Mukanov and Winitzki Chap 9 Notes
Hawking effect: Thermodynamics of black holes

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Refs:
Introduction to Quantum Effects in Gravity, Mukhanov, V., and Winitzki, S. (Cambridge, 2007)
Student Friendly Quantum Field Theory, Klauber, R.D., (Sandtrove 2015, 2nd ed, 3rd printing)

Typos

Pg. 110, un-numbered equation near top of page. The Boltzmann constant k should be in the denominator of the quantities in the middle and on the RHS.

Mid page 112, 3rd line below (9.11). at end of line change "and" to "or".

Chap 9 Preliminaries

By this time in your physics career, you are probably getting somewhat used to transforming from one coordinate system to another in order to facilitate analysis. This chapter offers a veritable blizzard of such transformations, one after the other. I'm afraid you will just have to settle in and get used to it. With the right frame of mind, you may, in time, even become comfortable with it.

Oveview of chapter

Section 9.1 Hawking Radiation without thermodynamics

First part: Heuristic look at Hawking radiation

Sect 9.1.1: Classical Schwarzchild solution to black holes in 2D in 3 different coordinate systems: i) spherical coords, ii) special (tortoise, by name) coords with radial coord r^*, and iii) tortoise light cone coords $\tilde{u}$ and $\tilde{v}$

Sect 9.1.2: Still classical in 2D with yet other different coordinate systems: iv) Kruskal-Szekeres light cone coords, u and v, v) special timelike T and spacelike R coords (which are unbounded), and vi) bounded coords $\varkappa$ and ν related to light cone coords, and vii) bounded coords η and χ, which are time like and spacelike.

Sect 9.1.3: Quantizing the classical Schwarzchild field of prior sections and seeing how Hawking radiation comes out.

Section 9.2 Thermodynamics of black holes

Sect 9.2.1. Black hole thermodynamics laws

9.1 Hawking radiation

Several statements in the first paragraph of this section are made without supplying support for them. For example, "for a rotating black hole there exist negative-energy states outside its horizon". And later, "a non-rotating black hole has no negative energy states outside its horizon." If you have not seen the formal analysis for the first statement, just accept it for now. Regarding the second statement, see the Appendix herein.

Additionally, the virtual particle/antiparticle pair production discussed has some issues associated with it, as discussed in Chap. 10 of Klauber. And M&W warn the reader in the second paragraph on page 110, that she/he should not take these arguments too seriously. But, they can help a bit as a lead-in to the real mathematics behind the Hawking effect.

One key point in that discussion is that negative energy states can exist *inside* the black hole, with positive states outside. So the presumed particle pairs near the horizon would have a negative state inside the horizon, which could not escape since it is inside, but a positive state outside, which can (but not necessarily would) escape. When the positive state outside does escape, it is Hawking radiation – at least in this heuristic model.

Top of pg 110

The de Broglie relations, with de Broglie wavelength λ, are

$$p = \frac{h}{\lambda} = \frac{2\pi\hbar}{\lambda} \quad \text{with} \quad k = \frac{2\pi}{\lambda} = \frac{p}{\hbar} . \tag{1}$$

For massless particles like photons and for massive particles[1] at very high energy (close to light speed),

$$E = \hbar\omega = hf = \frac{hc}{\lambda} \quad (\text{massless particles}) \qquad E \approx \frac{hc}{\lambda} \quad (\text{massive particles at very high energies}) . \tag{2}$$

At the black hole event horizon r_g, where

$$r_g = \frac{2GM}{c^2} , \tag{3}$$

(2) becomes

$$\text{for } \lambda \geq r_g \qquad E = \frac{hc}{\lambda} \leq \frac{hc}{r_g} = \frac{hc^3}{2GM} . \tag{4}$$

From the Boltzmann relation $E = kT$ we find the relation near the top of M&W pg. 110 (where M&W left out the Boltzmann constant k, which ultimately is OK since their relation is only a proportionality, but which could lead to confusion),

$$E = kT \qquad \rightarrow \qquad T \approx \frac{hc^3}{2kGM} \sim \frac{\hbar c^3}{GM} . \tag{5}$$

9.1.1 Schwarzschild solution

Comment on (9.1)

Note that in general relativity, units are often taken where $G = 1$ dimensionless, similar to what one does in QFT for natural units, where $c = \hbar = 1$ dimensionless. Thus in M&W (9.1),

$$\frac{2M}{r} \; (\text{natural units plus } G=1) \rightarrow \frac{2GM}{c^2 r} \; (\text{MKS units}) = \frac{2}{c^2} \times (\text{Newtonian potential}) . \tag{6}$$

