Part One

Mathematical Preliminaries

"Mathematics is the language in which God has written the universe."
Galileo Galilei

Chapter 2 *Draft vers: 5/25/2022 copyright of Robert D. Klauber*

Group Theory

"I believe that ideas such as absolute certitude, absolute exactness, final truth, etc. are figments of the imagination which should not be admissible in any field of science... This loosening of thinking seems to me to be the greatest blessing which modern science has given to us."

Max Born

2.0 Introduction

In Part One of this book, we cover topics that could be ignored for the canonical approach to QED (and were, in fact, ignored in Vol. 1), but play major roles elsewhere in QFT. These are

- group theory (this chapter), and
- other relevant math topics (Chapter 3).

The former is used throughout the theories of electroweak and strong interactions. The latter play essential roles in the path integral approach to all standard model (SM) interactions and include Green functions, Grassmann variables, and the generating functional, all of which are probably only names for most readers at this point.

Hopefully, for many, at least part of the group theory presentation will be a review of course work already taken. Also, hopefully, it will be sufficient for understanding, as treated in latter parts of this book, the structural underpinning that group theory provides for the SM of QFT.

As always, we will attempt to simplify, in the extreme, the presentations of these topics, without sacrificing accuracy. And we will only present the essential parts of group theory needed for QFT. For additional applications of the theory in physics (such as angular momentum addition in QM), presented in a pedagogic manner, I suggest Jeevanjee[1], McKenzie[2] and Schwichtenberg[3].

2.1 Overview of Group Theory

Areas of study: groups, fields, vector spaces, and algebras

Group theory is, in one sense, the simplest of the theories about mathematical structures known as groups, fields, vector spaces, and algebras, but in another sense includes all of these, as the latter three can be considered groups endowed with additional operations and axioms. Wholeness Chart 2-1 provides the basic defining characteristics of each of these types of structures and provides a few simple examples. The "In Common" column lists characteristics all structures share. "Particular" lists specific ones for that particular structure. Hopefully, there is not too much new in the chart for most readers, but even if there is, it, though compact, should be a relatively intelligible introduction.

Note that for algebras, the first operation has all the characteristics of a vector space. The second operation, on the other hand, does not necessarily have to be associative, have an identity element or inverse elements in the set, or be commutative.

Algebras may or may not be associative, unital, or commutative

An algebra with (without) the associative property for the second operation is called an associative (non-associative) algebra. An algebra with (without) an identity element for the second operation is called a unital (non-unital) algebra or sometimes a unitary (non-unitary) algebra. We will avoid the second term as it uses the same word (unitary) we reserve for probability conserving operations. A unital algebra is considered to possess an inverse element under the 2nd operation (for the first operation, it already has one, by definition), for every element in the set.

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[1] Jeevanjee, N., *An Introduction to Tensors and Group Theory for Physicists*, 2nd ed., (Birkhäuser/Springer 2015).
[2] McKenzie, D., An Elementary Introduction to Lie Algebras for Physicists, https://www.liealgebrasintro.com/
[3] Schwichtenberg, J., *Physics from Symmetry*, 2nd ed., (Springer 2018).

Wholeness Chart 2-1. Synopsis of Groups, Fields, Vector Spaces, and Algebras

Type of Structure	Elements	Main Characteristics	Examples (A, B, C, D = elements in set)	Other Characteristics In Common	Other Characteristics Particular
Group	Set of elements	1 (binary) operation "binary" = between two elements in the set	#1: Real numbers under addition. $\mathbf{A}+\mathbf{B}=\mathbf{C}$, e.g., 2+3 = 5 #2: 2D rotations (can be matrices) under multiplication. $\mathbf{AB}=\mathbf{C}$	Closure; associative; identity; inverse;	May or may not be commutative
Field	As above	2 (binary) operations Commonly, addition and multiplication	#1: Real numbers under addition & multiplication. 2•3 + 4 = 10 #2: Complex numbers under addition & multiplication. $(1+2i)(1-2i)+(2+4i)=7+4i$	1st operation: As in block above 2nd operation: All but inverse as in block above	Both operations commutative: 2nd distrib. over 1st 2nd operation: Inverse not required
Vector Space	Set of (vector) elements & 2nd set of scalars	1 (binary) operation & 1 scalar multiplication "scalar" = element of a field	#1: 3D vectors under vec addition & scalar multip. $3\mathbf{A}+2\mathbf{B}=\mathbf{C}$ #2: Elements are matrices with matrix addition & scalar multip. $3\mathbf{A}+2\mathbf{B}=\mathbf{C}$ #3: Hilbert space in QM	As in top block above	Commutative; distributive for scalar multip with vector operation; may have inner product, i.e., $\mathbf{A}\bullet\mathbf{B}$ = scalar
Algebra	As above	2 (binary) operations & 1 scalar multiplication	#1: 3D vectors under vec addition, vec cross product, scalar multip $3\mathbf{A}\times\mathbf{B}+2\mathbf{D}=\mathbf{C}$ #2: Matrices under matrix addition, matrix multip, scalar multip. $3\mathbf{AB}+2\mathbf{D}=\mathbf{C}$ #3: Matrices under matrix addition, matrix commut, scalar multip. $3[\mathbf{A},\mathbf{B}+\mathbf{D}]=\mathbf{C}$	1st operation (often addition): As in top block above 2nd operation: Closure	2nd operation distrib over 1st 1st operation: Commutative 2nd operation: Not required to be associative, have identity, have inverses, be commutative

Definitions (**A, B, C** represent any & all elements in set. ∘ denotes a binary operation)		**Examples**
Closure	All operations on set elements yield an element in the set. $\mathbf{A}\circ\mathbf{B}=\mathbf{C}$	All **C** in Examples column above are in original set of elements.
Associative	$\mathbf{A}\circ(\mathbf{B}\circ\mathbf{C})=(\mathbf{A}\circ\mathbf{B})\circ\mathbf{C}$	Real numbers, rotations, matrices, vectors, all under addition or multiplication.
Identity	There is an element **I** of the set with the property $\mathbf{A}\circ\mathbf{I}=\mathbf{I}\circ\mathbf{A}=\mathbf{A}$	Real number addition, **I** = 0. Matrix multiplication, **I** = identity matrix.
Inverse	For each **A** in the set, there is a unique element $\mathbf{A}^{-1}$ of the set with the property $\mathbf{A}\circ\mathbf{A}^{-1}=\mathbf{A}^{-1}\circ\mathbf{A}=\mathbf{I}$	Real number addition, $\mathbf{A}^{-1}=-\mathbf{A}$. Matrix multiplication, $\mathbf{A}^{-1}$ = matrix inverse of **A**
Commutative	$\mathbf{A}\circ\mathbf{B}=\mathbf{B}\circ\mathbf{A}$	Real number addition and multiplication. Vector addition. Non-commutative examples: 3D rotation, creation & destruction operators in QFT under multip.
Distributive	For two binary operations (✞ = another binary operation) $\mathbf{A}\circ(\mathbf{B}✞\mathbf{C})=(\mathbf{A}\circ\mathbf{B})✞(\mathbf{A}\circ\mathbf{C})$ and $(\mathbf{B}✞\mathbf{C})\circ\mathbf{A}=(\mathbf{B}\circ\mathbf{A})✞(\mathbf{C}\circ\mathbf{A})$	Real numbers: ✞ as addition, ∘ as multip. Matrices: ✞ as addition, ∘ as multip.

Doing **Problems 1 and 2** may help in understanding groups, fields, vector spaces, and algebras.

In this chapter, we will focus on a particular kind of group, called a Lie group, which will be defined shortly. We will then show how a Lie group can be generated from the elements of an associated algebra, called, appropriately, a Lie algebra. After that, we see how Lie groups and Lie algebras play a key role in the SM of QFT. For an overview chart of where we are going, check out Wholeness Chart 2-11 in the chapter summary on page 59. Don't worry too much about some of what is now undefined terminology in that chart. You will understand that terminology soon enough.

SM underlying structure → Lie groups and Lie algebras

2.1.1 A Set of Transformations as a Group

Consider the set of rotation transformations in 3D (in particular, it will be easier to think in terms of active transformations [see Vol. 1, pg. 164]), a typical element of which is symbolized by **A** herein. Such transformations can act as operations on a 3D vector, i.e., they rotate the 3D vector, which we designate by the symbol **v**. In the transformation, the vector **v** is rotated to a new position, designated **v′**. The transformations **A** comprise an abstract expression of rotation in physical space, i.e., **A** signifies rotation independent of any particular coordinate system. Any element **A** does the same thing to a given vector regardless of what coordinate system we choose to view the rotation operation from.

Rotation transformations = a set of elements

Now, if we select a given coordinate system, we can represent elements **A**, in one manner, as matrices, whose components depend on the coordinate system chosen. For practical applications, and for aid in teaching, we almost always have to express rotations as matrices.

One way to represent rotations is via matrices

$$\underbrace{\mathbf{A}\mathbf{v}=\mathbf{v}'}_{\substack{\text{Abstract}\\ \text{form}}} \xrightarrow[\text{and column vectors}]{\text{expressed as matrix}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix}. \tag{2-1}$$

Note that **A** has the characteristics delineated in Wholeness Chart 2.1 for a group. In particular, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are two members of this set of transformations, then rotating the vector first via $\mathbf{A}_1$ and then via $\mathbf{A}_2$ is symbolized by

$$\mathbf{A}_2 \circ \mathbf{A}_1 \mathbf{v} = \mathbf{v}'' = \mathbf{C}\mathbf{v} \qquad \text{where } \mathbf{C} = \mathbf{A}_2 \circ \mathbf{A}_1 \text{ is a member of the set of 3D rotations .} \tag{2-2}$$

Therefore, the rotation transformation set has closure and the binary operation, when the transformation is expressed in matrix form, is matrix multiplication. Further, the operation on set members (we are talking operation between matrices here [in the matrix representation], not the operation of matrices on vectors) is associative, and inverses ($\mathbf{A}^{-1}$ for each **A**) and an identity (**I**) exist. Further, there is no other operation, such as addition, involved for the members of the set. (In the matrix representation, two successive transformations involve matrix multiplication, not matrix addition.) Hence, the transformations **A** form a group.

Set of rotation transformations satisfy criteria to be a group

Note, this rotation example is a non-commutative group, since

$$\text{in general, } \mathbf{A}_2 \circ \mathbf{A}_1 \neq \mathbf{A}_1 \circ \mathbf{A}_2 . \tag{2-3}$$

You can prove this to yourself by rotating a book 90^o ccw (counterclockwise) along its binder axis first, then 90^o ccw along its lower edge second; and then starting from the same original book position and reversing the order of the rotation operations. The book ends up in different final positions.

A non-commutative group is denoted a non-Abelian group. Note that some pairs of elements in a non-Abelian group can still commute, just *not any and all* pairs. A group in which all elements commute is an Abelian group.

Set of 3D rotation transformations is non-Abelian (non-commutative)

2.1.2 Groups in QFT

As insight into where we are going with this, recall from Vol. 1 (see pg. 196, first row of eq. (7-49)) that the *S* operator in QFT transforms an initial (eigen) state $|i\rangle$ into a final (general) state $|F\rangle$ (that's what happens during an interaction).

In a similar way, the set of S operator transformations on QFT states forms a group

$$S\,|\,i\,\rangle = |\,F\,\rangle . \tag{2-4}$$

But that state could be further transformed (via another transformation) into another state $|\,F'\,\rangle$. So, for two such transformations S_1 and S_2, we would have

$$S_2 S_1 | i \rangle = S_3 | i \rangle = | F' \rangle . \qquad (2\text{-}5)$$

Recall also from Vol. 1 (pg. 195, eq. (7-43)) that the S operator could be represented by a matrix (S matrix) and the initial and final states by column vectors. The parallels between (2-1) and (2-4), and between (2-2) and (2-5), should allow us to surmise directly that the set of all transformations (interactions) in QFT form a group. And so, the mathematics of groups should help us (and it does help us as we will eventually see) in doing QFT.

2.1.3 Quick Summary

1st Bottom line: A set of transformations on column vectors (or on QM states) can form a group. We can apply group theory to them (with or without considering the column vectors [or QM states]).

2nd Bottom line: The column vectors (or QM states) can form a vector space.

Do **Problem 3** to show this.

Group operation between group elements differs from what some groups have as operation of a group element on a vector

So, the group elements act as operators on the vectors (or QM states). Discern between *operations* (which are transformations) by group members *on vector space members* from the *group operation between group members* (matrix multiplication in our sample representation.)

2.1.4 Notation

__Bold__ = abstract elements of group or vector space
Non-bold = matrix representation of elements

We will generally use bold capital letters, such as **A**, for abstract group elements (which could characterize some operation in physical space, such as rotation); and non-bold capital letters, such as A, for matrix representations of abstract group elements. We will generally use bold lower-case letters, such as **v**, for abstract vector elements in a vector space; and non-bold lower-case letters, such as v, for column matrix (or row matrix) representations of those vectors. The binary operation on abstract elements $\mathbf{A} \circ \mathbf{B}$, for matrix multiplication in the matrix representation, will be expressed simply as AB.

2.1.5 Our Focus: Matrix Representations of Groups

We focus on groups with elements represented by matrices

Group theory as the study of abstract groups is extensive, deep, and far from trivial. When restricted to representations of groups as matrices, it becomes easier. Since practically, for our purposes and in much of the rest of physics, matrix group representation theory covers the bases one wants to cover, we will focus on that. So, when we use terms such as "group", "group theory', "group element", etc. from here on out, they will generally mean "matrix group", "matrix group theory", "matrix group element", etc. unless otherwise stated or obvious.

2.2 Lie Groups

Lie group elements vary continuously and smoothly with a continuous, smooth parameter

A Lie group is a group whose elements are continuous, smooth functions of one or more variables (parameters) which vary continuously and smoothly. (See examples below.)

This definition is a bit heuristic, but will suffice for our work with matrix groups. There are fancier, more mathematically precise definitions, particularly for abstract groups.

Lie groups are named after the Norwegian mathematician Sophus Lie, who was among the first to develop them in the late 1800s.

2.2.1 Representations of Lie Groups

A representation of a Lie group, for us, is a set of matrices

A representation of an abstract Lie group, for our purposes, is a matrix that acts on a vector space and depends on one or more continuous, smoothly varying independent parameters, and that itself varies continuously and smoothly. (More formally, it is an action of a Lie group on a vector space.) The vector space is called representation space. The term representation is also used to mean *both* the vector space and the matrix operators (matrix group elements) together.

2.2.2 A One Parameter Lie Group

A simple example of a Lie group is rotation in 2D, which can be represented by a matrix $\hat{M}$ (with a "hat" because we will use the symbol M for something else later) that operates on a 2D vector, i.e., it rotates the vector counter clockwise,

$$\hat{M}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{2-6}$$

A simple example

This group is characterized by a single parameter, the angle of rotation θ. As (2-6) is just a special case of the 3D rotations of (2-1), it forms a group. And because all of its elements can be generated continuously and smoothly by a smooth, continuous variation of a parameter, it is a Lie group.

Note that for $\theta = 0$, $\hat{M}(0) = I$, and this is a property all of our Lie groups will share, i.e., when all parameters equal zero, the matrix equals the identity. Some authors include this as part of the definition of Lie groups, as virtually all groups that prove useful in physical theory have this property.

When $\theta = 0$, $\hat{M}(0) = I$ key for representing real world

An example of a non-Lie group would be the set of 2D rotations through increments of 90º. The set of matrix elements would be (take $\theta = 0^o, 90^o, 180^o, 270^o$ in (2-6))

$$\hat{M}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \hat{M}_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \hat{M}_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \hat{M}_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2-7}$$

Example of a non-Lie group: discrete elements

Obviously $\hat{M}_1$ is the identity. $\hat{M}_4$ is the inverse of $\hat{M}_2$, and $\hat{M}_1$ and $\hat{M}_3$ are each their own inverses. Additionally, (you can check if you like) any two of (2-7) multiplied together yield one of the other elements (closure). Further, matrix multiplication is always associative. So, (2-7) form a group under matrix multiplication. But since the elements are discrete and do not convert one into the other via a continuous variable parameter, it is not a Lie group.

A property of the particular Lie group (2-6) (for two values of θ such as α and ϕ) is

$$\hat{M}(\alpha)\hat{M}(\phi) = \hat{M}(\alpha + \phi). \tag{2-8}$$

Property of this particular Lie group

This should be evident from our general knowledge of 2D rotations. Rotating first through 30º, then second through 13º degrees is the same as rotating through 43º degrees. The parameter θ varies continuously and smoothly and so does $\hat{M}(\theta)$.

As an aside, the 2D rotation group is commutative (Abelian), as rotating first by 13º, then second by 30º is the same as rotating first by 30º and second by 13º.

