos\tilde{\eta} + cos\chi)}{(cos\tilde{\eta} + cos\chi)^2} + \frac{sin^2\tilde{\eta} d\tilde{\eta}}{(cos\tilde{\eta} + cos\chi)^2} + \frac{sin\tilde{\eta}\, sin\chi d\chi}{(cos\tilde{\eta} + cos\chi)^2} \\ &= \frac{\left(cos^2\tilde{\eta} + sin^2\tilde{\eta}\right)d\tilde{\eta} + cos\tilde{\eta}\, cos\chi d\tilde{\eta} + sin\tilde{\eta}\, sin\chi d\chi}{(cos\tilde{\eta} + cos\chi)^2} = \frac{d\tilde{\eta} + cos\tilde{\eta}\, cos\chi d\tilde{\eta} + sin\tilde{\eta}\, sin\chi d\chi}{(cos\tilde{\eta} + cos\chi)^2} . \end{aligned} \tag{14}$$

$$d\eta^2 = \frac{\overbrace{d\tilde{\eta}^2}^{\boxed{1}} + \overbrace{cos^2\tilde{\eta}\, cos^2\chi d\tilde{\eta}^2}^{\boxed{2}} + \overbrace{sin^2\tilde{\eta}\, sin^2\chi d\chi^2}^{\boxed{3}} + \overbrace{2\, cos\tilde{\eta}\, cos\chi d\tilde{\eta}^2}^{\boxed{4}} + \overbrace{2\, sin\tilde{\eta}\, sin\chi d\chi d\tilde{\eta}}^{\boxed{5}} + \overbrace{2\, cos\tilde{\eta}\, cos\chi\, sin\tilde{\eta}\, sin\chi d\chi d\tilde{\eta}}^{\boxed{6}}}{(cos\tilde{\eta} + cos\chi)^4} \tag{15}$$

$$r^2 = \frac{sin^2\chi}{(cos\tilde{\eta} + cos\chi)^2} \tag{16}$$

$$\begin{aligned} dr &= \frac{cos\chi d\chi}{cos\tilde{\eta} + cos\chi} - \frac{sin\chi(-sin\chi)d\chi}{(cos\tilde{\eta} + cos\chi)^2} - \frac{sin\chi(-sin\tilde{\eta})d\tilde{\eta}}{(cos\tilde{\eta} + cos\chi)^2} \\ &= \frac{(cos\tilde{\eta} + cos\chi)cos\chi d\chi}{(cos\tilde{\eta} + cos\chi)^2} + \frac{sin^2\chi d\chi}{(cos\tilde{\eta} + cos\chi)^2} + \frac{sin\chi\, sin\tilde{\eta} d\tilde{\eta}}{(cos\tilde{\eta} + cos\chi)^2} = \frac{d\chi + cos\tilde{\eta}\, cos\chi d\chi + sin\chi\, sin\tilde{\eta} d\tilde{\eta}}{(cos\tilde{\eta} + cos\chi)^2} . \end{aligned} \tag{17}$$

$$dr^2 = \frac{\overbrace{d\chi^2}^{\boxed{11}} + \overbrace{\cos^2\tilde{\eta}\cos^2\chi d\chi^2}^{\boxed{12}} + \overbrace{\sin^2\chi\sin^2\tilde{\eta} d\tilde{\eta}^2}^{\boxed{13}} + \overbrace{2\cos\tilde{\eta}\cos\chi d\chi^2}^{\boxed{14}} + \overbrace{2\sin\chi\sin\tilde{\eta} d\chi d\tilde{\eta}}^{\boxed{15}} + \overbrace{2\cos\tilde{\eta}\cos\chi\sin\chi\sin\tilde{\eta} d\chi d\tilde{\eta}}^{\boxed{16}}}{\left(\cos\tilde{\eta}+\cos\chi\right)^4} \tag{18}$$

Note for the part of M&W (7.5) [our (11)] where $d\eta^2 - dr^2$, terms $\boxed{5}$ and $\boxed{15}$ cancel, as do $\boxed{6}$ and $\boxed{16}$.

The other terms give us

$$d\eta^2 - dr^2$$

$$= \frac{\overbrace{d\tilde{\eta}^2}^{\boxed{1}} + \overbrace{\cos^2\tilde{\eta}\cos^2\chi d\tilde{\eta}^2}^{\boxed{2}} + \overbrace{\sin^2\tilde{\eta}\sin^2\chi d\chi^2}^{\boxed{3}} + \overbrace{2\cos\tilde{\eta}\cos\chi d\tilde{\eta}^2}^{\boxed{4}} - \overbrace{d\chi^2}^{\boxed{11}} - \overbrace{\cos^2\tilde{\eta}\cos^2\chi d\chi^2}^{\boxed{12}} - \overbrace{\sin^2\chi\sin^2\tilde{\eta} d\tilde{\eta}^2}^{\boxed{13}} - \overbrace{2\cos\tilde{\eta}\cos\chi d\chi^2}^{\boxed{14}}}{\left(\cos\tilde{\eta}+\cos\chi\right)^4}$$