Deducing (9.3)

To prove (9.3), start with (9.3) in r^* and deduce the un-numbered equation definition of dr^* at the bottom of pg. 110.

$$r^*(r) = r - r_g + r_g \, ln\left(\frac{r}{r_g} - 1\right) \qquad \text{M\&W (9.3).} \tag{7}$$

$$\begin{aligned} dr^* &= dr - 0 + r_g \frac{1}{\left(\frac{r}{r_g} - 1\right)} d\left(\frac{r}{r_g} - 1\right) = dr + \frac{1}{\left(\frac{r}{r_g} - 1\right)} dr = dr\left(1 + \frac{1}{\left(\frac{r}{r_g} - 1\right)}\right) \\ &= dr\left(\frac{\frac{r}{r_g} - 1 + 1}{\left(\frac{r}{r_g} - 1\right)}\right) = dr \frac{1}{\left(1 - \frac{r_g}{r}\right)} \qquad \text{Relation at bottom of pg. 110.} \end{aligned} \tag{8}$$

Note on (9.5) tortoise lightcone coordinates

The expression for r^* will be needed later.

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[1] From $p^\mu p_\mu = m^2 c^2$, $\frac{E^2}{c^2} - \mathbf{p}^2 = m^2 c^2$, and $E^2 = \frac{c^2 h^2}{\lambda^2} + m^2 c^4 = \frac{c^2 h^2}{\lambda^2}\left(1 + \frac{m^2 c^2 \lambda^2}{h^2}\right)$. At high energies (momenta), $\lambda \to 0$.

$$\tilde{u} \equiv t - r^* \qquad \tilde{v} \equiv t + r^* \qquad \rightarrow \qquad r^* = \frac{\tilde{v} - \tilde{u}}{2} \qquad t = \frac{\tilde{v} + \tilde{u}}{2}. \tag{9}$$

Note also

$$r\left(r^*\right) = r\left(\frac{\tilde{v} - \tilde{u}}{2}\right). \tag{10}$$

Finding (9.6)

This parallels the derivation of Klauber, M&W Chap 8 Notes on the Unruh effect, equations (3) to (7), with *r** here in place of *x* there. Simply find *dt* and *dr** from (10) and plug into M&W (9.4) to get (9.6),

$$ds^2 = \left(1 - \frac{r_g}{r\left(\tilde{u},\tilde{v}\right)}\right) d\tilde{u} d\tilde{v} \qquad \text{M\&W (9.6).} \tag{11}$$

9.1.2 Kruskal-Szekeres coordinates

First equation in section

To find the first (unlabeled) equation in this section, start with the RHS of the relation itself and use the *r** relation on the RHS of (9) along with (9.3) [our (7)].

$$\begin{aligned} &\frac{r_g}{r} e^{\left(1 - \frac{r}{r_g}\right)} e^{\left(\frac{\tilde{v} - \tilde{u}}{2r_g}\right)} = \frac{r_g}{r} e^{\left(1 - \frac{r}{r_g}\right)} e^{\left(\frac{r^*}{r_g}\right)} = \frac{r_g}{r} e^{\left(1 - \frac{r}{r_g}\right)} e^{\left(\frac{r - r_g + r_g \ln\left(\frac{r}{r_g} - 1\right)}{r_g}\right)} = \frac{r_g}{r} e^{\left(1 - \frac{r}{r_g}\right)} e^{\left(\frac{r}{r_g} - 1\right)} e^{\left(\ln\left(\frac{r}{r_g} - 1\right)\right)} \\ &= \frac{r_g}{r} e^{\left(\ln\left(\frac{r}{r_g} - 1\right)\right)} = \frac{r_g}{r}\left(\frac{r}{r_g} - 1\right) = 1 - \frac{r_g}{r} \;=\; \text{LHS of unlabeled eq on pg. 111.} \end{aligned} \tag{12}$$

Range of Kruskal-Szekeres ligthcone coords

Note $-\infty < u < 0$ and $0 < v < \infty$, as defined in (9.8).