2.2.3 Orthogonal vs Special Orthogonal Groups

Both groups (2-6) and (2-7) are what are termed <u>special orthogonal</u> groups. "<u>Orthogonal</u>" means the elements of the group (represented by the matrices) are real and the transpose of the matrix is the same as its inverse, i.e., $\hat{M}^T = \hat{M}^{-1}$. Recall from linear algebra that the magnitude of a vector remains unchanged under an orthogonal transformation (as in rotation). "<u>Special</u>" means the determinant of each matrix in the group is unity. det $\hat{M} = 1$.

Orthogonal, $O(n)$, $\hat{M}$ a real matrix $\hat{M}^T = \hat{M}^{-1}$ (magnitude of vector invariant under $\hat{M}$)

Do **Problem 4** to investigate a 2D orthogonal Lie group matrix that is *not* special orthogonal and to help understand the significance of special orthogonal transformations.

Special Orthogonal, $SO(n)$, Det $\hat{M} = 1$

In the above problem solution, an orthogonal matrix acting on a vector maintains the norm (magnitude) of the vector unchanged. This, as noted above, is a general rule. The solution also suggests that non-special orthogonal Lie matrices do not produce continuous rotation of a vector.

Note further, that for the matrix of Problem 4, when $\theta = 0$, we do not have the identity matrix, so its action on a vector would change the direction of the vector. It is more advantageous in representing real world phenomena if there is no change in a vector when the continuous parameter(s) on which it depends is (are) zero, as in (2-6).

Hopefully, from these examples, we can begin to see some of the advantages of working with special orthogonal matrix Lie groups which equal the identity when the independent parameter(s) is (are) zero.

The shorthand notation for our special orthogonal example groups (2-6) and (2-7) is $SO(2)$. If the rotations were in 3D instead of 2D, as in (2-1), we denote it an $SO(3)$ group. For n dimensional space rotations, we would have <u>$SO(n)$</u>. If the determinant were not constrained to be equal to positive unity, the group would be simply an <u>orthogonal group</u>, symbolized by <u>$O(n)$</u> .

The number <u>n</u> in $SO(n)$ and $O(n)$ is known as the <u>degree of the group</u>. In our examples above, it is equal to the dimension of the vector space upon which our matrices act. However, there are subtleties

n in $O(n)$ and $SO(n)$ is the degree of the group

involved regarding the actions of groups on particular dimension spaces, which we will address later in the chapter.

Note that orthogonal and special orthogonal groups do not have to be Lie groups. For example, (2-7) is special orthogonal, but not a Lie group. We define these more general terms in this Section 2.2, which is specifically on Lie groups, because it is easiest to understand them in the context of the examples presented herein.

$O(n)$ and $SO(n)$ groups can be Lie or non-Lie groups

2.2.4 Different Parametrizations of the Same Group

Note that we can represent the $SO(2)$ rotation group in a different way as

Same group can have different parametrizations

$$\hat{M}(x)=\begin{bmatrix}\sqrt{1-x^2} & -x \\ x & \sqrt{1-x^2}\end{bmatrix} \tag{2-9}$$

where $x = \sin\theta$. We say that (2-9) is another parametrization of the same $SO(2)$ group of (2-6). A third parametrization is the transpose of (2-6),

$$\hat{M}(\theta')=\begin{bmatrix}\cos\theta' & \sin\theta' \\ -\sin\theta' & \cos\theta'\end{bmatrix} \tag{2-10}$$

where $\theta' = -\theta$.

<u>Bottom line</u>: A particular group is an abstract structure (which can, as in the above example, characterize 2D rotations; in the example of (2-1), 3D rotations) that can be represented explicitly via matrices and by using different parameters in different ways, called <u>parametrizations</u>.

2.2.5 A Lie Group with More than One Parameter: SO(3)

Of course, there are many Lie groups with more than one parameter. As one example, let us express the $SO(3)$ group of 3D rotations (2-1) as a function of certain angles (successive ccw rotations about different axes) θ_1, θ_2, and θ_3. Typically, for a solid object with three orthogonal axes visualized as attached to the object, the first rotation is about the x_3 axis; the second, about the x_2 axis; the third, about the x_1 axis. In this perspective, it is an active transformation about each axis.

Consider **A** in (2-1) as an abstract group element (characterized simply in that it performs rotations), and A as the particular parametrized matrix representation under consideration.

Example of Lie group with more than one parameter → 3D rotation

$$A(\theta_1,\theta_2,\theta_3)v=v' \quad\rightarrow\quad \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix}=\begin{bmatrix}v_1'\\ v_2'\\ v_3'\end{bmatrix} \qquad a_{ij}=a_{ij}(\theta_1,\theta_2,\theta_3) \tag{2-11}$$

Any A can be expressed via different parametrization choices, and with an eye to the future we will choose to build A from the following particular building block parametrizations.

$$A_1(\theta_1)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_1 & -\sin\theta_1\\ 0 & \sin\theta_1 & \cos\theta_1\end{bmatrix} \quad A_2(\theta_2)=\begin{bmatrix}\cos\theta_2 & 0 & \sin\theta_2\\ 0 & 1 & 0\\ -\sin\theta_2 & 0 & \cos\theta_2\end{bmatrix} \quad A_3(\theta_3)=\begin{bmatrix}\cos\theta_3 & -\sin\theta_3 & 0\\ \sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{bmatrix}. \tag{2-12}$$

There are a number of ways the matrices (2-12) can be combined to embody different (equivalent) forms of $SO(3)$, but to be consistent with our above noted order of rotations, we will use (2-13). Note the operations proceed from the right side to the left side in (2-13). (I suggest you save yourself the time by just accepting the algebra involved, though you can prove it to yourself, if you wish.)

One of many ways to embody that group using building blocks

$$\begin{aligned}A(\theta_1,\theta_2,\theta_3)&=A_1(\theta_1)A_2(\theta_2)A_3(\theta_3)\\ &=\begin{bmatrix}\cos\theta_2\cos\theta_3 & -\cos\theta_2\sin\theta_3 & \sin\theta_2\\ \cos\theta_1\sin\theta_3+\sin\theta_1\sin\theta_2\cos\theta_3 & \cos\theta_1\cos\theta_3-\sin\theta_1\sin\theta_2\sin\theta_3 & -\sin\theta_1\cos\theta_2\\ \sin\theta_1\sin\theta_3-\cos\theta_1\sin\theta_2\cos\theta_3 & \sin\theta_1\cos\theta_3+\cos\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_1\cos\theta_2\end{bmatrix}.\end{aligned} \tag{2-13}$$

The matrix A varies continuously and smoothly with continuous, smooth variation in the three parameters θ_1, θ_2, and θ_3. We could, of course, define the order of operation on the RHS of (2-13) differently and have a different embodiment of the same $SO(3)$ rotation group. Similarly, we could define our building blocks with different parameters (similar to the x in (2-9) for $SO(2)$ rotations) and have yet other, different parametrizations. Again, we see that a group itself is an abstract structure (which can characterize 3D rotations here) that can be expressed mathematically in different ways.

Key point:

Note the A_i of (2-12) are not bases in the vector space sense of spanning the space of all possible 3X3 matrices. 'Basis' refers to addition and scalar multiplication but the matrices we are dealing with involve multiplication. However, all possible 3D rotations, expressed as matrices, can be obtained from the three A_i, so they are the foundation of the group.

Don't confuse building blocks of groups with bases of vector spaces. They are generally different.

The 3D rotation group matrices form a subset of all 3X3 real matrices, and the reader should be able to verify this by doing the problem referenced below. Further, any group element can be formed, in this representation, by matrix multiplication of three group building blocks. But in a typical vector space, any element can be expressed as a linear combination (adding, not multiplying) of basis vectors (which are matrices here).

Similarly, in 2D rotations, we only had one matrix, such as (2-6), which is a function of one parameter (θ in the referenced parametrization). For a basis for matrices in 2D, we would need four independent matrices. Don't confuse the building blocks of a Lie group with the basis vectors of a linear vector space.

Do **Problem 5** to help illustrate the difference between matrices as vector space elements and matrices as group elements.

Small Values of the Parameters

For future reference, we note that for small (essentially infinitesimal) parameters θ_i, (2-13) becomes

$$A(\theta_1,\theta_2,\theta_3) \approx \begin{bmatrix} 1 & -\theta_3 & \theta_2 \\ \theta_3 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix} \qquad |\theta_i| \ll 1 \ . \tag{2-14}$$

2.2.6 Lorentz Transformations Form a Lie Group

The Lorentz transformation, with $c = 1$ and boost velocity v in the direction of the x^1 coordinate, is (where we use the Einstein summation convention, as in Vol. 1)

$$\Lambda^{\alpha}{}_{\beta}(v)\,dx^{\beta} = \begin{bmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} & & \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \begin{bmatrix} dx'^0 \\ dx'^1 \\ dx'^2 \\ dx'^3 \end{bmatrix} = dx'^{\alpha} \ . \tag{2-15}$$

The set of transformations Λ^{α}_{β} satisfies our criteria for a group. The elements Λ^{α}_{β} are subject to a single operation (matrix multiplication) under which the set has an identity member (when $v = 0$), obeys closure, is associative, and possesses an inverse for every member. The Lorentz transformations constitute a group.

Lorentz transformations form a group

Due to the Minkowski metric in special relativity's 4D spacetime, things get a little tricky comparing the Lorentz transformation to matrices in Euclidean space. So, even though the magnitude (vector "length" as in (2-47), pg. 32, of Vol. 1, sometimes called the "Minkowski norm") of the four-vector $|dx^{\beta}|$ remains invariant under the transformation (see Section 2.2.3 above), the inverse of $\Lambda^{\alpha}_{\beta}(v)$ is not its transpose, but $\Lambda^{\alpha}_{\beta}(-v)$. (You can check this or save time by just taking my word for it, as this material is a bit peripheral.) In a special relativity sense, therefore, the Lorentz transformation is considered orthogonal. As the determinant of Λ^{α}_{β} is unity (you can check using (2-15) and the rules for calculating 4D matrix determinants), it is special. To discern the special relativistic nature of this particular kind of orthogonality, the Lorentz group is denoted by $SO(3,1)$ [for 3 dimensions of space, and one of time.]

Note that the 4D transformation matrix is a continuous function of v, the velocity between frames, whose possible values vary continuously. So, the Lorentz group is a Lie group. The addition property (2-8) holds, but for relativistic velocity addition, i.e.,

$$\Lambda^{\alpha}{}_{\beta}(v)\Lambda^{\beta}{}_{\delta}(v') = \Lambda^{\alpha}{}_{\delta}(v'') \quad (v'' = \text{relativistic velocity addition of } v \text{ and } v'). \tag{2-16}$$

Extending the form of (2-15) to include the more general cases of 3D coordinate axes rotation plus boosts in any direction leads to the same conclusions. Lorentz transformations, i.e., boosts plus all possible 3D rotations, comprise a Lie group $SO(3,1)$, the so-called Lorentz group.

In particular, a special orthogonal Lie group, $SO(3,1)$

2.2.7 Complex Groups: Unitary vs Special Unitary

So far, we have looked exclusively at groups represented by real matrices. But since QM is replete with complex numbers, we need to expand our treatment to include representations of groups using complex matrices. See Vol.1, Box 2-3, pg. 27, to review some differences and similarities between real, orthogonal transformations and complex, unitary transformations.

Unitary groups (symbol $U(n)$ for degree n) are effectively the complex number incarnation of (real number) orthogonal groups. A matrix group is unitary if for every element M in it, $M^{-1} = M^{\dagger}$. (Compare to orthogonal matrices, which are real, and for which $\hat{M}^{-1} = \hat{M}^{\mathrm{T}}$.)

Unitary, $U(n)$, M complex matrix $M^{\dagger} = M^{-1}$ (magnitude of vector invariant under M)

Special unitary groups (symbol $SU(n)$) are the complex number incarnation of special orthogonal groups. A matrix group is special if for every element M in it, Det $M = 1$.

Special Unitary, $SU(n)$ Det $M = 1$

The Simplest Unitary Lie Group $U(1)$

As a simple case of a representation of a unitary group of degree 1 ($n = 1$), consider the set of "matrices" (of dimension 1) for continuous, real θ,

$$U(\theta) = e^{i\theta} \quad (\text{a representation of } U(1)). \tag{2-17}$$

Do **Problem 6** to show that $U(1)$ of (2-17) forms a group.

U in (2-17) is unitary because $U^{\dagger}U = I$. It is not special because Det U does not equal 1 for all θ. The only 1X1 matrix for which the determinant equals one corresponds to $\theta = 0$, i.e., the trivial case where the only member of the (special unitary) group is $U = 1$. Hence, we won't have much use for $SU(1)$. We will, however, make good use of $U(1)$.

Also, similar to an orthogonal transformation on a real vector, a unitary transformation on a complex vector leaves the magnitude of the vector unchanged.

Do **Problem 7** to prove the last statement.

If the vector happens to be a normalized QM state, this means that the total probability (to find the quantum system in *some* quantum state) remains unity under the action of the transformation U.

Unitary transformations operating on quantum state vectors leave probability unchanged

Additionally, since the set elements of (2-17) vary continuously and smoothly with the continuous, smooth variation of the parameter θ, (2-17) comprises a Lie group.

Given that QFT teems with complex numbers (of which operators and states are composed) and the theory is inherently unitary (conservation of total probability = 1 under transformations), we can expect to be focusing herein on special unitary groups.

Problem 8 can help in understanding how some physical world phenomena can be described by different kinds of groups.

The $SU(2)$ Lie Group

The $SU(2)$ Lie group is a very important group in QFT, as it is intimately involved in theoretical descriptions of the weak force. We will examine the group as represented by matrices that are 2X2, have complex components, and may operate on two component vectors in a complex vector space.

The $SU(2)$ Lie group, important in physics

A general two-dimensional complex matrix M has form

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad m_{ij} \text{ complex, in general}. \tag{2-18}$$

2X2 *complex matrices*

But since the group we will examine is special unitary, it must satisfy

satisfying special unitary group requirements

$$M^{\dagger}M = I \qquad\qquad Det\, M = 1. \tag{2-19}$$

I submit (we will show it) that (2-20) below satisfies (2-19), where a and b are complex.

$$M = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} = \begin{bmatrix} a_{Re} + ia_{Im} & b_{Re} + ib_{Im} \\ -b_{Re} + ib_{Im} & a_{Re} - ia_{Im} \end{bmatrix} \qquad \text{with} \quad aa^* + bb^* = 1 \tag{2-20}$$

One parametrization that does satisfy them

The RHS (last part) of (2-19) is obvious from the constraint imposed at the end of (2-20). To save you the time and tedium, I show the LHS (first part) of (2-19) below.

$$M^{\dagger}M = \begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix}\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} = \begin{bmatrix} a^*a + b^*b & a^*b - a^*b \\ ab^* - ab^* & b^*b + a^*a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2-21}$$

Now consider M as a matrix representation of a Lie group, where the a and b are continuous and smoothly varying. Since a and b are complex numbers, there are four real number variables $a_{Re}, a_{Im}, b_{Re}, b_{Im}$, which vary continuously and smoothly. From the constraint at the end of (2-20), only three of these are independent. They are related by

The Det M = 1 requirement means 1 parameter dependent on other 3 independent

$$a_{Re}^2 + a_{Im}^2 + b_{Re}^2 + b_{Im}^2 = 1, \tag{2-22}$$

and we choose a_{Im}, b_{Re}, b_{Im} to be independent, and $a_{Re} = a_{Re}\left(a_{Im}, b_{Re}, b_{Im}\right)$. For future reference, we find the partial derivative of a_{Re} with respect to each of the independent variables via (2-22).

$$\begin{aligned} \frac{\partial a_{Re}}{\partial a_{Im}} &= \frac{\partial}{\partial a_{Im}}\left(1 - a_{Im}^2 - b_{Re}^2 - b_{Im}^2\right)^{\frac{1}{2}} = \tfrac{1}{2}\left(1 - a_{Im}^2 - b_{Re}^2 - b_{Im}^2\right)^{-\frac{1}{2}}\left(-2a_{Im}\right) \\ &= -\frac{a_{Im}}{\sqrt{1 - a_{Im}^2 - b_{Re}^2 - b_{Im}^2}} = -\frac{a_{Im}}{a_{Re}} \quad \text{and} \quad \frac{\partial a_{Re}}{\partial b_{Re}} = -\frac{b_{Re}}{a_{Re}} \qquad \frac{\partial a_{Re}}{\partial b_{Im}} = -\frac{b_{Im}}{a_{Re}} \end{aligned} \tag{2-23}$$

Evaluating derivatives of dependent parameter

Note that when $a_{Im} = b_{Re} = b_{Im} = 0$, $a_{Re} = 1$ (where we assume the positive value for the square root in (2-22)), and the partial derivatives in the last line of (2-23) all equal zero.

