Metrics in Homogeneous Isotropic Curved Spaces

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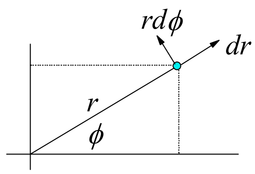
1 2D Spaces and Spherical Coordinates

1.1 Flat



$$dl^2 = dr^2 + r^2 d\phi^2 \tag{1}$$

Figure 1. Polar Coordinates, Flat Space

1.2 2-Sphere (Surface of 3-Ball)

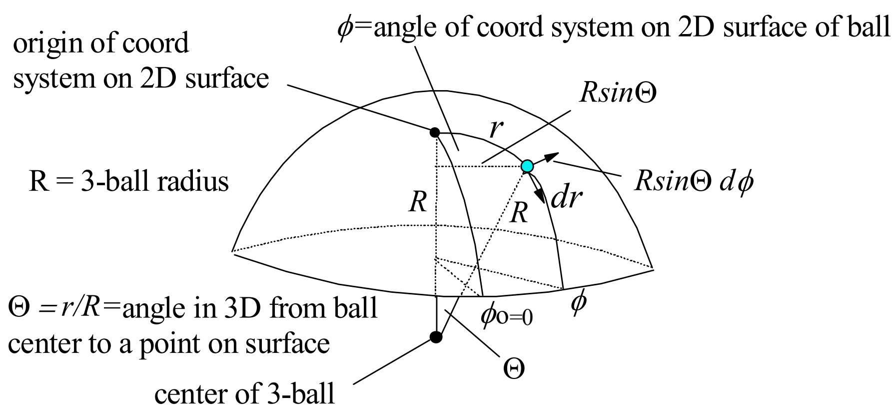


Figure 2. A 3-Ball with 2-Sphere Surface

The line element along the 2D surface of the ball, where a "hat" over a component symbolizes physical distance that would be measured with meter sticks in the direction of that component, is

$$dl^2 = dr^2 + d\hat{\phi}^2 = dr^2 + R^2 \sin^2 \Theta d\phi^2 = dr^2 + R^2 \sin^2 \frac{r}{R} d\phi^2 \,. \tag{2}$$

For a unit radius 3-ball ($R = 1$)

$$dl^2 = dr^2 + \sin^2 r d\phi^2 \qquad R = 1 \qquad 0 \le r \le \pi \tag{3}$$

For Susskind notation (https://www.youtube.com/watch?v=nJlWYDcGr8U&list=PLvh0vlLitZ7c8Avsn6gUaWX05uD5cedO-&index=3 minute 19 to 25)

For $l = \Omega_2$, $\phi = \Omega_1$, (3) becomes

$$d\Omega_2^2 = dr^2 + \sin^2 r d\Omega_1^2 \qquad R = 1 \qquad 0 \le r \le \pi \,. \tag{4}$$

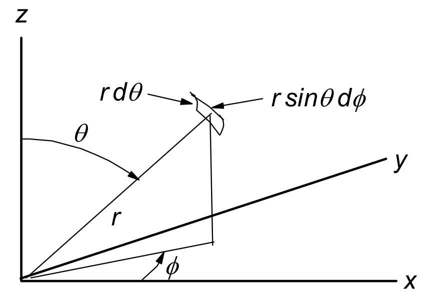
Further Note

For $R \to \infty$, the curved 2D space approaches a flat space. (2) becomes (1).

$$dl^2 = dr^2 + R^2 \sin^2 \frac{r}{R} d\phi^2 \to \quad dr^2 + R^2 \left(\frac{r}{R}\right)^2 d\phi^2 = dr^2 + r^2 d\phi^2 \tag{5}$$

2 3D Spaces and Spherical Coordinates

2.1 Flat



$$dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$
$$= dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \quad (6)$$

Figure 3. Polar Coordinates, Flat Space

2.2 3-Sphere (3D "surface" of a 4-ball)

3D "surface" (a volume, really, to us 3D creatures) of a 4D ball, where all dimensions are spatial. We are not talking 4D spacetime.

Noe that in going from (1) to (2), we took r, the lower dimensional radial coordinate over to $R\sin r/R$, where R is the radius from the higher dimensional space coordinate system origin to a point in the lower dimensional space. (Note that Θ in Sect. 1.2 is best thought of as r/R, because in (6) we are now taking θ as part of the lower dimensional coordinate system. In Sect. 1.2, Θ was part of the higher dimensional coordinate system. Θ and θ are two different things.)

So, in going from flat to curved space in 3D, we parallel what we did in going from flat to curved space in 2D. That is, we take

$$r \xrightarrow[\text{curved space}]{\text{flat to}} R\sin\frac{r}{R} \qquad \begin{array}{l} R = \text{radial distance to center of 4-ball in 4D} \\ r = \text{distance from origin in curved 3D space} \end{array} \quad (7)$$

Then (6) becomes

$$dl^2 = dr^2 + R^2 \sin^2\frac{r}{R}\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (8)$$

which is what Ryden (*Introduction to Cosmology*, Cambridge 2002) shows in her (3.31), pg. 40.

<u>For Unit Radius 4D Ball (R = 1)</u>

On a 4D unit radius ball, where R = 1, (8) becomes

$$dl^2 = dr^2 + \sin^2 r\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad R = 1 \quad 0 \le r \le \pi \quad (9)$$

<u>Susskind Notation</u>

For a 3-sphere (4-ball), Susskind has Ω_3 for l in (9), and Ω_2 for what we call l of a 2-sphere, as shown in (4), and Ω_1 for what we call ϕ. Using these relations in (9), we get

$$d\Omega_3^2 = dr^2 + \sin^2 r\left(d\theta^2 + \sin^2\theta d\Omega_1^2\right). \quad (10)$$

Yet, Susskind shows

$$d\Omega_3^2 = dr^2 + \sin^2 r d\Omega_2^2 = dr^2 + \sin^2 r\left(dr^2 + \sin^2 r d\Omega_1^2\right). \quad (11)$$

(11) does not equal (10). He seems to have confused Θ with θ, and taken $\theta = r/R$, with R = 1, in (10). But, that does not seem justified. Θ is an angle outside the lower dimensional space, but θ is inside the lower dimensional space.

Ryden keeps θ for the 3-sphere (4-ball "surface") as a separate thing from Θ = r/R, as in (8).