Finding (9.9)

To find (9.9), start with the RHS of (9.9) and show it equals (9.7). The only part that is different between (9.9) and (9.7) is *dudv* in (9.9), which we need to show equal to $e^{-\frac{\tilde{u}}{2r_g}} e^{\frac{\tilde{v}}{2r_g}} d\tilde{u}\, d\tilde{v}$ of (9.7). From (9.8), we find *du* and *dv*.

$$u = -2r_g e^{-\frac{\tilde{u}}{2r_g}} \qquad v = 2r_g e^{\frac{\tilde{v}}{2r_g}} \qquad \text{M\&W (9.8).} \tag{13}$$

$$du = -2r_g\left(-\frac{d\tilde{u}}{2r_g}\right) e^{-\frac{\tilde{u}}{2r_g}} = d\tilde{u}\, e^{-\frac{\tilde{u}}{2r_g}} \qquad dv = 2r_g\left(\frac{d\tilde{v}}{2r_g}\right) e^{\frac{\tilde{v}}{2r_g}} = d\tilde{v}\, e^{\frac{\tilde{v}}{2r_g}} \tag{14}$$

Substituting (14) into (9.9)

$$ds^2 = \frac{r_g}{r} e^{\left(1 - r/r_g\right)} du dv \qquad \text{M\&S (9.9)} \tag{15}$$

yields (9.7).

Extending range of *u* and *v*

We can simply choose to extend the range of Kruskal coordinates *u* and *v* so they range from – ∞ to + ∞. Then they will also cover the interior of the black hole.

Finding Relation (9.10)

From (9.8) [our (13)], then using (12),

$$
\begin{aligned}
uv &= -2r_g e^{-\frac{\tilde{u}}{2r_g}} 2r_g e^{\frac{\tilde{v}}{2r_g}} = -4r_g^2 e^{\frac{\tilde{v}-\tilde{u}}{2r_g}} = -4r_g^2 e^{\frac{r^*}{r_g}} = -4r_g^2 e^{\frac{r-r_g+r_g\, ln\left(\frac{r}{r_g}-1\right)}{r_g}} = -4r_g^2 e^{\frac{r}{r_g}-1+r_g\, ln\left(\frac{r}{r_g}-1\right)} \\
&= -4r_g^2 e^{\frac{r}{r_g}-1} e^{ln\left(\frac{r}{r_g}-1\right)} = -4r_g^2 e^{\frac{r}{r_g}-1}\left(\frac{r}{r_g}-1\right) \qquad \text{M\&W (9.10).}
\end{aligned} \tag{16}
$$

Finding Relation (9.11)

From (9.8) and the last relation in (9), one gets (9.11).

Figure 9.1 with comments

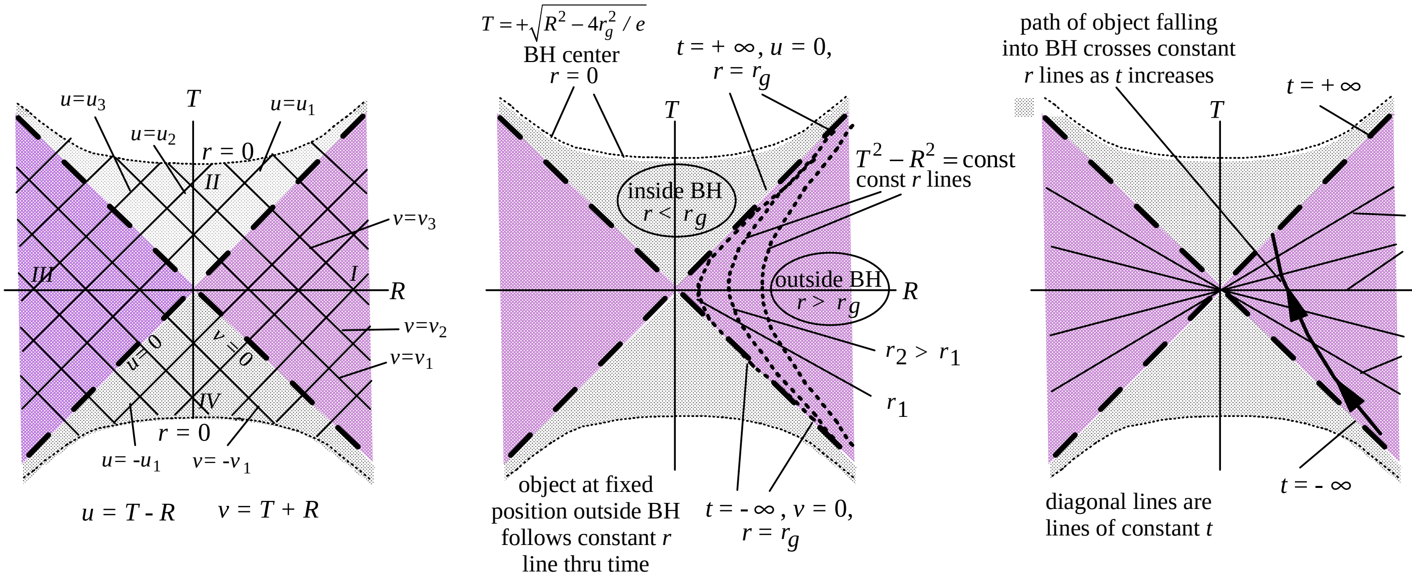

Figure 1. M&W Fig. 9.1 (reproduced 3 times) with Comments[2]