For the Lie group, when the continuously variable independent parameters are all zero, nothing has changed, so we must have the identity element, and with (2-22), this is true for (2-20), i.e.,

$$M\left(a_{Im} = b_{Re} = b_{Im} = 0\right) = I. \tag{2-24}$$

For all independent parameters = 0, M = I

Thus, we have shown that M of (2-20) represents the $SU(2)$ Lie group of three real parameters.

Do **Problem 9** to show that M obeys the group closure property.

Using the *Re* and *Im* subscripts helped us keep track of which real parameters went where in the 2X2 matrix, but it will help us in the future if we change symbols, such that $a_{Re} = \alpha_0$, $b_{Im} = \alpha_1$, $b_{Re} = \alpha_2$, and $a_{Im} = \alpha_3$. Then,

$$M = \begin{bmatrix} a_{Re} + ia_{Im} & b_{Re} + ib_{Im} \\ -b_{Re} + ib_{Im} & a_{Re} - ia_{Im} \end{bmatrix} = \begin{bmatrix} \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{bmatrix} = i\begin{bmatrix} -i\alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -i\alpha_0 - \alpha_3 \end{bmatrix}. \tag{2-25}$$

<u>The $SU(3)$ Lie Group</u>

The $SU(3)$ Lie group is also an important group in QFT, as it is intimately involved in theoretical descriptions of the strong force. We will examine the group as represented by matrices, which are 3X3, have complex components, and may operate on three-component vectors in a vector space. As you may be surmising, in strong interaction theory, the three components of the vectors will represent three quark eigenstates, each with a different color charge (eigenvalue). More on that later in the book.

The $SU(3)$ Lie group relevant to the strong interaction

A general three-dimensional complex matrix N has form

$$N=\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{23} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} \qquad n_{ij} \text{ complex, in general} \tag{2-26}$$

Unitary and special properties a 3X3 matrix must have to represent SU (3)

Since the group we will examine is special unitary, it must satisfy

$$N^\dagger N = I \qquad\qquad Det\, N = 1 \tag{2-27}$$

Using (2-27) with (2-26) would lead us, in similar fashion to what we did in $SU(2)$, to one dependent real variable and eight independent ones upon which the dependent one depends. But doing so requires an enormous amount of complicated, extremely tedious algebra, and there are many possible choices we could make for independent variables. Further, with eight such variables, the final result would be far more complicated than (2-13) for $SO(3)$ or (2-25) for $SU(2)$, both of which have only three (where α_0 depends on the other α_i in (2-25)).

SU (3) Lie group relations parallel those of SO (3)

So, we will instead jump to a result obtained by Murray Gell-Mann in the mid-20th century. That result uses a particular set of independent variables that end up working best for us in QFT. Additionally, it is a much-simplified version of N that applies only in cases where the independent variables are very small, essentially infinitesimal. By so restricting it, a whole lot of cumbersome terms are dropped. (Compare with the parallel situation of (2-13) simplified to (2-14).) Fortunately, the small independent variable case will be all we need for our work with the SM.

The Gell-Mann $SU(3)$ matrix is

Gell-Mann SU(3) matrix useful in QFT, $|\alpha_i| << 1$

$$N(\alpha_i)=i\begin{bmatrix} -i+\alpha_3+\frac{\alpha_8}{\sqrt{3}} & \alpha_1-i\alpha_2 & \alpha_4-i\alpha_5 \\ \alpha_1+i\alpha_2 & -i-\alpha_3+\frac{\alpha_8}{\sqrt{3}} & \alpha_6-i\alpha_7 \\ \alpha_4+i\alpha_5 & \alpha_6+i\alpha_7 & -i-\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix} \qquad |\alpha_i| \ll 1 \tag{2-28}$$

Note the similarity between (2-28) for $SU(3)$ and (2-25) for $SU(2)$. However, (2-25) is good globally, whereas (2-28) is only good locally, essentially because $SU(2)$ is a simpler theory than $SU(3)$.

Note that $N(0) = I$. Note also (where "*HOT*" means higher order terms) that

$$Det\, N \text{ for } |\alpha_i| \ll 1$$
$$\approx 1-i\left(-\alpha_3+\frac{\alpha_8}{\sqrt{3}}\right)-i\left(-\frac{2\alpha_8}{\sqrt{3}}\right)-i\left(\alpha_3+\frac{\alpha_8}{\sqrt{3}}\right)-i\left(-\frac{2\alpha_8}{\sqrt{3}}\right)-i\left(\alpha_3+\frac{\alpha_8}{\sqrt{3}}\right)-i\left(-\alpha_3+\frac{\alpha_8}{\sqrt{3}}\right)+HOT \approx 1, \tag{2-29}$$

and

$$\underset{|\alpha_i|\ll 1}{N^\dagger N \approx} \begin{bmatrix} i+\alpha_3+\frac{\alpha_8}{\sqrt{3}} & \alpha_1-i\alpha_2 & \alpha_4-i\alpha_5 \\ \alpha_1+i\alpha_2 & i-\alpha_3+\frac{\alpha_8}{\sqrt{3}} & \alpha_6-i\alpha_7 \\ \alpha_4+i\alpha_5 & \alpha_6+i\alpha_7 & i-\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -i+\alpha_3+\frac{\alpha_8}{\sqrt{3}} & \alpha_1-i\alpha_2 & \alpha_4-i\alpha_5 \\ \alpha_1+i\alpha_2 & -i-\alpha_3+\frac{\alpha_8}{\sqrt{3}} & \alpha_6-i\alpha_7 \\ \alpha_4+i\alpha_5 & \alpha_6+i\alpha_7 & -i-\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix}$$

$$\approx \begin{bmatrix} 1+i\alpha_3+i\frac{\alpha_8}{\sqrt{3}}-i\alpha_3-i\frac{\alpha_8}{\sqrt{3}}+HOT & i\alpha_1+\alpha_2-i\alpha_1-\alpha_2+HOT & i\alpha_4+\alpha_5-i\alpha_4-\alpha_5+HOT \\ -i\alpha_1+\alpha_2+i\alpha_1-\alpha_2+HOT & 1-i\alpha_3+i\frac{\alpha_8}{\sqrt{3}}+i\alpha_3-i\frac{\alpha_8}{\sqrt{3}}+HOT & i\alpha_6+\alpha_7-i\alpha_6-\alpha_7+HOT \\ -i\alpha_4+\alpha_5+i\alpha_4-\alpha_5+HOT & -i\alpha_6+\alpha_7+i\alpha_6-\alpha_7+HOT & 1-i\frac{2\alpha_8}{\sqrt{3}}+i\frac{2\alpha_8}{\sqrt{3}}+HOT \end{bmatrix} \tag{2-30}$$

$$\approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So, for small values of the 8 parameters, N is special unitary and equals the identity when all parameters are zero

Note also that taking all imaginary terms in N as zero, we should get an $SO(3)$ group. Doing so with (2-28), we get A of (2-14) (with a relabeling of variables such that $\theta_3 = -\alpha_2$, $\theta_2 = \alpha_5$, and $\theta_1 = -\alpha_7$.)

2.2.8 Direct, Outer, and Tensor Products

<u>The Concept</u>

In group theory applications, one commonly runs into a type of multiplication known as the direct product, which is also common in linear algebra.

Consider an example where elements of two groups are represented by matrices, and although they could have the same dimension, in our case the first matrix A has dimension 3, and the second matrix B, dimension 2. Note the <u>symbol X</u> in (2-31) represents <u>direct product</u>.

$$\mathbf{A}\times\mathbf{B}\xrightarrow[\text{represented by matrices}]{\text{direct product of groups}} A\times B=\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}\times\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}$$

$$=\begin{bmatrix}A_{11}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix} & A_{12}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix} & A_{13}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}\\ A_{21}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix} & A_{22}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix} & A_{23}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}\\ A_{31}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix} & A_{32}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix} & A_{33}\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}\end{bmatrix} \tag{2-31}$$

Essentially, this is a component-wise operation wherein components live side-by-side and do not mix. The direct product is also called the <u>Cartesian product</u>. In index notation,

Direct product definition

$$A\times B\xrightarrow[\text{notation}]{\text{in index}} = A_{ij}B_{k'l'}\,. \tag{2-32}$$

Direct product indices = indices of constituent matrices

The result (2-32) has four different indices, 2 indices for A_{ij} plus 2 indices for $B_{k'l'}$, with different dimensions for each index. The i index dimension is 3; the j dimension is 3; the k' dimension is 2; and the l' dimension is 2. The total number of components is $3 \times 3 \times 2 \times 2 = 36$. Said another way, it is the total number of components of A times the total number of components of B, i.e., $9 \times 4 = 36$.

Number of components = product of number of components of constituent matrices

Note that, in this example, we use primes for the indices of the $B_{k'l'}$ in (2-32) to make it clear that generally the $B_{k'l'}$ components live in a completely different world from the components of A_{ij}. That is, the factors with unprimed subscripts in each element in (2-32) are usually (but not always) of different character from the factors with primed subscripts. Keep this in mind in future work, whether or not we employ primed subscripts to differentiate component factors.

We have already used the direct product concept in Vol. 1 (see (4-123), pg. 115, reproduced as (2-33) below), and there it was called the *oute*r product (of spinor fields).

$$\underbrace{\psi\bar{\psi}}_{\substack{\text{not writing}\\ \text{out spinor}\\ \text{indices}}} = \underbrace{\psi_\alpha\bar{\psi}_\beta}_{\substack{\text{with spinor}\\ \text{indices}\\ \text{written}}} = \psi_\alpha\psi^\dagger_\delta\gamma^0_{\delta\beta} = X_{\alpha\beta} = \text{outer product, a matrix quantity in spinor space} \tag{2-33}$$

Similar example from prior work

Here, the outer product is symbolized by having the adjoint (complex conjugate transpose) factor on the RHS. (The inner product is symbolized by having it on the LHS and yields a scalar, rather than a matrix quantity.) Spinors are essentially vectors (4D in QFT) in spinor space.

Outer product of two vectors = a matrix

Note that a direct product can be carried out between many different types of mathematical objects. (2-31) [i.e., (2-32)] is a direct product of matrix *group* representations. (2-33) is a direct product of *spinors* (vectors in spinor space). For historical reasons, the latter is more commonly called the outer product.

Direct and outer prods similar in concept, but former term used for matrices, the latter typically for vectors

The general direct product principle, regardless of the mathematical entities involved, is this. We get a separate component in the result for each possible multiplication of one component in the first entity times one component in the second.

In tensor analysis, tensors are commonly represented by matrices, and one often runs into what could be called a direct product of tensors, which looks just like (2-31) [i.e., (2-32)]. However, that term rarely seems to be used, and the almost universally used one is tensor product. For tensors, ⊗ is typically employed instead of X.

Tensor product like direct product but doesn't go by that name, and usually employs different symbol

Note that a vector is a 1st rank tensor. (A scalar is a zeroth rank tensor. What we usually think of as a tensor is a 2nd rank tensor.) In that context, the operation of what we might expect to be referred to as the direct product of two vectors is instead commonly labeled the outer product or the tensor product. For two vectors **w** and **y**, it is denoted by $\mathbf{w} \otimes \mathbf{y}$.

Note further that tensor products do not generally commute. $\mathbf{w} \otimes \mathbf{y}$ does not equal $\mathbf{y} \otimes \mathbf{w}$, except for special cases.

The various terminologies are probably a little confusing. (They were for me while learning it.) But generally, the term direct product is conventionally applied to groups and matrices, and employs the symbol X. The term tensor product is typically used for any tensors, including vectors, and commonly employs the symbol ⊗. As noted, another, very common term for the tensor product of vectors, in particular, is outer product.

Summary of the nomenclature: direct product, tensor product, outer product

A Hypothetical Example

Now consider **A** and **B** being operators that operate on vectors **v** in a vector space. Imagine the quantities represented by vectors have two characteristics. They are colored, and they are charged. The colors are red, green, blue and the charges are + and –. We represent a given vector **v** as a tensor product (which is an outer product) of a 3D vector **w** representing color (*r, g, b* symbolizing the particular color component in the vector) and a 2D vector **y** representing charge (*p* symbolizing the positively charged component; *n*, the negatively charged).

A hypothetical example for illustrative purposes

$$\mathbf{v} = \mathbf{w} \otimes \mathbf{y} \quad \xrightarrow[\text{by column and row matrices}]{\text{tensor product represented}} \quad v = \begin{bmatrix} r \\ g \\ b \end{bmatrix} \begin{bmatrix} p & n \end{bmatrix} = \begin{bmatrix} rp & rn \\ gp & gn \\ bp & bn \end{bmatrix} \quad \text{or} \quad v_{jl} = w_j y_l \,. \qquad (2\text{-}34)$$

In one sense, the vector **v** is a matrix (2 columns, 3 rows), but in the abstract vector space sense it is a vector in a vector space, where that space is comprised of 3X2 matrices.

The **A** operator here is related to color and thus acts only on the **w** part of **v**. It is blind to the **y** part. The **B** operator is related to charge and acts only on the **y** part. So, given (2-31) and (2-32),

A particular operator commonly acts on only one of the constituent vectors of a tensor product element

$$(\mathbf{A} \times \mathbf{B})\mathbf{v} = (\mathbf{A} \times \mathbf{B})(\mathbf{w} \otimes \mathbf{y}) = \mathbf{A}\mathbf{w} \otimes \mathbf{B}\mathbf{y} = \mathbf{v}'' \quad \xrightarrow[\text{notation}]{\text{in index}} \quad A_{ij} B_{k'l'} v_{jl'} = A_{ij} B_{k'l'} w_j y_{l'} = v''_{ik'} \quad (2\text{-}35)$$

Do **Problem 10** to show (2-35) in terms of matrices.

You are probably already considering the color part of the (state) vector **v** above in terms of the strong force, and that, in fact, is why I choose color as a characteristic for this example. Similarly, the same state represented symbolically by **v** may have a particular weak interaction charge.

When we get to weak interactions, we will see that operators like **B** act on a two-component state (such as a quark or lepton). We can imagine a particular **B** operator, a weak charge operator, acting on the state that would yield an eigenvalue equal to its weak charge.

With strong interactions, we will see that operators like **A** act on a 3-component state (such as a quark). For example, a quark state having a given color *r* (red) would be in a color eigenstate (with zero values for *g* and *b* in w_j), and we could imagine a particular **A** operator, a color operator, acting on it that would yield an eigenvalue corresponding to red. This is all very rough around the edges, and not completely accurate, since we are trying to convey the general idea as simply as possible. Many details and modifications will be needed (in pages ahead) to make it correct and more complete.

Direct Products and the Standard Model

In fact, the famous $SU(3)$X$SU(2)$X$U(1)$ relation of the SM symbolizes the action of three groups (strong/color interaction operators in $SU(3)$, weak interaction operators in $SU(2)$, and QED operators in $U(1)$) whose operators act on fields (vectors in the mathematical sense here). Each field has a separate part (indices) in it for each of the three interaction types. And each such part of the field is acted upon only by operators associated with that particular interaction. More on this, with examples, later in this chapter.

SM: direct product of 3 group reps $SU(3)$X$SU(2)$X$U(1)$

To Summarize

The direct product of group matrix representations acts on the tensor product of vector spaces. Each original group acts independently on each individual vector space.

An aside

An aside on spin and group theory

The following will not be relevant for our work, but I mention it in case you run across it in other places (such as the references in the footnotes on pg. 8). Do not spend too much time scratching your head over this for now, but save it to come back to if and when you run into it elsewhere.

In some applications, the two parts of the state vector, such as **w** and **y** in (2-34), respond to the same operator(s). For example, a spin 1 particle interacting with a spin ½ particle in NRQM can both be operated on by a spin operator. When two such particles interact to form a bound system, that total system has six possible spin eigenstates (four states with $J_{tot} = 3/2$, $J_z = 3/2, 1/2, -1/2, -3/2$, and two with $J_{tot} = 1/2$, $J_z = 1/2, -1/2$). The state vector of the system is the outer product of the spin 1 state multiplied by the spin ½ state. Both parts of the system state vector relate to spin and both parts are acted on by a spin operator.

The two parts of the system state vector have, respectively, 3 spin components (spin 1 particle has eigenstates $J_z = 1, 0, -1$) and 2 spin components (spin ½ particle has eigenstates $J_z = 1/2, -1/2$). The spin operator acts in the 3D space of the spin 1 particle and also in the 2D space of the spin ½ particle.

In that case, instead of a 3X2 state vector matrix with six components representing the system, one can formulate the math using a six-component column vector for the system. And then the spin operator for the system becomes a 6X6 matrix, instead of a 3X3 matrix group direct product multiplied by a 2X2 matrix group.

This is commonly done and can be confusing when one considers direct products defined as in (2-31) (equivalently, (2-32)), as we do here.