$$= \frac{\overbrace{d\tilde{\eta}^2}^{\boxed{1}} - \overbrace{d\chi^2}^{\boxed{11}} + \left(\overbrace{\cos^2\tilde{\eta}\cos^2\chi d\tilde{\eta}^2}^{\boxed{2}} - \overbrace{\sin^2\chi\sin^2\tilde{\eta} d\tilde{\eta}^2}^{\boxed{13}} + \overbrace{2\cos\tilde{\eta}\cos\chi d\tilde{\eta}^2}^{\boxed{4}}\right) + \left(-\overbrace{\cos^2\tilde{\eta}\cos^2\chi d\chi^2}^{\boxed{12}} + \overbrace{\sin^2\tilde{\eta}\sin^2\chi d\chi^2}^{\boxed{3}} - \overbrace{2\cos\tilde{\eta}\cos\chi d\chi^2}^{\boxed{14}}\right)}{\left(\cos\tilde{\eta}+\cos\chi\right)^4}$$

$$= \frac{d\tilde{\eta}^2 - d\chi^2 + \left(\cos^2\tilde{\eta}\cos^2\chi - \sin^2\chi\sin^2\tilde{\eta} + 2\cos\tilde{\eta}\cos\chi\right)d\tilde{\eta}^2 + \left(-\cos^2\tilde{\eta}\cos^2\chi + \sin^2\tilde{\eta}\sin^2\chi - 2\cos\tilde{\eta}\cos\chi\right)d\chi^2}{\left(\cos\tilde{\eta}+\cos\chi\right)^4}$$

$$= \frac{d\tilde{\eta}^2 - d\chi^2 + \left(\cos^2\tilde{\eta}\cos^2\chi - \sin^2\chi\sin^2\tilde{\eta} + 2\cos\tilde{\eta}\cos\chi\right)d\tilde{\eta}^2 - \left(\cos^2\tilde{\eta}\cos^2\chi - \sin^2\tilde{\eta}\sin^2\chi + 2\cos\tilde{\eta}\cos\chi\right)d\chi^2}{\left(\cos\tilde{\eta}+\cos\chi\right)^4}$$

Or,

$$d\eta^2 - dr^2 = \frac{d\tilde{\eta}^2 - d\chi^2 + \left(\cos^2\tilde{\eta}\cos^2\chi - \sin^2\chi\sin^2\tilde{\eta} + 2\cos\tilde{\eta}\cos\chi\right)\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta}+\cos\chi\right)^4}$$

$$= \left(1 + \cos^2\tilde{\eta}\cos^2\chi - \sin^2\tilde{\eta}\sin^2\chi + 2\cos\tilde{\eta}\cos\chi\right)\frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta}+\cos\chi\right)^4} \tag{19}$$

$$= \left(1 + \cos^2\tilde{\eta}\left(1 - \sin^2\chi\right) - \sin^2\tilde{\eta}\sin^2\chi + 2\cos\tilde{\eta}\cos\chi\right)\frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta}+\cos\chi\right)^4}$$

$$= \left(1 + \cos^2\tilde{\eta} - \cos^2\tilde{\eta}\sin^2\chi - \sin^2\tilde{\eta}\sin^2\chi + 2\cos\tilde{\eta}\cos\chi\right)\frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta}+\cos\chi\right)^4} \tag{20}$$

$$= \left(1 + \cos^2\tilde{\eta} - \underbrace{\left(\cos^2\tilde{\eta} + \sin^2\tilde{\eta}\right)}_{1}\sin^2\chi + 2\cos\tilde{\eta}\cos\chi\right)\frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta}+\cos\chi\right)^4}$$

$$= \left(\overbrace{1 - \sin^2\chi}^{\cos^2\chi} + \cos^2\tilde{\eta} + 2\cos\tilde{\eta}\cos\chi\right)\frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta}+\cos\chi\right)^4} \tag{21}$$

$$= \overbrace{\left(\cos^2\chi + \cos^2\tilde{\eta} + 2\cos\tilde{\eta}\cos\chi\right)}^{\left(\cos\tilde{\eta}+\cos\chi\right)^2}\frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta}+\cos\chi\right)^4}$$

So, finally

$$d\eta^2 - dr^2 = \frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta} + \cos\chi\right)^2}. \tag{22}$$

Using (22) in (7.5) [our (11)], along with the values for $\eta$ and $r$ of (7.6) [our(13)], we have

$$\begin{aligned} ds^2 &= \frac{1}{H_\Lambda^2 \eta^2}\left(d\eta^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi\right)\right) = \frac{1}{H_\Lambda^2\eta^2}\left(\frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta} + \cos\chi\right)^2} - r^2\left(d\theta^2 + \sin^2\theta d\phi\right)\right) \\ &= \frac{1}{H_\Lambda^2}\frac{\left(\cos\tilde{\eta} + \cos\chi\right)^2}{\sin^2\tilde{\eta}}\left(\frac{\left(d\tilde{\eta}^2 - d\chi^2\right)}{\left(\cos\tilde{\eta} + \cos\chi\right)^2} - \frac{\sin^2\chi}{\left(\cos\tilde{\eta} + \cos\chi\right)^2}\left(d\theta^2 + \sin^2\theta d\phi\right)\right) \qquad (23) \\ &= \frac{1}{H_\Lambda^2 \sin^2\tilde{\eta}}\left(d\tilde{\eta}^2 - d\chi^2 - \sin^2\chi\left(d\theta^2 + \sin^2\theta d\phi\right)\right). \qquad \text{M\&W } (7.7)\text{, pg.86} \end{aligned}$$

MORE TO COME AT SOME POINT