Note that in the RHS of Fig. 1, the path of an object falling into the black hole takes an infinite amount of time t to reach and pass over the horizon at $r = r_g$, although a finite amount of the Kruskal timelike coordinate T. t, of course, is the time that would pass on physical clocks of observers at infinity. So, essentially, if we were to observe a particle falling into a distant black hole, it would seem to us to take forever (literally) to do so. To determine the time passing on the particle itself (particle's proper time τ), consider M&W (9.9) [our (15)] with $r \rightarrow r_g$.

$$ds^2 = d\tau^2 \rightarrow \; dudv \qquad \left(\text{approaching BH horizon } r_g\right). \tag{17}$$

From M&W (9.12),

$$u = T - R \qquad v = T + R \qquad \text{M\&W (9.12),} \tag{18}$$

we have
$$du = dT - dR \qquad dv = dT + dR \qquad \rightarrow \qquad dudv = dT^2 - dR^2\,, \tag{19}$$

and from (17),

$$d\tau^2 = dT^2 - dR^2 \qquad \rightarrow \qquad d\tau = \sqrt{dT^2 - dR^2} \qquad \text{close to horizon.} \tag{20}$$

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[2] Take care when consulting other texts, such as Misner, Thorne, and Wheeler's *Gravitation* (pg. 835) or J.B. Hartle's *Gravity: An Introduction to Einstein's General Relativity* (pg. 271), on Kruskal-Szekeres coordinates. These two texts, and probably others, use u and v for what M&W designate as R and T. M&W use u and v as light cone coordinates at 45° angles to what these other texts label as u and v axes.

The integral of path length of a particle close to the horizon in a Kruskal-Szekeres diagram, via (20), is the proper time on the particle.

More generally, at any distance from the black hole, M&W (9.9) [our (15)], along with the LHS of (19), gives us

$$ds^2 = d\tau^2 = \frac{r_g}{r} e^{\left(1-r/r_g\right)} du dv = \frac{r_g}{r} e^{\left(1-r/r_g\right)} \left(dT^2 - dR^2\right). \qquad \text{anywhere} \qquad (21)$$

The integral of (21) on a path in *T-R* (Kruskal-Szekeres) space will be finite for all *r*. So the particle falling in, from its point of view, does so in finite time.

Note further that Fig. 1 represents only the radial direction *r* in physical space, so the particle trajectory shown is for a particle falling straight in along a radial line. Particles in orbit or passing by the black hole in a curved trajectory cannot be represented in the 2D (time plus one space dimension) of Fig. 1.

<u>Finding (9.13)</u>

From (9.12) [our (18)],

$$uv = T^2 - R^2. \qquad (22)$$

Using (9.10) with $r = 0$ and (22), one gets (9.13).

<u>Note on lower part of pg. 113</u>

Given the form of M&W (9.11), the sign on *v/u* can be positive or negative for a given value of *t*. With an eye towards the *u* and *v* axes of the LHS of our Fig. 1, this means we can plot a given constant *t* line on the LHS of the vertical axis in either of the two ways shown below in Fig. 2 (LHS vs middle). M&W chose the middle way in Fig. 2, and by doing so, both *t* and *T* increase in the vertical direction, anywhere on the plot.