End of aside

Things to Note

In NRQM, we have already used the concept of separate operators acting independently on state vectors, where we have free particle states like

NRQM example of different operators operating on different parts of state vector (i.e., on an outer (tensor) product)

$$\text{NRQM spin up state} \qquad \psi_{state} = Ae^{-i(Et-\mathbf{k}\cdot\mathbf{x})}\begin{bmatrix}1\\0\end{bmatrix}. \tag{2-36}$$

The Hamiltonian operator $H = i\frac{\partial}{\partial t}$ acts on the $e^{-i(Et-\mathbf{k}\cdot\mathbf{x})}$ part of the wave function and does nothing to the 2-component spinor part. The spin operator S_z operates on the spinor part, but not the $e^{-i(Et-\mathbf{k}\cdot\mathbf{x})}$ part.

We commonly write the outer (tensor) products of two vectors as a column vector on the left times a row vector on the right, as in (2-34). An inner product, conversely, is denoted by a row vector on the left with a column vector on the right, as most readers have been doing for a long time.

Often write inner product as row on left, column on right; outer product as column on left, row on right

$$\mathbf{w}_1\cdot\mathbf{w}_2 \xrightarrow[\text{row on left and column on right}]{\text{inner product represented by}} = \begin{bmatrix} r_1 & g_1 & b_1\end{bmatrix}\begin{bmatrix}r_2\\g_2\\b_2\end{bmatrix} = r_1r_2 + g_1g_2 + b_1b_2 \tag{2-37}$$

But index notation is most foolproof

However, as noted in the solutions book, in the answer to Problem 10, the row vs column vector methodology can become cumbersome, and even a bit confusing, at times. Of all the choices, the most foolproof notation is the index notation, as on the RHS of (2-34) and (2-35).

Additionally, be aware that the tensor product is defined for tensors of all ranks, not just rank 1 (vectors). For example, for two 2nd rank tensors expressed as matrices T_{ij} and S_{kl}, the tensor product is $T_{ij}S_{kl} = Z_{ijkl}$, a rank 4 tensor.

Finally, and likely to cause even more confusion, the literature is not 100% consistent with use of the term "direct product". Further, take care that the symbol ⊗ is often used for it, and sometimes X is used for tensor product.

2.2.9 Summary of Types of Operations Involving Groups

Wholeness Chart 2-2 lists the types of operations associated with groups that we have covered.

Wholeness Chart 2-2. Types of Operations Involving Groups and Vector Spaces

Operation	What	Type	Relevance	In Matrix Representation
$\mathbf{A} \circ \mathbf{B}$	Group operation (binary)	Between 2 elements **A** and **B** of the group	Defining characteristic of the group	Matrix multiplication, AB
$\mathbf{Av}$	Group action on a vector space	A group element **A** operates on a vector **v**	Some, but not all groups, may do this.	Matrix multiplication with column vector, Av
$\mathbf{A}\times\mathbf{B}$	Direct product of group elements	Combining groups (**A** & **B** here symbolize entire groups **A** & **B**.)	Larger group formed from 2 smaller ones	Larger group $C_{ijk'l'} = A_{ij}B_{k'l'}$ has indices of A and indices of B*
$\mathbf{w} \otimes \mathbf{y}$	Outer (tensor) product of vectors	Combining vectors	Composite formed from 2 vectors	Composite v_{ij} has index of w_i and index of y_j

* The direct product in group matrix representations, in some applications (see "An Aside" section on pg. 20), can instead be re-expressed as a matrix with the same number of indices as each of A and B (in the example, two indices), but of dimension equal to the product of the dimensions of A and B.

2.2.10 Overview of Types of Groups

Infinite vs finite groups

The types of groups we have encountered are summarized in Wholeness Chart 2-3, along with one (first row) we have yet to mention, infinite groups, which simply have an infinite number of group elements. One example is all real numbers with the group operation being addition. Another is continuous 2D rotations (see (2-6)), which is infinite because there are an infinite number of angles θ through which we can rotate (even when θ is constrained to $0 \leq \theta < 2\pi$, since θ is continuous). As a result, Lie groups are infinite groups, as they have one or more continuously varying parameters. On the other hand, a finite group has a finite number of elements. One example is shown in (2-7), which has only four group members.

Note the various types of groups are not mutually exclusive. For example, we could have an $SO(n)$, direct product, Abelian, Lie group. Or many other different combinations of group types.

Wholeness Chart 2-3. Overview of Types of Groups

Type of Group	Characteristic	Symbols	Matrix Representation
Infinite (vs. finite)	Group has an infinite number of elements.		Example: 2D rotation matrices as function of θ
Abelian (vs Non-abelian)	All elements commute	**AB = BA**	Some groups of matrices Abelian, but generally no.
Lie (vs Non-Lie)	Elements continuous smooth functions of continuous, smooth variable(s) θ_i	$\mathbf{A}=\mathbf{A}(\theta_i)$	Example: rotation matrices as function of rotation angle(s)
Orthogonal, $O(n)$	Under **A**, magnitude of vector unchanged. All group elements real.	$\lvert\mathbf{Av}\rvert = \lvert\mathbf{v}\rvert$	$A^{-1} = A^{T}$ $\lvert\text{Det } A\rvert = 1$
Special Orthogonal, $SO(n)$	As in block above	As in block above	As in block above, but Det $A = 1$
Unitary, $U(n)$	Under **A**, magnitude of vector unchanged. At least some group elements complex.	$\lvert\mathbf{Av}\rvert = \lvert\mathbf{v}\rvert$	$A^{-1} = A^{\dagger}$ $\lvert\text{Det } A\rvert = 1$
Special Unitary, $SU(n)$	As in block above	As in block above	As in block above, but Det $A = 1$
Direct product	Composite group is formed by direct product of two groups	$\mathbf{C} = \mathbf{A}\times\mathbf{B}$	Example: $C_{ijk'l'} = A_{ij}B_{k'l'}$

2.2.11 Same Physical Phenomenon Characterized by Different Groups

Different mathematical groups can characterize the same physical phenomenon

It is interesting that certain natural phenomena can be characterized by different groups. For example, consider 2D rotation. Mathematically, we can characterize rotation by the *SO*(2) group, represented by (2-6). (Some other parametrizations are shown in (2-9) and (2-10).) This group rotates a vector, such as the position vector (x,y), through an angle θ.

But we can also characterize 2D rotation via

One example

$$U(\theta)=e^{i\theta} \quad \left(\text{a representation of } U(1)\right), \qquad \text{Repeat of (2-17)}$$

where the unitary group (2-17) rotates a complex number $x + iy$ though an angle θ. (See Problem 8.)

SO(2) and U(1) are different groups, not different representations of the same group

Note, the *SO*(2) group and the *U*(1) group above are different groups (here characterizing the same real world phenomenon), and *not* different representations of the same group[1].

That is why we prefer to say a particular group is a *characterization* of a given natural phenomenon and not a "representation" of the phenomenon. (The word "characterization" is employed by me in this text to help avoid confusion with the term "representation", but it is not generally used by others.)

In a similar way, which we will look at very briefly later on, both *SO*(3) and *SU*(2) can characterize 3D rotation. In fact, *SU*(2) is a preferred way of handling spin (which is a 3D angular momentum vector) for spin ½ particles in NRQM, as seen from different orientations (z axis up, x axis up, y axis up, or other orientations). Many QM textbooks show this.[2]

2.2.12 Subgroups

A subgroup is a group unto itself inside a larger group

A subgroup is a subset of elements of a group, wherein the subset elements under the group binary operation satisfy the properties of a group. For one, all binary operations between elements of the subset result in another element in the subset. That is, the subgroup has closure, within itself.

As examples, the group *SO*(n) is a subgroup of *O*(n), and *SU*(n) is a subgroup of *U*(n). So is rotation in 2D a subgroup of rotation in 3D, where rotation in 2D can be represented by an *SO*(3) 3X3 matrix with unity as one diagonal component and zeros for the other components of the row and column that diagonal component is in. (See (2-12).) And (2-7) is a subgroup of (2-6). As a relativistic example, the set of 3D rotations *SO*(3) is a subgroup of the Lorentz group *SO*(3,1). And the Lorentz group is itself a subgroup of the Poincaré group, which consists of the Lorentz group plus translations (in 4D).

The most important example for us is the group covering the standard model, i.e., *SU*(3)X*SU*(2)X *U*(1), which is a direct product of its subgroups *SU*(3), *SU*(2), and *U*(1),

Other than working individually with the subgroups of the SM, we will not be doing much with subgroups, but mention them in passing, as you will no doubt see the term in the literature.

Note that a matrix subgroup is not the same thing as a submatrix. The latter is obtained by removing one or more rows and/or columns from a larger (parent) matrix, so it would have fewer rows and/or columns than its parent. A matrix subgroup, on the other hand, is comprised of elements of the larger matrix group, so each matrix of the subgroup must have the same number of rows and columns as the matrices of the larger group.

2.2.13 Most Texts Treat Group Theory More Formally Than This One

We have purposefully not used formal mathematical language in our development of group theory, in keeping with the pedagogic principles delineated in the preface of Vol. 1. In short, I think that, for most of us, it is easier to learn a theory introduced via concrete examples than via more abstract presentations, as in some other texts.

But, for those who may consult other books, Table 2-1 shows some symbols you will run into in more formal treatments, and what they mean in terms of what we have done here.

For example, where we said that the closure property of a group means the result of the group operation between any two elements in the group is also an element of the group, the formal notation for this would be

---

[1] In mathematics lingo, *SO*(2) and *U*(1) are said to be "isomorphic". A group isomorphism is a function between two groups that sets up a one-to-one correspondence between the elements of the groups in a way that respects the given group operations.

[2] For one, see Merzbacher, E. *Quantum Mechanics*, 2nd ed, (Wiley, 1970), pg. 271 and Chap. 16.

$$\forall \mathbf{A}, \mathbf{B} \in G, \quad \mathbf{A} \circ \mathbf{B} \in G \ . \tag{2-38}$$

Note, however, that different authors can use different conventions. For example, the symbol $\subseteq$ is sometimes used to designate "is a sub*group* of". So, terminology usage requires some vigilance.

Brief look at more formal treatment of group theory

Table 2-1.
Some Symbols Used in Formal Group Theory

Symbol	Use	Meaning
$\in$	$\mathbf{A} \in G$	**A** is a member of group (or set) G
$\notin$	$\mathbf{A} \notin G$	**A** is not a member of group (or set) G
$\subseteq$	$\mathbf{A} \subseteq S$	**A** is a subset of set S
$\leq$	$\mathbf{A} \leq G$	**A** is a subgroup of group G
$\forall$	$\forall\ \mathbf{A}$	For any and all elements **A**
$\mathbb{R}$		Set of real numbers (for some authors, **R**)
$\mathbb{C}$		Set of complex numbers (for some, **C**)
$\mathbb{R}^3$		3D space of real numbers (for some, $\mathbf{R}^3$)

2.3 Lie Algebras

Lie algebra intro

Different from a group, a Lie algebra, like any algebra, has *two binary operations* (in a matrix representation, matrix addition plus a matrix multiplication type operation) between elements of the algebra and also a *scalar operation* (which will be multiplication of matrix elements by scalars in our applications). But, as we will see, a *Lie* algebra, in addition, can be used to generate an associated Lie group, in ways that prove to be advantageous, particularly in QFT. An $SO(2)$ group can be generated from its particular associated Lie algebra; an $SU(2)$ group, from its particular associated Lie algebra; an $SU(3)$ group, from its Lie algebra; etc.

The set element multiplication type operation used for matrix Lie algebras is not simple matrix multiplication (which won't actually work) such as AB, but a matrix commutation operation. That is,

2nd binary operation for the algebra is commutation

$$\text{common 2}^{\text{nd}} \text{ operation in physics} \rightarrow [A, B] = AB - BA = C \qquad C \text{ an element of the algebra,} \tag{2-39}$$

where (2-39) is commonly called the Lie bracket. (In more abstract Lie algebras, the Lie bracket does not have to be a commutator. In the special case we are dealing with, it is.)

As with Lie groups, a precise mathematical definition of Lie algebras, particularly abstract (vs matrix) ones, is beyond the level of this book. For our purposes, where all our groups and algebras have elements composed of matrices, the following heuristic simplification will do.

Suitable definition of Lie algebra for our purposes

A Lie algebra is an algebra with binary operations between elements, consisting of addition and commutation, that can generate an associated Lie group (in a manner yet to be shown).

In practice, this will mean, because of their connection to Lie groups (with the continuous, smooth nature of their elements dependence on parameters) that Lie algebra elements vary smoothly and continuously as functions of continuous, smooth parameters (scalars).

Our goal now is to deduce the relationship between Lie algebras and Lie groups, and then show how it applies to certain areas of physics.

2.3.1 Relating a Lie Group to a Lie Algebra: Simple Example of SO(2)

Consider the $SO(2)$ 2D rotation group representation of (2-6), reproduced below for convenience.

One parameter, real Lie group (2D rotation)

$$\hat{M}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad \text{Repeat of (2-6)}$$

We can express $\hat{M}(\theta)$ as a Taylor expansion around $\theta = 0$.

$$\hat{M}(\theta) = \hat{M}(0) + \theta \hat{M}'(0) + \frac{\theta^2}{2!}\hat{M}''(0) + \frac{\theta^3}{3!}\hat{M}'''(0) + \ldots \ , \tag{2-40}$$

which for small θ ($|\theta| << 1$) becomes (where the factor of i is inserted at the end because it will make things easier in the future)

$$\hat{M}(\theta) \approx \hat{M}(0) + \theta \hat{M}'(0) = \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix} + \theta \begin{bmatrix} -\sin 0 & -\cos 0 \\ \cos 0 & -\sin 0 \end{bmatrix} \\ = I + \theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = I + i\theta \hat{X} \qquad |\theta| \ll 1 \, , \tag{2-41}$$

Taylor expansion

where[1]

$$\hat{X} = -i\hat{M}'(0) = -i \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \tag{2-42}$$

The single matrix $\hat{X}$ and the continuous scalar field θ represent a Lie algebra, where the elements are all of form $\theta\hat{X}$, the first operation is matrix addition, and the second operation is matrix commutation (Commutation, as mentioned earlier, is necessary, and we will look at why a little later.)

Lie algebra for $SO(2)$ has one matrix $\hat{X}$, and elements $\theta\hat{X}$

Do **Problem 11** to show that (2-42) is an algebra and gain a valuable learning experience. If you have some trouble, it will help to continue reading to the end of Sect. 2.3.3, and then come back to this.

A Lie algebra such as this one (i.e., $\theta\hat{X}$) is often called the <u>tangent space</u> of the group (here, $\hat{M}(\theta)$) because the derivative of a function, when plotted, is tangent to that function at any given point. $\hat{X}$ in the present case is the first derivative of $\hat{M}$ at $\theta = 0$. Actually, it is i times $\hat{X}$. We have inserted the imaginary factor because doing so works best in physics, especially in quantum theory, where operators are Hermitian. Here, $\hat{X}$ is Hermitian, i.e., $\hat{X}^\dagger = \hat{X}$, and we will see the consequences of that later in this chapter. Mathematicians typically don't use the i factor, but the underlying idea is the same.

<u>Generating *SO*(2) Lie Group from Its Lie Algebra</u>

Given (2-41), and knowing $\hat{X}$, we can actually reproduce the group $\hat{M}$ from the Lie algebra $\theta\hat{X}$. We show this using (2-40) with (2-6) and its derivatives, along with (2-42). That is,

From $\hat{X}$, can generate $SO(2)$ group via expansion

$$\hat{M}(\theta) = \underbrace{\begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}}_{I} + \theta \underbrace{\begin{bmatrix} -\sin 0 & -\cos 0 \\ \cos 0 & -\sin 0 \end{bmatrix}}_{i\hat{X}} + \frac{\theta^2}{2!} \underbrace{\begin{bmatrix} -\cos 0 & \sin 0 \\ -\sin 0 & -\cos 0 \end{bmatrix}}_{-I} + \frac{\theta^3}{3!} \underbrace{\begin{bmatrix} \sin 0 & \cos 0 \\ -\cos 0 & \sin 0 \end{bmatrix}}_{-i\hat{X}} + .. \tag{2-43}$$

Realizing that every even derivative factor in (2-40) is the identity matrix (times either 1 or – 1), and every odd derivative factor is $i\hat{X}$ (times 1 or – 1), we can obtain the group element for any θ directly from the Lie algebra matrix $\hat{X}$, as illustrated by re-expressing (2-43) as

$$\hat{M}(\theta) = \begin{bmatrix} 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} .. & -\theta + \frac{\theta^3}{3!} - \frac{\theta^5}{5!} + .. \\ \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} .. & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + .. \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{2-44}$$

<u>Exponentiation of Lie Algebra to Generate *SO* (2) Lie Group</u>

More directly, as shown below, we can simply exponentiate $\theta\hat{X}$ to get $\hat{M}$. From (2-40) and (2-43), where we note $\hat{X}\hat{X} = I$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \hat{M}(\theta) = \hat{M}(0) + \theta\hat{M}'(0) + \frac{\theta^2}{2!}\hat{M}''(0) + \frac{\theta^3}{3!}M'''(0) + ... \\ = I + \theta(i\hat{X}) + \frac{\theta^2}{2!}(-I) + \frac{\theta^3}{3!}(-i\hat{X}\, I) + ... \\ = I + (i\theta\hat{X}) + \frac{(i\theta\hat{X})^2}{2!} + \frac{(i\theta\hat{X})^3}{3!} + ... = e^{i\theta\hat{X}} \, . \tag{2-45}$$

or via $SO(2) = e^{i\theta\hat{X}}$

---

[1] Some authors, usually non-physicists, define $\hat{X}$ as simply $\hat{M}'(0)$ without a factor of i.