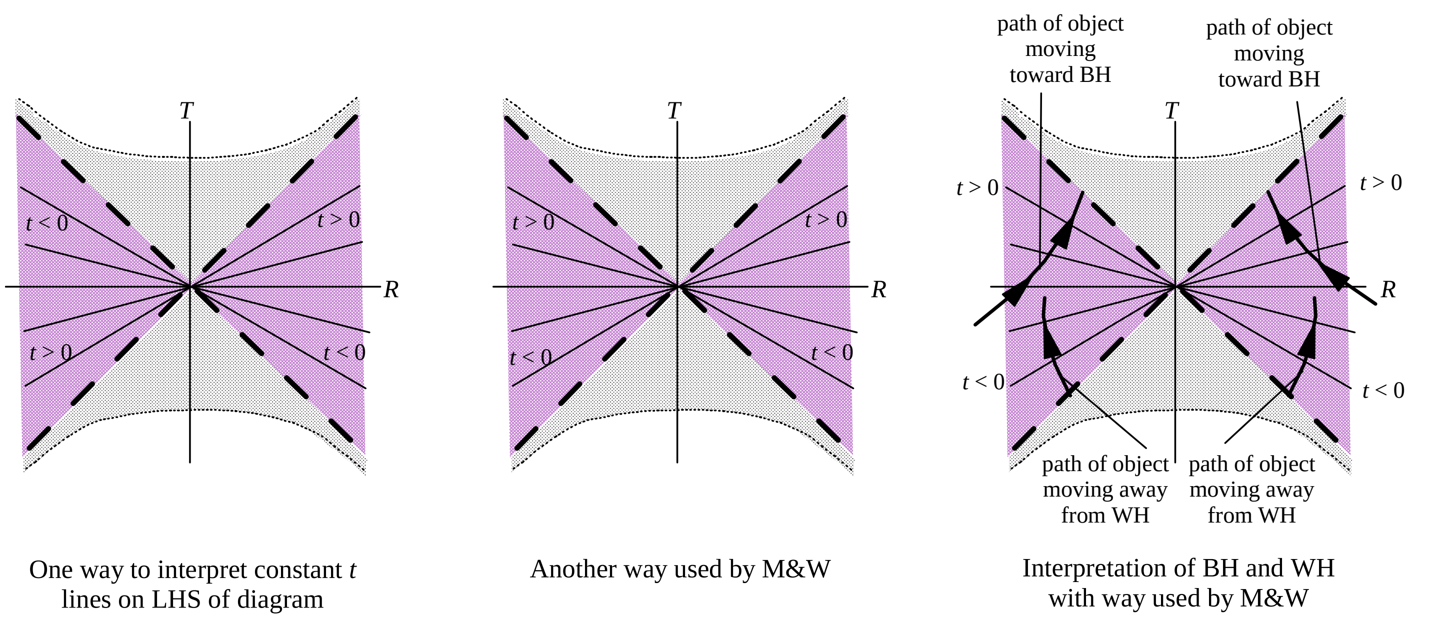


Figure 2. Options on Constant *t* Lines in M&W Fig. 9.1 and Resulting Interpretation for Black Holes and White Holes

Thus, the statements in this paragraph regarding past and future apply to either real time *t*, or Kruskal time *T*.

On the RHS of Fig. 2, we show that paths of typical free falling particles in spacetime as seen on the Kruskal-Szekeres plot. Note that as time increases, the uppermost particles shown move closer to the black hole (smaller *r* values as seen in the middle part of Fig. 1, approaching the $u = 0$ line on the right and the $v = 0$ line on the left). Particles in the lower part of the figure behave in the same manner, but this effectively means they are free falling away from the v = 0 axis on the right and the u = 0 axis on the left. Hence (I believe this is what M&W mean), one could interpret the two lower regions near the v = 0 line on the RHS and the u =0 on the LHS as white holes (particles repelled from them rather than attracted to them.)

However, we generally stick to regions *I* and *II* (see region numbering in LHS of Fig. 1 and M&W Fig. 9.1), as they cover everything for a black hole in our universe.

<u>Deriving (9.15)</u>

To get (9.15), start with (9.9) [our (15)]. Then, expressing (9.14) slightly differently,

$$\tan\varkappa = u \qquad \tan\nu = v \qquad \text{M\&S (9.14) rearranged} \tag{23}$$

$$du = \frac{1}{\cos^2\varkappa}d\varkappa \qquad dv = \frac{1}{\cos^2\nu}d\nu\,. \tag{24}$$

Using (24) in (9.9) [our (15)], we have

$$ds^2 = \frac{r_g}{r}e^{\left(1-r/r_g\right)}\frac{1}{\cos^2\varkappa\cos^2\nu}d\varkappa\, d\nu \qquad \text{M\&S (9.15).} \tag{25}$$

<u>Null geodesics</u>

For null geodesics, ds^2 = 0. In inertial frames using Minkowski coordinates, null geodesics mean (2 dimensions suppressed) $ds^2 = dt^2 - dx^2 = 0 \rightarrow dt = \pm\, dx$. That is, a null geodesic is a straight line at 45° angle to both the t and x axes. Since, in natural units, the speed of light c = 1, and on a null geodesic, $dx/dt = \pm 1$, null geodesics represent the path of light through spacetime. With one more spatial dimension, one gets the "light cone" with the tip on the origin.

The question arises as to what a null geodesic would look like in the Kruskal-Szekeres coordinates of M&W Fig. 9.1 [our Fig. 1]. To answer, note from M&W (9.9) [our (15)], that

$$ds^2 = 0 \quad \text{if} \quad du = 0 \quad \text{or} \quad dv = 0 \qquad (\text{i.e., } u \text{ or } v = \text{ constant})\,. \tag{26}$$

That is, lines of u or v = constant represent null geodesics, the path of light. From the LHS of our Fig. 1, we can see that null geodesics in a Kruskal-Szekeres coordinate diagram make 45° angles to both the T and R axes.

This tends to make life easier, as we can consider 45° lines as null geodesics in both Minkowski and Kruskal-Szekeres coordinate spacetime plots. It also makes it easier to track causality, because any event in a Kruskal-Szekeres plot in a can only affect another event if the second event is inside the light cone of the first event (just like in Minkowski space plots).

Question/problem: Are null geodesics in Schwarzchild coordinate (t vs r) plots at 45° angles to the horizontal and vertical axes? Consider M&S (9.2) with ds^2 = 0. (The answer is no. And, the angle of null geodesics in a t-r plot varies with r.)

2nd Question/problem: In tortoise coordinates (M&W unlabeled equation at bottom of pg. 110 to (9.4)) do null geodesics make 45° angles with the t and r^* axes? Consider M&S (9.4) with ds^2 = 0. (The answer is yes.)

3rd Question/problem: In the deformed Kruskal coordinates (M&W (9.14) to (9.16) with Fig. 9.2), do null geodesics make 45° angles with the η and χ axes? Consider M&S (9.15) [our (25)] to see that $\varkappa$ = constant or ν = constant means ds^2 = 0. From M&S (9.16) one then sees that such constant lines form 45° angles with the η and χ axes. (So, the answer is yes.)

9.1.3 Field quantization and Hawking radiation

The hard part of this chapter is behind us. The main point of the chapter is made in this section by comparing Unruh radiation (Chap. 8) to black holes (this Chap. 9) to see how Hawking radiation comes out of parallel mathematics. This can all be made simpler with the aid of Fig. 3, which should be fairly self-explanatory.

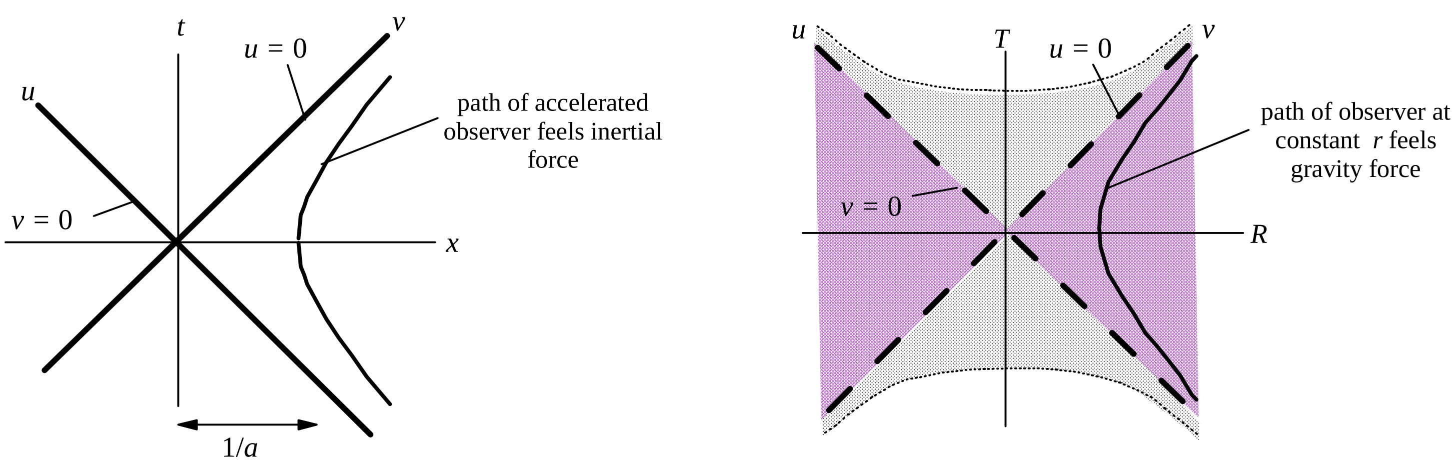
With Fig. 3 as a guide, note the following relations from Chap. 8 and Chap. 9.

$$u = -\frac{1}{a}e^{-a\tilde{u}} \qquad v = \frac{1}{a}e^{a\tilde{v}} \qquad \tilde{u} = \xi^0 - \xi^1 \qquad \tilde{v} = \xi^0 + \xi^1 \qquad \text{M\&W (8.25) for accelerating observer} \qquad (27)$$

$$u = -2r_g e^{-\frac{\tilde{u}}{2r_g}} \qquad v = 2r_g e^{\frac{\tilde{v}}{2r_g}} \qquad \tilde{u} = t - r^* \qquad \tilde{v} = t + r^* \qquad \text{M\&W (9.8) [our (13)] for black hole observer} \quad (28)$$

In the LHS of Fig. 3 the path of the accelerated observer has ξ^1 fixed and the path in *u*-*v* coordinates is found from (27). In the RHS of Fig. 3 the path of the observer fixed at a spot outside the black hole has *r** fixed (which means *r* is fixed) and the path through spacetime in *u*-*v* coordinates is found from (28).