In essence, $\hat{X}$ can generate $\hat{M}$ (via exponentiation as in the last part of (2-45)). The matrix $\hat{X}$ is called the generator of the group $\hat{M}$ (or of the Lie algebra). It is a basis vector in the Lie algebra vector space. Actually, it is *the* basis vector in this case, as there is only one basis matrix, and that is $\hat{X}$.

$\hat{X}$ called the generator of the Lie group/algebra

Note that the exponentiation to find $\hat{M}$ can be expressed via addition with $\hat{X}$ and I, as in the LHS of the last row of (2-45).

As an aside, inserting the i in the last step of (2-41) to define $\hat{X}$ as in (2-42) led to (2-45).

Key point

Knowing the generator of the Lie algebra, we can construct (generate) the associated Lie group simply by exponentiation of the generator (times the scalar field). Knowing the associated Lie algebra is (almost) the same as knowing the group.

Knowing generator essentially = knowing the group

2.3.2 Starting with a Different Parametrization of SO(2)

Let's repeat the process of the prior section, but start with a different parametrization of the same $SO(2)$ group, i.e., (2-9), which we repeat below for convenience.

$$\hat{M}(x) = \begin{bmatrix} \sqrt{1-x^2} & -x \\ x & \sqrt{1-x^2} \end{bmatrix} \qquad \text{Repeat of (2-9)}$$

Different form of SO(2)

We calculate the generator in similar fashion as before.

$$\hat{X} = -i\hat{M}'(x)_{x=0} = -i\begin{bmatrix} \dfrac{-2x}{2\sqrt{1-x^2}} & -1 \\ 1 & \dfrac{-2x}{2\sqrt{1-x^2}} \end{bmatrix}_{x=0} = -i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \qquad (2\text{-}46)$$

It has same generator as other form

But generally, different forms of group have different generators

(2-46) is the same as (2-42), but generally different parametrizations of the same group can have different generators, even though the underlying abstract associated Lie algebra is the same. We will see an example of this shortly in a problem related to $SO(3)$.

We could, if we wished, obtain the original group by expanding (2-9), similar to what we did in (2-45), and the terms in the expansion would involve only the matrices I and $\hat{X}$.

2.3.3 Lie Algebra for a Three Parameter Lie Group: SO(3) Example

The 2D rotation example above was a one parameter (i.e., θ) Lie group. Consider the 3D rotation group matrix representation A of (2-11) to (2-13) with the three parameters θ_1, θ_2, and θ_3. The multivariable Taylor expansion, parallel to (2-45), is

3D rotation = SO(3) group

3 parameters

$$\begin{aligned} &A(\theta_1,\theta_2,\theta_3) = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3) = e^{i\theta_1\hat{X}_1}e^{i\theta_2\hat{X}_2}e^{i\theta_3\hat{X}_3} = \\ &\left(A_1\big|_{\theta_1=0} + \theta_1 A_1'\big|_{\theta_1=0} + \tfrac{1}{2!}\theta_1^2 A_1''\big|_{\theta_1=0} + \dots\right)\left(A_2\big|_{\theta_2=0} + \theta_2 A_2'\big|_{\theta_2=0} + \tfrac{1}{2!}\theta_2^2 A_2''\big|_{\theta_2=0} + \dots\right)\times \\ &\qquad\qquad \left(A_3\big|_{\theta_3=0} + \theta_3 A_3'\big|_{\theta_3=0} + \tfrac{1}{2!}\theta_3^2 A_3''\big|_{\theta_3=0} + \dots\right) = \qquad (2\text{-}47) \\ &\left(I + (i\theta_1\hat{X}_1) + \frac{(i\theta_1\hat{X}_1)^2}{2!} + \dots\right)\left(I + (i\theta_2\hat{X}_2) + \frac{(i\theta_2\hat{X}_2)^2}{2!} + \dots\right)\left(I + (i\theta_3\hat{X}_3) + \frac{(i\theta_3\hat{X}_3)^2}{2!} + \dots\right). \end{aligned}$$

Taylor expansion in 3 parameters

Parallel to what we did for the one parameter case, we can find three generators

$$\hat{X}_i = -i\frac{\partial A_i}{\partial \theta_i}\bigg|_{\theta_i=0} \qquad i = 1,2,3 \ (\text{no sum}), \qquad (2\text{-}48)$$

which, more explicitly expressed (taking first derivatives of components in (2-12)), are

$$\hat{X}_1 = i\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \qquad \hat{X}_2 = i\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \hat{X}_3 = i\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (2\text{-}49)$$

3 generators $\hat{X}_i$

Note, for future reference, the commutation relations between the $\hat{X}_i$. For example,

$$\left[\hat{X}_1,\hat{X}_2\right]=i^2\left(\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&1\\0&-1&0\end{bmatrix}\right)$$
$$=-\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}0&1&0\\-1&0&0\\0&0&0\end{bmatrix}=-i\hat{X}_3\ . \qquad (2\text{-}50)$$

In general, where ε_{ijk} is the Levi-Civita symbol, and as you can prove to yourself by calculating the other two commutations involved (or just take my word for it),

Commutation relations for generators

$$\left[\hat{X}_i,\hat{X}_j\right]=-i\varepsilon_{ijk}\hat{X}_k \qquad i,j,k \text{ each take on value } 1,2, \text{ or } 3\ . \qquad (2\text{-}51)$$

Scalar field multiplication of the θ_i by their respective $\hat{X}_i$, under matrix addition and matrix commutation, comprise a Lie algebra. We now show that this is indeed an algebra.

Demonstrating (2-49) with θ_i yield elements of an algebra

$\hat{X}_i$ and θ_i yield an algebra

An algebra has two binary operations (matrix addition and commutation of two matrices here) and one scalar multiplication operation. We need to check that the elements in a set created using (2-49) as a basis and employing these operations satisfy the requirements for an algebra, as given in Wholeness Chart 2-1, pg. 9.

Showing it

We consider the set for which every possible (matrix) element in the set can be constructed by addition of the $\hat{X}_i$, each multiplied by a scalar. That is, for any element of the set $\hat{X}_a$

$$\hat{X}_a=\theta_{a1}\hat{X}_1+\theta_{a2}\hat{X}_2+\theta_{a3}\hat{X}_3=\theta_{ai}\hat{X}_i. \qquad \hat{X}_i \text{ denote the three elements of (2-49)}$$

For example, the special case $\hat{X}_1$ has $\theta_{11}=1$, and $\theta_{12}=\theta_{13}=0$. More general elements can have any real values for the θ_{ai}.

First, we look at the 1st binary operation of matrix addition with scalar field multiplication.

1st binary operation satisfies group properties

Closure: It may be obvious that by our definition of set elements above, every addition of any two elements yields an element of the set. To show it explicitly, for any two elements, we can write

$$\hat{X}_a=\theta_{a1}\hat{X}_1+\theta_{a2}\hat{X}_2+\theta_{a3}\hat{X}_3 \qquad \hat{X}_b=\theta_{b1}\hat{X}_1+\theta_{b2}\hat{X}_2+\theta_{b3}\hat{X}_3,$$

$$\hat{X}_c=\hat{X}_a+\hat{X}_b=(\theta_{a1}+\theta_{b1})\hat{X}_1+(\theta_{a2}+\theta_{b2})\hat{X}_2+(\theta_{a3}+\theta_{b3})\hat{X}_3==\theta_{c1}\hat{X}_1+\theta_{c2}\hat{X}_2+\theta_{c3}\hat{X}_3$$

So, the sum of any two elements of the set is in the set, and we have closure.

Associative: $\hat{X}_a+(\hat{X}_b+\hat{X}_c)=(\hat{X}_a+\hat{X}_b)+\hat{X}_c$. Matrix addition is associative.

Identity: For $\theta_{di}=0$ with i values = 1,2,3, $\hat{X}_b+\hat{X}_d=\hat{X}_b$. So, $\hat{X}_d=[0]_{3X3}$ is the identity element for matrix addition.

Inverse: Each element $\hat{X}_a$ of the set has an inverse $(-\hat{X}_a)$, since $\hat{X}_a+(-\hat{X}_a)=[0]_{3X3}$ (the identity element).

Commutation: $\hat{X}_a+\hat{X}_b=\hat{X}_b+\hat{X}_a$. Matrix addition is commutative.

1st binary operation commutative, so we have a vector space (the vectors are matrices)

Thus, under addition and scalar multiplication, the set of all elements $\hat{X}_a$ comprises a vector space and satisfies the requirements for one of the operations of an algebra.

Second, we look at the 2nd binary operation of matrix commutation with scalar field multiplication.

2nd binary operation satisfies closure requirement of an algebra

Closure: $[\hat{X}_a,\hat{X}_b]=[\theta_{ai}\hat{X}_i,\theta_{bj}\hat{X}_j],=\theta_{ai}\theta_{bj}[\hat{X}_i,\hat{X}_j]$, which from (2-51) yields the following.

$$[\hat{X}_a,\hat{X}_b]=-i\theta_{ai}\theta_{bj}\varepsilon_{ijk}\hat{X}_k=-i\theta_{ck}\hat{X}_k \qquad \text{where } \theta_{ck}=\theta_{ai}\theta_{bj}\varepsilon_{ijk}.$$

At this point, recalling that all of the scalars, such as θ_{ai} and θ_{bj}, are *real,* and ε_{ijk} is real too, we *cannot* write $[\hat{X}_a,\hat{X}_b]=\hat{X}_c$, because of the imaginary factor i. That is, the RHS of the previous relation $(-i\theta_{ck}\hat{X}_k)$ is real and *not* an element of the set (which has all imaginary elements), so there is no closure. However, we can easily fix the situation by defining the binary commutation operation to include a factor of i, as follows.

Definition of 2nd binary operation: $i[\hat{X}_a, \hat{X}_b]$ (2-52)

Given (2-51), we find (2-52) yields $i[\hat{X}_a, \hat{X}_b] = \theta_{ai}\theta_{bj}\,\varepsilon_{ijk}\hat{X}_k = \theta_{ck}\hat{X}_k$, which is in the set. Therefore, under the 2nd operation of (2-52), there is closure.

Third, we look at both binary operations together.

Distributive: *From* Wholeness Chart 2-1, **A**∘(**B** ✣ **C**) = (**A**∘**B**) ✣ (**A**∘**C**) for us, is distributive, *if* we have $i[A, B+C] = i[A,B] + i[A,C]$ or simply $[A, B+C] = [A,B] + [A,C]$.

2nd binary operation distributive over 1st: satisfies requirement of an algebra

Now $[A, B+C] = [\hat{X}_a, \hat{X}_b + \hat{X}_c] = \theta_{ai}\hat{X}_i\,(\theta_{bj}\hat{X}_j + \theta_{cj}\hat{X}_j) - (\theta_{bj}\hat{X}_j + \theta_{cj}\hat{X}_j)\,\theta_{ai}\hat{X}_i$

or $= \theta_{ai}\hat{X}_i\theta_{bj}\hat{X}_j - \theta_{bj}\hat{X}_j\theta_{ai}\hat{X}_i + \theta_{ai}\hat{X}_i\theta_{cj}\hat{X}_j - \theta_{cj}\hat{X}_j\theta_{ai}\hat{X}_i$

$= [\hat{X}_a,\hat{X}_b] + [\hat{X}_a,\hat{X}_c] = [A,B] + [A,C]$.

So, the commutation operation is distributive over the addition operation.

<u>Conclusion:</u> The set of the $\hat{X}_a$'s under matrix addition, the matrix commutation operation (2-52), and scalar field multiplication is an algebra. It is a Lie algebra because we can use it to generate a Lie group (via (2-47)). Note that every element in the set is a smooth, continuous function of the smooth, continuous (real) variables $\theta_{\alpha\iota}$.

So, we have an algebra, a Lie algebra

Further, regarding the 2nd binary operation, one sees from the analysis below that this particular algebra is non-associative, non-unital, and non-Abelian.

Associative: General relation **A**∘(**B**∘**C**) = (**A**∘**B**)∘**C.** For us, it is associative, *if* we have $i\,[A, i[B,C]] = i\,[i[A,B],C]$ or simply $[A,[B,C]] = [[A,B],C]$.

Now: $[A, [B,C]] = [\hat{X}_a, [\hat{X}_b,\hat{X}_c]] = \hat{X}_a(\hat{X}_b\hat{X}_c - \hat{X}_c\hat{X}_b) - (\hat{X}_b\hat{X}_c - \hat{X}_c\hat{X}_b)\hat{X}_a$

$= \hat{X}_a\hat{X}_b\hat{X}_c - \hat{X}_a\hat{X}_c\hat{X}_b - \hat{X}_b\hat{X}_c\hat{X}_a + \hat{X}_c\hat{X}_b\hat{X}_a$

$= \theta_{ai}\theta_{bj}\theta_{ck}(\hat{X}_i\hat{X}_j\hat{X}_k - \hat{X}_i\hat{X}_k\hat{X}_j - \hat{X}_j\hat{X}_k\hat{X}_i + \hat{X}_k\hat{X}_j\hat{X}_i)$

And: $[[A,B], C] = [[\hat{X}_a,\hat{X}_b],\hat{X}_c] = (\hat{X}_a\hat{X}_b - \hat{X}_b\hat{X}_a)\,\hat{X}_c - \hat{X}_c\,(\hat{X}_a\hat{X}_b - \hat{X}_b\hat{X}_a)$

$= \hat{X}_a\hat{X}_b\hat{X}_c - \hat{X}_b\hat{X}_a\hat{X}_c - \hat{X}_c\hat{X}_a\hat{X}_b + \hat{X}_c\hat{X}_b\hat{X}_a$

$= \theta_{ai}\theta_{bj}\theta_{ck}(\hat{X}_i\hat{X}_j\hat{X}_k - \hat{X}_j\hat{X}_i\hat{X}_k - \hat{X}_k\hat{X}_i\hat{X}_j + \hat{X}_k\hat{X}_j\hat{X}_i)$

These relations are not equal. The middle terms differ because the $\hat{X}_i$ do not commute. So, the second binary operation is non-associative and we say that this Lie algebra is non-associative.

This algebra is non-associative

Identity: An element I would be the identity element under commutation relation (2-52), if and only if, $i\,[I, \hat{X}_a] = \hat{X}_a$ where $\hat{X}_a$ is any element in the set. As shown by doing Problem 12, there is no such I. Since no identity element exists, this algebra is non-unital.

and non-unital

Inverse: If there is no identity element, there is no meaning for an inverse.

Commutative: General relation **A**∘**B** = **B**∘**A** needed for all elements, for the binary operation ∘. For us, ∘ is commutation, so we need commutation of the commutation operation. That is, we need, in general, $[A,B] = [B,A]$. Thus, as one example, $i[\hat{X}_1, \hat{X}_2] = i[\hat{X}_2, \hat{X}_1]$. But from (2-51), or simply from general knowledge of commutation, this is not true (we are off by a minus sign), so there are elements in the set that do not commute under the 2nd binary operation (2-52) (which is itself a commutation relation). This 2nd binary operation is non-Abelian, and thus, so is the algebra.

and non-Abelian

<u>End of demo</u>

Do **Problem 12** to show there is no identity element for the 2nd operation (2-52) in the *SO*(3) Lie algebra.

Do **Problem 13** to see why we took matrix commutation as our second binary operation for the Lie group, rather than the simpler alternative of matrix multiplication.

The commutation relations embody the structure of the Lie algebra and Lie group, and tell us almost everything one needs to know about both the algebra and the group. Because of this

Constants in the generator commutation relations are called "structure constants". These structure the group (contain the key info about the group)

"structuring", the $-\varepsilon_{ijk}$ of (2-51) are often called the structure constants. We will see that other groups have their own particular (different) structure constants, but in every case, they tell us the properties of the algebra and associated group.

Another Choice of Parametrization

Consider an alternative parametrization for $SO(3)$. where instead of θ_i, we use $\theta'_i = -\theta_i$. This could be considered rotation (of a vector for example) in the cw direction around each axis.

Do **Problem 14** to find the generator commutation relations in $SO(3)$. where $\theta_i \rightarrow \theta'_i = -\theta_i$ in (2-12).

With this alternative set of parameters, we find different generators $\hat{X}'_i$, where

$$\left[\hat{X}'_i, \hat{X}'_j\right] = i\varepsilon_{ijk}\hat{X}'_k \qquad \text{for parametrization } \theta'_i = -\theta_i \tag{2-53}$$

We have a different set of structure constants, which differ from those in (2-51) by a minus sign. So, the structure "constants" are not really constant in the sense that they can change for different choices of parametrization. The fact that they were the same for different choices of parametrization in $SO(2)$ [see Section 2.3.2, pg. 25], was a coincidence. More generally, they are not the same.

But structure "constants" change with different parametrizations

In what follows, we will stick with the original parametrization of θ_i, as in (2-12).

Quick intermediate summary for $SO(3)$

For $SO(3)$, and our original choice of parametrization, the

$$\text{generators obey } \left[\hat{X}_i, \hat{X}_j\right] = -i\varepsilon_{ijk}\hat{X}_k \quad i,j,k \text{ each take on a value } 1, 2, \text{ or } 3, \qquad \text{repeat of (2-51)}$$

and, for any parametrization, the Lie algebra operations are addition and

$$\text{the 2}^{\text{nd}} \text{ binary operation is } \quad i[\hat{X}_a, \hat{X}_b]\,. \qquad \text{repeat of (2-52)}$$

Summary of $SO(3)$:
1) 3 generators
2) 3 commutation rels
3) binary operations: addition & commutation

2.3.4 Generating SO(3) from Its Lie Algebra

As with the $SO(2)$ Lie group, one can generate the $SO(3)$ group from its generators, via the expansion in the last line of (2-47). We won't go through all the algebra involved, as the steps for each factor parallel those for $SO(2)$, and the actual doing of it is fairly straightforward (though tedious).

2.3.5 Exponentiation Relating SO(3) Lie Group to Its Lie Algebra

General Case is Tricky

For a one parameter Lie group such as (2-6) in θ, the relationship between it and the associated Lie algebra $\theta\hat{X}$ (see (2-42)) was simple exponentiation (2-45). One can generate the group via $M(\theta) = e^{i\theta\hat{X}}$. For a Lie group of more than one parameter, however, things get a little trickier, because one must use the Baker-Campbell-Hausdorf relationship for exponentiation of operators,

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}\left(X,[X,Y]+[Y,[Y,X]]\right)+\ldots}, \tag{2-54}$$

where we imply the infinite series of nested commutators after the second commutator relation[1]. If X and Y commute (as numbers do), we get the familiar addition of exponents relation. When they don't, such as with many operators, things get more complicated.

In our example of $SO(3)$ (2-47), one might naively expect to obtain the Lie group from the Lie algebra using the exponentiation relation on the RHS of (2-55), but due to (2-54) one cannot.

Exponential addition law for operators makes exponentiation of generators to get $SO(3)$ Lie group not simple

$$A(\theta_1,\theta_2,\theta_3) = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3) = e^{i\theta_1\hat{X}_1}e^{i\theta_2\hat{X}_2}e^{i\theta_3\hat{X}_3} \neq e^{i(\theta_1\hat{X}_1+\theta_2\hat{X}_2+\theta_3\hat{X}_3)} = e^{i\theta_i\hat{X}_i} \tag{2-55}$$

So, if you have a particular Lie algebra element $\theta_i\hat{X}_i$ (some sum of the generators), you do not use the RHS of (2-55) to generate the Lie group (2-47). You have to use the relationship in the middle of

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[1] To be precise, (2-54) only holds if X and Y are "sufficiently small", where defining that term mathematically would take us too far afield. Simply note that all operators we will work with will be sufficiently small.

(2-55). Conversely, if you have a Lie group element in terms of three θ_i, such as A on the LHS of (2-55), you cannot assume the associated Lie algebra element to be exponentiated is $\theta_i\hat{X}_i$.

To get the Lie algebra element, call it $\hat{\theta}_i \hat{X}_i$, associated with a given Lie group element in terms of θ_i, we need to use (2-54). That is, we need to find the $\hat{\theta}_i$ values in

$$A\left(\theta_1,\theta_2,\theta_3\right) = A_1\left(\theta_1\right)A_2\left(\theta_2\right)A_3\left(\theta_3\right) = e^{i\,\hat{\theta}_i\hat{X}_i} \qquad \hat{\theta}_i = \hat{\theta}_i\left(\theta_j\right). \tag{2-56}$$

As a simple example, consider the case where $\theta_3 = 0$, so the total group action amounts to a rotation through θ_2 followed by a rotation through θ_1. (Operations act from right to left in (2-56).) Using (2-54), we find

$$\begin{aligned} A\left(\theta_1,\theta_2,\theta_3=0\right) &= A_1\left(\theta_1\right)A_2\left(\theta_2\right) = e^{i\theta_1\hat{X}_1}e^{i\theta_2\hat{X}_2} \\ &= e^{i\theta_1\hat{X}_1 + i\theta_2\hat{X}_2 + \frac{1}{2}\left[i\theta_1\hat{X}_1, i\theta_2\hat{X}_2\right] + \frac{1}{12}\left(\left[i\theta_1\hat{X}_1,\left[i\theta_1\hat{X}_1, i\theta_2\hat{X}_2\right]\right] + \left[i\theta_2\hat{X}_2,\left[i\theta_2\hat{X}_2, i\theta_1\hat{X}_1\right]\right]\right) + \ldots} \\ &= e^{i\,\hat{\theta}_i\hat{X}_i}\;. \end{aligned} \tag{2-57}$$

So, using the defining commutation relation of the Lie algebra (2-51), we find

But we can still generate the group from $\hat{X}_i$ using the generator commutation relations

$$\begin{aligned} i\hat{\theta}_i\hat{X}_i &= i\theta_1\hat{X}_1 + i\theta_2\hat{X}_2 + \tfrac{1}{2}\left[i\theta_1\hat{X}_1, i\theta_2\hat{X}_2\right] + \tfrac{1}{12}\left[i\theta_1\hat{X}_1,\left[i\theta_1\hat{X}_1, i\theta_2\hat{X}_2\right]\right] + \tfrac{1}{12}\left[i\theta_2\hat{X}_2,\left[i\theta_2\hat{X}_2, i\theta_1\hat{X}_1\right]\right] .. \\ &= i\theta_1\hat{X}_1 + i\theta_2\hat{X}_2 + \tfrac{1}{2}\theta_1\theta_2\left(i\hat{X}_3\right) + i\tfrac{1}{12}\theta_1^2\theta_2\left[\hat{X}_1, i\hat{X}_3\right] + i\tfrac{1}{12}\theta_1\theta_2^2\left[\hat{X}_2, -i\hat{X}_3\right] + \ldots \\ &= i\theta_1\hat{X}_1 + i\theta_2\hat{X}_2 + i\tfrac{1}{2}\theta_1\theta_2\hat{X}_3 - i\tfrac{1}{12}\theta_1^2\theta_2\hat{X}_2 - i\tfrac{1}{12}\theta_1\theta_2^2\hat{X}_1 + .. \end{aligned} \tag{2-58}$$

At second order, $\hat{\theta}_i\hat{X}_i \approx \theta_1\hat{X}_1 + \theta_2\hat{X}_2 + \frac{1}{2}\theta_1\theta_2\hat{X}_3$, so $\hat{\theta}_1 \approx \theta_1$, $\hat{\theta}_2 \approx \theta_2$, $\hat{\theta}_3 \approx \frac{1}{2}\theta_1\theta_2$. In principle, we can find the $\hat{\theta}_i$ at any order by using all terms in (2-58) up to that order. And for cases where $\theta_3 \neq 0$, one just repeats the process one more time using the results of (2-58) with (2-54) and the third operator in the exponent $\theta_3\hat{X}_3$.

Do **Problem 15** to obtain the third order $\hat{\theta}_i$ values for our example.

A key thing to notice is that any two group elements $A(\theta_{A1}, \theta_{A2}, \theta_{A3})$ and $B(\theta_{B1}, \theta_{B2}, \theta_{B3})$ of form like (2-56), when multiplied together via the group operation of matrix multiplication, are also in the group, i.e., $AB = C$, where C is in the group. That is, due to the commutation relations (2-51) used in (2-54) we will always get a result equal to the exponentiation of $\underset{\sim}{\theta}_i\hat{X}_i$, i.e., $C = e^{i\,\underset{\sim}{\theta}_i\hat{X}_i}$, where the $\underset{\sim}{\theta}_i$ can be determined. That is, every group operation on group elements yields a group element, and that group element has an associated Lie algebra element $\underset{\sim}{\theta}_i\hat{X}_i$. All of this is only because each of the commutation relations (2-51) used in (2-54) [and thus, (2-58)] yields one of the Lie algebra basis matrices $\hat{X}_i$.

Group property of $AB=C$ (with A, B, C in group) still holds

<u>Infinitesimal Scalars θ_i Case is Simpler</u>

For small θ_i in (2-58), at lowest order $\hat{\theta}_i\hat{X}_i \approx \theta_1\hat{X}_1 + \theta_2\hat{X}_2$, so $\hat{\theta}_1 \approx \theta_1$, $\hat{\theta}_2 \approx \theta_2$, $\hat{\theta}_3 \approx \theta_3 = 0$. It is common to simply consider the group and the algebra to be local (small values of θ_i), so orders higher than the lowest are negligible, and one can simply identify $\hat{\theta}_i \approx \theta_i$. Then, we find (2-57) becomes

$$A\left(\theta_1,\theta_2,\theta_3=0\right) = A_1\left(\theta_1\right)A_2\left(\theta_2\right) = e^{i\theta_1\hat{X}_1}e^{i\theta_2\hat{X}_2} \approx e^{i\theta_1\hat{X}_1 + i\theta_2\hat{X}_2} \qquad \left|\theta_1\right|,\left|\theta_2\right| \ll 1\;, \tag{2-59}$$

and for the more general case,

$$A\left(\theta_1,\theta_2,\theta_3\right) = A_1\left(\theta_1\right)A_2\left(\theta_2\right)A_3\left(\theta_3\right) = e^{i\theta_1\hat{X}_1}e^{i\theta_2\hat{X}_2}e^{i\theta_3\hat{X}_3} \approx e^{i\theta_i\hat{X}_i} \approx I + i\,\theta_i\hat{X}_i \qquad \left|\theta_i\right| \ll 1 \tag{2-60}$$

Exponentiation to get SO(3) group is simpler in infinitesimal case

In principle, we can generate the global (finite) Lie group by taking $\theta_i \rightarrow d\theta_i$ in (2-60) and carrying out step-wise integration. And of course, we can always generate the finite group with the first part of (2-55), $A\left(\theta_1,\theta_2,\theta_3\right) = e^{i\theta_1\hat{X}_1}e^{i\theta_2\hat{X}_2}e^{i\theta_3\hat{X}_3}$.

2.3.6 Summary of $SO(n)$ Lie Groups and Algebras

Summary of special orthogonal Lie groups in top part of Wholeness Chart 2-5

The first three rows of Wholeness Chart 2-5, pg. 37, summarize special orthogonal Lie groups, their associated Lie algebras, and the use of exponentiation of the latter to generate the former. Note, we have introduced the symbol $\hat{N}$ (with a caret) as a surrogate for what we have been calling the 3D rotation matrix A. The symbol A is common in the literature for that matrix, whereas in the chart, we use carets for real matrices and no carets for complex ones (which are yet to be treated herein).

The number of generators for an $SO(n)$ Lie algebra is deduced from what we found for $SO(2)$ and $SO(3)$. The result can be proven, but is not central to our work, so we will not go through that proof here. Also, we use the symbol $\hat{\theta}_i\left(\theta_j\right)$ for both $SO(3)$ and $SO(n)$, in order to emphasize the parallel between the two cases, though in general, the functional form of $\hat{\theta}_i$ will be different in each case.

You may wish to follow along with the rest of the chart as we develop $SU(2)$ and $SU(3)$ theory in the next sections.

2.3.7 The $SU(2)$ Associated Lie Algebra

We find the Lie algebra generators for the $SU(2)$ Lie group (2-20) [with notation of (2-25)], repeated below for convenience, in similar fashion to what we did for $SO(2)$ and $SO(3)$.

$$M=\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}=\begin{bmatrix} \alpha_0+i\alpha_3 & \alpha_2+i\alpha_1 \\ -\alpha_2+i\alpha_1 & \alpha_0-i\alpha_3 \end{bmatrix}=i\begin{bmatrix} -i\alpha_0+\alpha_3 & \alpha_1-i\alpha_2 \\ \alpha_1+i\alpha_2 & -i\alpha_0-\alpha_3 \end{bmatrix} \qquad \text{Repeat of (2-25)}$$

That is, from the multivariable Taylor expansion

Expanding $SU(2)$ group of 3 independent parameters to get generators

$$\begin{aligned} M\left(\alpha_i\right)=\underbrace{M\left(0,0,0\right)}_{I}+\alpha_1\left.\frac{\partial M}{\partial\alpha_1}\right|_{\alpha_i=0}+\alpha_2\left.\frac{\partial M}{\partial\alpha_2}\right|_{\alpha_i=0}+\alpha_3\left.\frac{\partial M}{\partial\alpha_3}\right|_{\alpha_i=0} \\ +\frac{\left(\alpha_1\right)^2}{2!}\left.\frac{\partial^2 M}{\partial\alpha_1^2}\right|_{\alpha_i=0}+\frac{\alpha_1\alpha_2}{2!}\left.\frac{\partial^2 M}{\partial\alpha_1\partial\alpha_2}\right|_{\alpha_i=0}+\ldots, \end{aligned} \tag{2-61}$$

the generators are

$$X_1=-i\left.\frac{\partial M}{\partial\alpha_1}\right|_{\alpha_i=0} \qquad X_2=-i\left.\frac{\partial M}{\partial\alpha_2}\right|_{\alpha_i=0} \qquad X_3=-i\left.\frac{\partial M}{\partial\alpha_3}\right|_{\alpha_i=0}. \tag{2-62}$$

Evaluating (2-62) for (2-25), we find, with (2-23),

The 3 generators X_i

$$\begin{aligned} X_1=-i\left.\frac{\partial M}{\partial\alpha_1}\right|_{\alpha_i=0}=-i\left(\frac{\partial}{\partial\alpha_1}\begin{bmatrix} \alpha_0+i\alpha_3 & \alpha_2+i\alpha_1 \\ -\alpha_2+i\alpha_1 & \alpha_0-i\alpha_3 \end{bmatrix}\right)_{\alpha_i=0}=-i\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ X_2=-i\left.\frac{\partial M}{\partial\alpha_2}\right|_{\alpha_i=0}=\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad X_3=-i\left.\frac{\partial M}{\partial\alpha_3}\right|_{\alpha_i=0}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \end{aligned} \tag{2-63}$$

The 3 generators X_i are the Paul matrices

with the Pauli matrices commutation relations

which are the Pauli matrices, and which have the commutation relations

$$\left[X_i,X_j\right]=i2\varepsilon_{ijk}X_k \xrightarrow[\text{symbols}]{\text{more common}} \left[\sigma_i,\sigma_j\right]=i2\varepsilon_{ijk}\sigma_k \quad \text{or} \quad \left[\frac{\sigma_i}{2},\frac{\sigma_j}{2}\right]=i\,\varepsilon_{ijk}\frac{\sigma_k}{2}. \tag{2-64}$$

We will not take the time to show that the X_i along with the three scalar field multipliers comprise an algebra under the binary operations of addition and commutation. We have done that twice before for other algebras and should be able to simply accept it here. I assure you it is indeed an algebra.

$SU(2)$ generators can be taken as $\frac{1}{2}\sigma_i$, with structure constants ε_{ijk}

Note that had we defined M with $\alpha_i \rightarrow \frac{1}{2}\alpha_i$ (i = 1,2,3), we would have found $X_i \rightarrow \frac{1}{2}X_i = \frac{1}{2}\sigma_i$. Then, the commutation relations would have been as in the RHS of (2-64), and we would have the structure constants ε_{ijk}. So, ε_{ijk} are the structure constants if we take our Lie algebra X_i as $\frac{1}{2}\sigma_i$ (which is common in QFT); $2\varepsilon_{ijk}$ are the structure constants if we take our Lie algebra X_i as σ_i.

$SU(2)$ happens to be similar to what we found for $SO(3)$

The Lie algebra X_i for the $SU(2)$ group has the same number of generators, and for $X_i = \frac{1}{2}\sigma_i$, has the same commutation relation we found for one parametrization of the $SO(3)$ group [(2-53)]. The two different groups $SU(2)$ and $SO(3)$ have similar structure and are similar in many ways. For one, which we won't get into in depth here[1], as it doesn't play much role in the standard model of QFT, the 3-dimensional (pseudo) vector of angular momentum can be treated under either the 3D rotation group $SO(3)$ or (non-relativistically) under the 2D $SU(2)$ group. As you may have run into in other studies, spin angular momentum in NRQM is often analyzed using a 2D complex column vector with the top component representing spin up (along z axis) and the lower component representing spin down. The Pauli matrices, via their operations on the column vector, play a key role in all of that.