Thus, we can see a clear parallel between the two analyses, where acceleration *a* in the first case is analogous to $1/2r_g$ in the second, as noted in the summary chart on the bottom of M&W pg. 116. The quantity $1/2r_g = \kappa$ is called the *surface gravity*.



Accelerating observer in flat spacetime in Minkowski coordinates

Observer fixed in space near black hole spacetime in Kruskal-Szekeres coordinates

Figure 3. Comparing Accelerating Observer in Flat Space to Fixed Observer Near Black Hole

<u>Expressing the Comparison in Terms of *t*,*x* and *T*,*R*</u>

(27) and (28) are expressed in terms of lightcone coordinates *u*,*v* in the two systems. Expressing them, instead, in terms of *t*,*x* and *T*,*R* coordinates can be insightful.

For the <u>accelerated frame</u> (ε^0 and ε^1 are timelike and spacelike coordinates for accelerating non-inertial observer)

$$t = \frac{1}{a}e^{a\varepsilon^1}\sinh a\varepsilon^0 \qquad x = \frac{1}{a}e^{a\varepsilon^1}\cosh a\varepsilon^0 \qquad \text{M\&W (8.27) pg. 102} \qquad (29)$$

For the <u>black hole frame</u> (*T* and *R* are timelike and spacelike coordinates for stationary non-inertial observer)

From M&W (9.12) [our (18)], we have

$$T = \frac{v+u}{2} \qquad R = \frac{v-u}{2}. \qquad (30)$$

Substituting (28) into (30), we get

$$T = 2r_g\left(\frac{e^{\frac{\tilde{v}}{2r_g}} - e^{-\frac{\tilde{u}}{2r_g}}}{2}\right) = 2r_g\left(\frac{e^{\frac{t+r^*}{2r_g}} - e^{-\frac{t+r^*}{2r_g}}}{2}\right) = 2r_g e^{\frac{r^*}{2r_g}}\left(\frac{e^{\frac{t}{2r_g}} - e^{-\frac{t}{2r_g}}}{2}\right) = 2r_g e^{\frac{1}{2r_g}r^*}\sinh\frac{1}{2r_g}t \qquad (31)$$

$$R=2r_g\left(\frac{e^{\frac{\tilde{v}}{2r_g}}+e^{-\frac{\tilde{u}}{2r_g}}}{2}\right)=2r_g\left(\frac{e^{\frac{t+r^*}{2r_g}}+e^{-\frac{t+r^*}{2r_g}}}{2}\right)=2r_g e^{\frac{r^*}{2r_g}}\left(\frac{e^{\frac{t}{2r_g}}+e^{-\frac{t}{2r_g}}}{2}\right)=2r_g e^{\frac{1}{2r_g}r^*}\cosh\frac{1}{2r_g}t\,. \tag{32}$$

Comparing (31) and (32) to (29), a fixed position (r, or equivalently r^*) near a black hole corresponds to a fixed position ε^1 in a constant acceleration system; and coordinate time t for the black hole (time measured by a standard clock at infinity), corresponds to coordinate time ε^0 in the accelerating system (measured by a standard clock in the accelerating frame).

Thus, for the appropriate corresponding parameters, the curve on the LHS of Fig. 3 is the same as the curve on the RHS, each a function of its particular horizontal and vertical axis coordinates.

That comparison of (31) and (32) with (29) leads to the same parallel we found from u,v coordinates, i.e.,

$$a \;\rightarrow\; \frac{1}{2r_g}=\kappa\,, \quad \text{the } \textit{surface gravity} \text{ for a black hole} \tag{33}$$

Note on the term "surface gravity"

Pretend the acceleration at the surface of a black hole can be determined by the Newtonian relation

$$a=\frac{F}{m}=\frac{GM}{r^2} \quad\rightarrow\quad \frac{M}{r^2}\ \left(\text{in typical GR where } G=1\right). \tag{34}$$

At the Schwarchild radius, $r = r_g$, which is the radius on the black hole surface, via (34), we have, with $M=r_g/2$,

$$a=\frac{M}{r_g^2}=\frac{r_g}{2r_g^2}=\frac{1}{2r_g}\,. \tag{35}$$

(35) is what we have been calling the surface gravity, and via this (Newtonian) analysis we can see why that may be appropriate terminology.