Take caution that $SO(3)$ and $SU(2)$ are *not* different representations of the same group, even though they may, under certain parametrizations, share the same structure constants (same commutation relations). They are different groups, but they can characterize the same physical phenomenon. This is similar to the relationship between 2D rotation group $SO(2)$ and the $U(1)$ group we looked at in Problem 8.

$SU(2)$ and $SO(3)$ are different groups, but can both characterize 3D rotation. They have similar Lie algebra structures

2.3.8 Generating the SU(2) Group from the SU(2) Lie Algebra

Do Problem 16 to prove to yourself that the X_i above generate the $SU(2)$.

From the results of Problem 16, we see that (2-61) can be expressed as

$$M(\alpha_i) = I + i\alpha_1 X_1 + i\alpha_2 X_2 + i\alpha_3 X_3 - \frac{\alpha_1^2}{2!}I - \frac{\alpha_2^2}{2!}I - \frac{\alpha_3^2}{2!}I + \frac{\alpha_1\alpha_2}{2!}[0] + \ldots \tag{2-65}$$

Expressing $SU(2)$ elements in terms of generators and independent parameters

2.3.9 Exponentiation of the SU(2) Lie Algebra

<u>Finite Parameter Case</u>

One can obtain the Lie group from the Lie algebra via the expansion (2-61) along with (2-62), expressed in (2-65). One can also obtain it in a second (related) way, which involves exponentiation, in a manner similar to what we saw earlier with $SO(2)$ and $SO(3)$. However, we would find doing so to be a mathematical morass, so we will simply draw parallels to what we saw with earlier groups.

Exponentiating Lie algebra to get $SU(2)$ Lie group is a nightmare

We illustrate how it is done in principle

Consider a general Lie algebra element

$$X = \alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3, \tag{2-66}$$

where one could exponentiate it as

$$e^{iX} = e^{i(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3)}. \tag{2-67}$$

and where we note, in passing, that (see (2-54) and (2-55))

$$e^{i(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3)} \neq e^{i\alpha_1 X_1} e^{i\alpha_2 X_2} e^{i\alpha_3 X_3}. \tag{2-68}$$

We would like to explore whether (2-67) equals (2-25) [equivalently, (2-65)],

$$\begin{aligned} e^{i(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3)} &\overset{?}{=} M(\alpha_i) \\ &= I + i\alpha_1 X_1 + i\alpha_2 X_2 + i\alpha_3 X_3 - \frac{\alpha_1^2}{2!}I - \frac{\alpha_2^2}{2!}I - \frac{\alpha_3^2}{2!}I + \frac{\alpha_1\alpha_2}{2!}[0] + \ldots \end{aligned} \tag{2-69}$$

By expanding the top row LHS of (2-69) around $\alpha_i = 0$, we could see whether or not it matches the expansion of M in (2-69), 2nd row. We will not go through all that tedium, but draw instead on our knowledge of the other multiple parameter case $SO(3)$, where we found the equal sign with the question mark in (2-69) is actually a $\neq$ sign. If we wished, however, we could, with a copious amount of labor, find a matrix function to exponentiate that would give us M. That is, similar to (2-56),

$$M(\alpha_i) = e^{i\beta_i X_i} \neq e^{i(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3)} \qquad \beta_i = \beta_i(\alpha_i). \tag{2-70}$$

Do Problem 17 to help in what comes next**.**

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[1] See footnote references on pg. 8 or almost any text on group theory.

Infinitesimal Parameter Case

However, for small values of $|\alpha_i|$, (<< 1), as can be found by doing Problem 17,

$$e^{i(\alpha_1 X_1+\alpha_2 X_2+\alpha_3 X_3)} \approx I+i\alpha_1 X_1+i\alpha_2 X_2+i\alpha_3 X_3 \approx M(\alpha_i) \qquad |\alpha_i| \ll 1. \tag{2-71}$$

Exponentiation in infinitesimal case is simpler

As with prior cases, one could generate the global (finite) $SU(2)$ group by step-wise integration over infinitesimal α_i.

2.3.10 Another Parametrization of SU(2)

(2-72) below is a different parametrization of $SU(2)$ with three different parameters.

Do **Problem 18** to prove it.

Another parametrization of SU(2)

$$M(\phi_1,\phi_2,\phi_3)=M(\phi_i)=\begin{bmatrix} cos\,\phi_1 e^{i\phi_2} & sin\,\phi_1 e^{i\phi_3} \\ -sin\,\phi_1 e^{-i\phi_3} & cos\,\phi_1 e^{-i\phi_2}\end{bmatrix} \qquad a=cos\,\phi_1 e^{i\phi_2} \qquad b=sin\,\phi_1 e^{i\phi_3} \tag{2-72}$$

Note that (2-6) is a special case (subgroup, actually) of (2-72) where $\phi_2 = \phi_3 = 0$ (and here, $\phi_1 = -\theta$).

We find the generators for (2-72) in the same way as we did for (2-25).

The Lie algebra generators of (2-73) below turn out to be somewhat different from the earlier example (2-63) in that they are switched around, and X_3 is not found via the simple derivative with all $\phi_i = 0$, as we had earlier. There are subtleties of group theory involved here, and we don't want to go off on too much of a tangent from our fundamental goal, so we will leave it at that.

As we noted earlier, different parametrizations generally lead to different generators. In any vector space (such as our Lie algebra) one can have different basis vectors (matrices are the vectors here). So, the generators for any matrix Lie group depend on what choice we make for the independent parameters.

The generators for this form

$$X_1=-i\left.\frac{\partial M}{\partial\phi_1}\right|_{\phi_i=0}=-i\begin{bmatrix} -sin\,\phi_1 e^{i\phi_2} & cos\,\phi_1 e^{i\phi_3} \\ -cos\,\phi_1 e^{-i\phi_3} & -sin\,\phi_1 e^{-i\phi_2}\end{bmatrix}_{\phi_i=0}=-i\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

$$X_2=-i\left.\frac{\partial M}{\partial\phi_2}\right|_{\phi_i=0}=-i\begin{bmatrix} i\,cos\,\phi_1 e^{i\phi_2} & 0 \\ 0 & -i\,cos\,\phi_1 e^{-i\phi_2}\end{bmatrix}_{\phi_i=0}=-i\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

$$-i\left.\frac{\partial M}{\partial\phi_3}\right|_{\phi_i=0}=-i\begin{bmatrix} 0 & i\,sin\,\phi_1 e^{i\phi_3} \\ i\,sin\,\phi_1 e^{-i\phi_3} & 0\end{bmatrix}_{\phi_i=0}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} \quad \text{Not relevant}$$

$$-i\left.\frac{\partial^2 M}{\partial\phi_1\partial\phi_3}\right|_{\phi_i=0}=-i\begin{bmatrix} 0 & i\,cos\,\phi_1 e^{i\phi_3} \\ i\,cos\,\phi_1 e^{-i\phi_3} & 0\end{bmatrix}_{\phi_i=0}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \text{needed in expansion of } M$$

$$X_3=-i\left.\frac{\partial M(\phi_1,0,\pi/2)}{\partial\phi_1}\right|_{\phi_1=0}=-i\begin{bmatrix} -sin\,\phi_1 & cos\,\phi_1 e^{i\pi/2} \\ -cos\,\phi_1 e^{-i\pi/2} & -sin\,\phi_1\end{bmatrix}_{\phi_1=0}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad \text{3rd generator}$$

(2-73)

For small values of the parameters in (2-72),

$$M(\phi_1,\phi_2,\phi_3)\approx\begin{bmatrix}1+i\phi_2 & \phi_1\\ -\phi_1 & 1-i\phi_2\end{bmatrix} \qquad |\phi_i| \ll 1 \tag{2-74}$$

and we can express the group matrix as

$$M(\phi_i)\approx I+if_j(\phi_i)X_j\approx e^{if_j(\phi_i)X_j}\approx e^{i(\phi_1 X_1+\phi_2 X_2)} \qquad |\phi_i|\ll 1 \quad f_1=\phi_1,\, f_2=\phi_2,\, f_3=0 \tag{2-75}$$

However, we cannot express $M(\phi_i)$ globally as either a simple finite sum or as an exponentiation of $i(\phi_1 X_1 + \phi_2 X_2)$ as in (2-75). We could express it globally as an infinite sum, similar in concept to (2-61). We could also express it globally (after a whole lot of algebra) as an exponentiation of some function $\tilde{\beta}_j(\phi_i)$ of the ϕ_i as in

$$M(\phi_i) = e^{i\tilde{\beta}_j(\phi_i)Xj} \qquad \text{any size } \phi_i,\ \tilde{\beta}_j(\phi_i) \text{ deduced to make it work .} \tag{2-76}$$

Bottom line:

(Finite parameters α_i) For an $SU(2)$ parametrization of particular form (2-25), we can obtain a global expression of the group 1) as a finite sum of terms linear in the generators and the identity, and 2) as a (complicated) exponentiation of the generators (see (2-70)).

(Other finite parameters e.g., ϕ_i) For other parametrizations, 1) above needs an infinite sum of terms.

(Any infinitesimal parameters) In any parametrization, we can find a local expression of the group as 1) a finite sum of terms linear in the generators and the identity or 2) as an exponential [see (2-75) and (2-71)]. This is the usual approach to the Lie algebra as the tangent space (local around the identity) to the Lie group. Locally, finding the group from the Lie algebra via exponentiation is relatively easy. Globally, it is generally horrendous.

Note that, in general, the generators for different parametrizations can be different, and thus so are the structure constants. However, we can find linear combinations of the generators from one parametrization that equal the generators from another parametrization. In other words, the vector space of the abstract Lie algebra for a given Lie group is the same for any parametrization, even though we generally get different generators (basis vectors [= matrices]) for different parametrizations.

(2-72) has value in analyzing spin. (2-25) has value in QFT. Different parametrizations work better for different applications.

This form good for spin; prior form better for QFT

Digression for Brief Look at Spin and $SU(2)$ in NRQM

However, we will now digress briefly to show how (2-72) can be used for spin analysis. Recall the wave function in NRQM had a two-component column vector representing spin.

Brief look at how this form of $SU(2)$ can handle spin

$$|\psi\rangle_{\substack{spin\\up}} = Ae^{-ikx}\begin{bmatrix}1\\0\end{bmatrix} \ \text{spin in } +z \text{ direction} \qquad |\psi\rangle_{\substack{spin\\down}} = Ae^{-ikx}\begin{bmatrix}0\\1\end{bmatrix} \ \text{spin in } -z \text{ direction} \tag{2-77}$$

Consider the case where we rotate the spin down particle to a spin up orientation (or conversely, rotating our observing coordinate system in the opposite direction). In physical space we have rotated the z axis 180° and could use the $SO(3)$ rotation group (2-47) to rotate the 3D (pseudo) vector for spin angular momentum through 180°. However, for the manner in which we represent spin in (2-77), that would not work, as spin there is represented by a two-component vector, not a three-component one. But, consider the $SU(2)$ parametrization (2-72) where, in this case, $\phi_2 = \phi_3 = 0$, and the ϕ_1 is a rotation about the x axis (which effectively rotates the z axis around the x axis). We actually need to take $\phi_1 = \phi/2$, where ϕ is the actual physical angle of rotation, in order to make it work, as we are about to see.

Then note what (2-72) does to the spin down wave function on the RHS of (2-77).

3D rotation in $SU(2)$ via $\phi_1 = \phi_{3D}/2$.

$$M|\psi\rangle_{\substack{spin\\down}} = \begin{bmatrix}\cos\phi_1 & \sin\phi_1\\ -\sin\phi_1 & \cos\phi_1\end{bmatrix} Ae^{-ikx}\begin{bmatrix}0\\1\end{bmatrix} = Ae^{-ikx}\underbrace{\begin{bmatrix}\cos\frac{\phi}{2} & \sin\frac{\phi}{2}\\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2}\end{bmatrix}}_{\text{for } \phi=180^\circ}\begin{bmatrix}0\\1\end{bmatrix} \tag{2-78}$$

$$= Ae^{-ikx}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = Ae^{-ikx}\begin{bmatrix}1\\0\end{bmatrix} = |\psi\rangle_{\substack{spin\\up}} .$$

So, we see that the spinor (two-component column vector) lives in a 2D vector space, on which the elements of the $SU(2)$ group operate. And rotations in 3D space can be characterized, in a 2D complex space, by the $SU(2)$ group. Because ϕ_1 in the 2D complex space of $SU(2)$ ranges over 360°, while ϕ in the physical space of $SO(3)$ ranges over 720°, we say $SU(2)$ is the double cover of $SO(3)$.

As noted earlier this has wide ranging application in analyzing spin, but we will leave further treatment of this topic to other sources, such as those cited previously.

End of digression

We do note that the matrix operation of (2-78) is sometimes referred to as a raising operation as it raises the lower component into the upper component slot. Conversely, when an operation transfers an upper component to a lower component slot, it is called a lowering operation. We will run into these concepts again in QFT.

Raising operation: column vector component up one level. Lowering operation: down one

2.3.11 Summary of Parametrizations and Characterizations

Wholeness Chart 2-4 summarizes what we have found for different relationships between Lie groups and their respective Lie algebras.

Wholeness Chart 2-4. Lie Group Parametrizations and Characterizations

Lie Group Relationships	Examples	Matrices	Lie Algebra
Same group, different parametrizations	$SU(2)$ of (2-25) and (2-72)	Different forms for matrices, but same dimension	Same abstract Lie algebra. May (or may not) have same basis matrices (generators) with same structure constants
Different groups characterizing same physical world phenomenon	$SO(3)$ and $SU(2)$, both for 3D rotation. $SO(2)$ and $U(1)$, both for 2D rotation	Different forms for matrices, different dimensions	Different abstract Lie algebra. May (or may not) have same structure constants

2.3.12 Shortcut Way to Generate the First Parametrization

Note that because of its particular form, our first parametrization (2-25) of the $SU(2)$ representation can be found rather easily from the Lie algebra simply by adding the generators and the identity matrix, multiplied by their associated parameters. That is,

1st SU(2) form is easy to generate from the generators

$$
\begin{aligned}
M &= \begin{bmatrix} \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{bmatrix} = \alpha_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + i\alpha_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i\alpha_2 \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + i\alpha_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\
&= \alpha_0 I + i\alpha_1 X_1 + i\alpha_2 X_2 + i\alpha_3 X_3 = \sqrt{1-\alpha_1^2-\alpha_2^2-\alpha_3^2}\, I + i\alpha_1 X_1 + i\alpha_2 X_2 + i\alpha_3 X_3 \qquad \text{(2-79)} \\
&= I + i\alpha_1 X_1 + i\alpha_2 X_2 + i\alpha_3 X_3 - \frac{\alpha_1^2}{2!} I - \frac{\alpha_2^2}{2!} I - \frac{\alpha_3^2}{2!} I + \frac{\alpha_1 \alpha_2}{2!}[0] + \ldots ,
\end{aligned}
$$

where the last line, in which we expand the dependent variable α_0 in terms of the independent variables, is simply our original expansion (2-61), which in terms of the generators is (2-65).

So, in this particular parametrization, going back and forth between the Lie group and the Lie algebra (plus the identity matrix) is relatively easy.

However, it is not so easy and simple with the second parametrization (2-72). In the expansion of $M(\phi_i)$ (which we didn't do), one gets terms of form $\phi_i X_i$ in the infinite summation, but the original matrix had functions of $sin\phi_1$, $cos\phi_1$, $e^{\pm i\phi_2}, e^{\pm i\phi_3}$ multiplied by one another. That gets complicated in a hurry.

2nd SU(2) form is hard to generate from the generators

As noted, in NRQM, we deal with the 2nd parametrization (2-72). In QFT, we deal with the 1st. So, in this sense, QFT is easier. (But, probably only in that sense.)

2.3.13 The SU(3) Lie Algebra

We repeat (2-28), the most suitable form of $SU(3)$ for our purposes, below

$$N = i\begin{bmatrix} -i+\alpha_3+\dfrac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -i-\alpha_3+\dfrac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -i-\dfrac{2\alpha_8}{\sqrt{3}} \end{bmatrix} \qquad \text{Repeat of (2-28)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + i\begin{bmatrix} \alpha_3+\dfrac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -\alpha_3+\dfrac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -\dfrac{2\alpha_8}{\sqrt{3}} \end{bmatrix} \qquad |\alpha_i| \ll 1.$$

The Gell-Mann matrices λ_i, which are convenient to work with in $SU(3)$ theory, are

Using these λ_i matrices, we can construct (or generate) N

$$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \qquad (2\text{-}80)$$

Then, with (2-80) and (2-28), we find

$$\begin{aligned} N &= I + i\left(\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 + \alpha_4\lambda_4 + \alpha_5\lambda_5 + \alpha_6\lambda_6 + \alpha_7\lambda_7 + \alpha_8\lambda_8\right) \\ &= I + i\alpha_i\lambda_i \qquad |\alpha_i| \ll 1, \end{aligned} \qquad (2\text{-}81)$$

which parallels the second line of (2-79). By doing Problem 19, you can show that the λ_i are the Lie algebra generators of N.