It turns out that the acceleration at the Schwarzchild radius in a full general relativistic analysis turn out to be the same as that of (35) from a simple Newtonian analysis. We won't go into that here, but the interested reader can check out https://en.wikipedia.org/wiki/Surface_gravity.

The rest of the chapter

The remainder of the chapter follows fairly directly from the parallel between the two analyses above of an accelerating observer and an observer fixed outside a black hole.

A note on black hole area and entropy

The *generalized second law of thermodynamics*,

$$\delta S_{\text{total}}=\delta S_{\text{matter}}+\delta S_{\text{BH}}\geq 0\,, \qquad \text{(M\&W pg. 122 unlabeled equation)} \tag{36}$$

or, in other words,

$$S_{\underset{\text{later}}{\text{total}}}\geq S_{\underset{\text{earlier}}{\text{total}}}\,. \tag{37}$$

(37) must also be true for two black holes merging, i.e.,

$$S_{\text{merged}}\geq S_{\text{BH1}}+S_{\text{BH2}}\,. \tag{38}$$

But if black hole entropy is proportional to black hole area, as in

$$S_{\text{BH}}=\frac{1}{4}A \qquad \text{LHS of M\&W (9.32),} \tag{39}$$

then (38) must mean

$$A_{\text{merged}} \geq A_{\text{BH1}} + A_{\text{BH2}} \,. \tag{40}$$

<u>Proof of (40)</u>

To prove (40), note

$$A_{\text{BH}_1} = 4\pi r_{g_1}^2 = 4\pi\left(2M_1\right)^2 = 16\pi M_1^2 \qquad A_{\text{BH}_2} = 16\pi M_2^2 \qquad A_{\text{merged}} = 16\pi M_{\text{merged}}^2 \,, \tag{41}$$

$$M_{\text{merged}} = M_1 + M_2 \,. \tag{42}$$

So,

$$\begin{aligned} A_{\text{merged}} &= 16\pi\left(M_1 + M_2\right)^2 = 16\pi M_1^2 + 16\pi M_1^2 + 32\pi M_1 M_2 = A_{\text{BH1}} + A_{\text{BH2}} + 32\pi M_1 M_2 \\ &\geq A_{\text{BH1}} + A_{\text{BH2}} \,. \end{aligned} \tag{43}$$

<u>End of proof</u>

Appendix (for these notes, not of M&S Chap. 9)

Note that in a gravitational field, for a particle of (rest) mass m,

$$\text{invariant} \quad m^2 = g_{\mu\nu} p^\mu p^\nu \,. \tag{44}$$

If the gravitational field is a Schwarzchild field, then from

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2\left(d\theta^2 + d\phi^2 \sin^2\theta\right) \qquad \text{M\&W (9.1), pg. 110,} \tag{45}$$

and we have (2D part only with other dimensions suppressed),

$$g_{\mu\nu} = \begin{bmatrix} 1 - \frac{2M}{r} & 0 \\ 0 & \frac{1}{1 - \frac{2M}{r}} \end{bmatrix} \qquad \text{Schwarzchild metric in 2D} \,. \tag{46}$$

From (45), at a fixed location ($dr = d\theta = d\phi = 0$),

$$ds = d\tau = \sqrt{1 - \frac{2M}{r}}\, dt \,. \tag{47}$$

Thus,

$$p^\mu = mu^\mu = m\begin{bmatrix} \frac{dt}{d\tau} \\ \frac{dr}{d\tau} \end{bmatrix} = m\begin{bmatrix} \frac{1}{\sqrt{1 - \frac{2M}{r}}} \\ 0 \end{bmatrix} \quad \rightarrow \quad u^\mu = \begin{bmatrix} \frac{1}{\sqrt{1 - \frac{2M}{r}}} \\ 0 \end{bmatrix} \,, \tag{48}$$

and plugging (48) and (46) into (44) proves (44).

Now consider the stress-energy tensor in a Schwarchild field for mass at rest (where ρ is rest mass-energy density)

$$T^{\mu\nu} = \rho u^\mu u^\nu = \rho \frac{1}{1 - \frac{2M}{r}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \rightarrow \quad \text{mass-energy density} = T^{00} \,. \tag{49}$$

Note T^{00} is positive outside the black hole horizon (Schwarzchild radius $r = r_g = 2M$), but negative inside.

The analysis is a bit more complicated if the particle has velocity, and even more complicated if the black hole is rotating, but the final conclusion is similar (though the line dividing negative and positive mass-energy density may shift from the Schwarzchild radius.)