Do **Problem 19** to show that λ_i are the Lie algebra generators of N. This problem is important for a sound understanding of $SU(3)$.

Then do **Problem 20** to help in what comes next.

It turns out, if one cranks all the algebra using (2-80), that the following commutation relations exist between the Lie algebra generators (basis vector matrices), similar to (2-64) in $SU(2)$ theory,

SU(3) generators' commutation relations

$$\left[\lambda_i, \lambda_j\right] = i2 f_{ijk}\lambda_k \qquad \text{or} \qquad \left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i\, f_{ijk}\frac{\lambda_k}{2}, \qquad (2\text{-}82)$$

and where repeated indices, as usual, indicate summation. Similar to what we found with $SU(2)$ in (2-64), we can take our $SU(3)$ generators as $\frac{1}{2}\lambda_i$ just as readily as we can take them to be λ_i. For the former, the structure constants are f_{ijk}; for the latter, $2f_{ijk}$. The f_{ijk} are not, however, as simple as the structure constants for $SU(2)$, which took on the values ±1 of the Levi-Civita symbol ε_{ijk}.

SU(3) generators can be taken as $\frac{1}{2}\lambda_i$, with structure constants f_{ijk}

So that you are not confused, note that some authors define another tensor $F_i = \lambda_i/2$, use that to construct N, and refer when needed to the explicit form of the Gell-Mann matrices λ_i. The choice is, of course, conventional, but I think it easier and clearer to stick to one set of matrices (the λ_i), and that is what we do herein.

The f_{ijk} do turn out to be totally anti-symmetric, like the ε_{ijk}.

f_{ijk} are fully anti-symmetric

$$f_{ijk} = -f_{jik} = f_{jki} = -f_{kji} = f_{kij} = -f_{ikj}, \qquad (2\text{-}83)$$

and they take on the specific values shown in Table 2-2 (some of which you can check via your solution to Problem 20).

Table 2-2
Values of *SU*(3) Structure Constants

ijk	123	147	156	246	257	345	367	458	678	Others
f_{ijk}	1	½	−½	½	½	½	−½	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0

Note the form of (2-28) parallels that of (2-25). The form of (2-28) meshes with the conventional, widely used definition of (2-80).

As with $SU(2)$, we will not go through all the steps to show the λ_i with the α_i comprise a Lie algebra, as they parallel what we did for $SO(2)$ and $SO(3)$. One can see that the λ_i matrices and the scalars α_i form a vector space, and that with commutation as the second binary operation, defined via (2-82), there is closure.

The λ_i with α_i satisfy requirements of a Lie algebra

2.3.14 Generating the SU(3) Group from the SU(3) Lie Algebra

We include this section to keep the parallel with the $SU(2)$ group (Section 2.3.8 on pg. 31), although it should be fairly obvious that for $|\alpha_i| \ll 1$, we can generate the $SU(3)$ group from its Lie algebra via (2-81).

2.3.15 Exponentiation of the SU(3) Lie Algebra

General Case

One can generate the $SU(3)$ Lie group from its Lie algebra via a second, related way, which involves exponentiation, in a manner similar to what we saw earlier with $SO(2)$, $SO(3)$, and $SU(2)$. As we found before, doing so would be an algebraic nightmare, so we will once again simply outline the steps that would be taken.

Due to Baker-Campbell-Hausdorf, exponentiating Lie algebra to get SU(3) Lie group would be a mathematical morass

Consider a general $SU(3)$ Lie algebra element

$$\Lambda = i\alpha_i\lambda_i \qquad \text{any size } |\alpha_i|, \tag{2-84}$$

where, as we found with other groups earlier,

$$e^{i\alpha_i\lambda_i} \neq N(\alpha_i) \qquad \text{any size } |\alpha_i|. \tag{2-85}$$

A Taylor expansion of the LHS of (2-85) would not give us the RHS.

However, again as discussed before for other groups, we could, with a whole lot of effort, deduce other parameters, call them β_i, for which

$$e^{i\beta_i\lambda_i} = N(\alpha_i) \qquad \beta_i = \beta_i(\alpha_i). \tag{2-86}$$

But in principle, we could do it with $e^{i\beta_i\lambda_i}$ and eight β_i as functions of eight α_i.

The functions $\beta_i(\alpha_i)$ here are, in general, different functions from $\beta_i(\alpha_i)$ of (2-70), as they must be since there are different numbers of independent variables in the two cases. We use the same symbol to emphasize the parallels between $SU(2)$ and $SU(3)$. We won't be taking the time and effort to find the β_i of (2-86) here.

Infinitesimal Case

But, as with other groups, the non-equal sign in (2-85) becomes approximately equal for small parameters, i.e., $\alpha_i \to 0$, as the higher-order terms are dwarfed by the lower-order ones. And essentially, the equal sign replaces the non-equal sign for infinitesimal values of α_i. So,

Infinitesimal case is easier, as then $\beta_i = \alpha_i$.

$$\beta_i \to \alpha_i \qquad \text{as } \alpha_i \to 0, \tag{2-87}$$

and

$$e^{i\beta_i\lambda_i} \approx e^{i\alpha_i\lambda_i} \approx N(\alpha_i) \approx I + i\alpha_i\lambda_i \qquad |\alpha_i| \ll 1. \tag{2-88}$$

Generalizing Exponentiation

Wholeness Chart 2-5 below summarizes, and extrapolates to groups of degree n, what we have learned about exponentiation of the Lie algebra generators to generate the group.

Note that we have deduced the number of generators for a group of degree n, for each group type, from what we've learned about 2nd and 3rd degree groups. The number of generators for an nXn $SO(n)$ matrix equals the number of components above and to the right of the diagonal, i.e., one half of the total number of off-diagonal components. The number of generators for an nXn $SU(n)$ matrix equals the total number of components minus one.

Summary of n degree Lie groups shown in Wholeness Chart 2-5.

In the chart, and generally from here on throughout this text, we employ carets for real matrices and no carets for complex ones. Take caution, however, that this is not a commonly used convention. In particular, A is typically used to denote the 3D physical rotation matrix, but in the chart, we use $\hat{N}$.

And as noted earlier for special orthogonal groups, in order to draw clear parallels, we use the same symbol (i.e., $\hat{\theta}_i$ for real groups, β_i for complex ones) as functions of the independent variables in the 3 and n degree cases, but the functional dependence is generally different. That is $\beta_i(\alpha_i)$ is a different function of α_i for 3D matrices than it is for nD matrices (for $n \neq 3$).

Wholeness Chart 2-5. Exponentiation of Lie Algebra Generators

Lie Algebra	**Generators** Preferred Parametrization	**Lie Group Representation** Finite Parameters	 Infinitesimal Parameters
$SO(2)$	$\hat{X}$ of (2-42)	$\hat{M} = e^{i\theta\hat{X}}$	$\hat{M} = e^{i\theta\hat{X}}$ $= I + i\theta\hat{X}$
$SO(3)$	3 $\hat{X}_i$ matrices of (2-49) $i = 1, 2, 3$	$\hat{N}(\theta_i) = A(\theta_1,\theta_2,\theta_3) = e^{i\hat{\theta}_i\hat{X}_i}$ $= e^{i\theta_1\hat{X}_1}e^{i\theta_2\hat{X}_2}e^{i\theta_3\hat{X}_3}$ $\hat{\theta}_i = \hat{\theta}_i(\theta_j)$	$\hat{N} = e^{i\hat{\theta}_i\hat{X}_i} = e^{i\theta_i\hat{X}_i}$ $= I + i\theta_i\hat{X}_i$ $= I + i\hat{\theta}_i\hat{X}_i$ $\quad\hat{\theta}_i = \theta_i$
$SO(n)$	$\frac{n^2-n}{2}$ matrices $\hat{Y}_i$ $i = 1, 2, \ldots, \frac{n^2-n}{2}$	$\hat{P}(\theta_1,\theta_2,\ldots,\theta_n) = e^{i\hat{\theta}_i\hat{Y}_i}$ $= e^{i\theta_1\hat{Y}_1}e^{i\theta_2\hat{Y}_2}\ldots e^{i\theta_n\hat{Y}_n}$ $\hat{\theta}_i = \hat{\theta}_i(\theta_j)$	$\hat{P} = e^{i\hat{\theta}_i\hat{Y}_i} = e^{i\theta_i\hat{Y}_i}$ $= I + i\theta_i\hat{Y}_i$ $= I + i\hat{\theta}_i\hat{Y}_i$ $\quad\hat{\theta}_i = \theta_i$
$SU(2)$	3 Pauli matrices* $X_i = \sigma_i$ of (2-63) $i = 1, 2, 3$	$M(\alpha_i) = e^{i\beta_i X_i}$ $\beta_i = \beta_i(\alpha_i)$	$M = e^{i\beta_i X_i} = e^{i\alpha_i X_i}$ $= I + i\alpha_i X_i$ $= I + i\beta_i X_i$ $\quad\beta_i = \alpha_i$
$SU(3)$	8 Gell-Mann matrices* $X_i = \lambda_i$ of (2-80) $i = 1, 2, \ldots 8$	$N(\alpha_i) = e^{i\beta_i X_i}$ $\beta_i = \beta_i(\alpha_j)$	$N = e^{i\beta_i F_i} = e^{i\alpha_i X_i}$ $= I + i\alpha_i X_i$ $= I + i\beta_i X_i$ $\quad\beta_i = \alpha_i$
$SU(n)$	$n^2 - 1$ matrices Y_i $i = 1, 2, \ldots, n^2 - 1$	$P(\alpha_i) = e^{i\beta_i Y_i}$ $\beta_i = \beta_i(\alpha_j)$	$P = e^{i\beta_i Y_i} = e^{i\alpha_i Y_i}$ $= I + i\alpha_i Y_i$ $= I + i\beta_i Y_i$ $\quad\beta_i = \alpha_i$
* We can, instead, as is common in QFT, take $X_i = ½\sigma_i$ in $SU(2)$ and $X_i = ½\lambda_i$ in $SU(3)$. This would simply mean our arbitrary parameters β_i and α_i above would be multiplied by 2.			

2.3.16 Other Parametrizations of SU(3)

As with $SU(2)$, there are other parametrizations of $SU(3)$ [than (2-28)], but we won't delve into those at this point.

Other forms of $SU(3)$ exist, but we won't look at them here

2.4 "Rotations" in Complex Space and Associated Symmetries

2.4.1 Conceptualizing SU(n) Operations on Vectors

As we have noted in Vol. 1 (pg. 27, Box 2-3, and top half of pg. 199), a unitary operator operating on a generally complex state vector keeps the "length" (absolute magnitude) of that vector unchanged. This is parallel to real orthogonal matrices operating on real column vectors.

For real matrices and vectors, the operation of the matrix on the vector corresponds to a rotation of the vector (for the active operation interpretation), or alternatively, to a rotation of the coordinate axes from which the (stationary) vector is viewed (passive interpretation). (See Vol. 1, Section 6.1.2, pg. 164.)

$SU(n)$ group operations complex, but analogous to rotations in real spaces

For complex vector spaces and complex matrix operators acting on them, one can conceptualize, in an abstract sense, a unitary operation as a "rotation" in complex space of the complex vector (for the active interpretation), or alternatively, as a "rotation" of the abstract vector space coordinate "axes" from which the (stationary) vector is viewed (passive interpretation). Having this perspective in the back of your mind can often help in following the mathematical procedures involved in carrying out such operations.

2.4.2 *Symmetries Under SU(n)* Operations

We should know that a real scalar invariant, such as the length of a vector, remains unchanged under a special orthogonal transformation, i.e., a rotation. That is, for nD space, where $\hat{P}$ (see (2-6) for the 2D case) is the rotation operation and $[v]$ is the column vector form of the abstract vector **v**,

$SO(n)$ group operations = rotations with vector length invariant

$$|\mathbf{v}|^2 = \mathbf{v}\bullet\mathbf{v} = [v]^T[v] \qquad [v'] = [\hat{P}][v] \qquad \rightarrow \qquad |\mathbf{v}'|^2 = \mathbf{v}'\bullet\mathbf{v}' = [v']^T[v'] = [v]^T[\hat{P}]^T[\hat{P}][v] \tag{2-89}$$

For an orthogonal matrix, $[\hat{P}]^T = [\hat{P}]^{-1}$, so

$$|\mathbf{v}'|^2 = [v]^T[\hat{P}]^T[\hat{P}][v] = [v]^T[v] = |\mathbf{v}|^2, \tag{2-90}$$

and the length of the vector after rotation is unchanged, perhaps not such a surprising result to most readers. The key points are that 1) the length of the vector is invariant under the transformation (we say the scalar length is invariant or symmetric) and 2) we can carry this concept over to complex spaces.

So, now, consider an nD complex column vector represented by $[w]$ and an $SU(n)$ matrix operator P. (See (2-28) and Problem 7 for $n = 2$). We have

$SU(n)$ group operations keep complex vector magnitude invariant, i.e., like "rotations"

$$|\mathbf{w}|^2 = \mathbf{w}\bullet\mathbf{w} = [w]^\dagger[w] \qquad [w'] = [P][w] \qquad \rightarrow \qquad |\mathbf{w}'|^2 = \mathbf{w}'\bullet\mathbf{w}' = [w']^\dagger[w'] = [w]^\dagger[P]^\dagger[P][w]. \tag{2-91}$$

For a unitary matrix, $[P]^\dagger = [P]^{-1}$, so

$$|\mathbf{w}'|^2 = [w]^\dagger[P]^\dagger[P][w] = [w]^\dagger[w] = |\mathbf{w}|^2, \tag{2-92}$$

and the "length" (absolute magnitude) of the vector after the unitary operation is unchanged. As this is the hallmark of pure rotation of a vector, i.e., with no stretching/compression, in real vector spaces, it is natural to think of a special unitary operation as a kind of "rotation" in complex vector space.

Bottom line: The absolute magnitude of a real (complex) vector is symmetric (invariant) under an orthogonal (unitary) transformation.

Reminder note: Recall from Chap. 6, Sect. 6.1.2, of Vol. 1. (pgs.164-166) that for a scalar to be symmetric it must have both 1) the same functional form in transformed (primed for us) variables as it had in the original (non-prime for us) variables, and 2) the same value after the transformation.

A symmetric scalar (like vector length) has same value and functional form after transformation

As examples, in (2-89) and (2-90), and also in (2-91) and (2-92), we had

$$|\mathbf{v}|^2 = \underbrace{v_1v_1 + v_2v_2 + \ldots + v_nv_n}_{\text{original functional form}} = \underbrace{v_1'v_1' + v_2'v_2' + \ldots + v_n'v_n'}_{\substack{\text{transformed functional form}\\ \text{same as original}}} = |\mathbf{v}'|^2 \quad \begin{matrix}\text{equal sign means}\\ \text{same numeric value}\end{matrix}, \tag{2-93}$$

$$|\mathbf{w}|^2 = \underbrace{w_1^*w_1 + w_2^*w_2 + \ldots + w_n^*w_n}_{\text{original functional form}} = \underbrace{w_1'^*w_1' + w_2'^*w_2' + \ldots + w_n'^*w_n'}_{\substack{\text{transformed functional form}\\ \text{same as original}}} = |\mathbf{w}'|^2 \quad \begin{matrix}\text{equal sign means}\\ \text{same numeric value}\end{matrix}. \tag{2-94}$$

2.4.3 Applications in Quantum Theory

In General

In theories of quantum mechanics, a state vector can be expressed as a superposition of orthogonal basis state vectors, e.g.,

$$\underbrace{|\psi\rangle}_{\substack{\text{general}\\ \text{state}}} = A_1 \underbrace{|\psi_1\rangle}_{\substack{\text{basis}\\ \text{state}}} + A_2 \underbrace{|\psi_2\rangle}_{\substack{\text{basis}\\ \text{state}}} + A_3 |\psi_3\rangle + \ldots . A_n |\psi_n\rangle = A_i |\psi_i\rangle . \tag{2-95}$$

For the usual normalization, the probability of measuring (observing) the system to be in any given state i is $|A_i|^2$ and the "length" (absolute magnitude) of the state vector is unity, i.e.,

QM complex. States are SU(n) vectors with total (normalized) probability ("length") =1

$$\text{total probability} = \text{"length" of state vector} = |A_1|^2 + |A_2|^2 + |A_3|^2 + \ldots . |A_n|^2 = \sum_i |A_i|^2 = 1 . \tag{2-96}$$

The complex vector space state (2-95) can be represented as a column vector i.e.,

$$|\psi\rangle = \psi_{state} \xrightarrow[\text{basis}]{\text{in chosen}} = \begin{bmatrix} \langle\psi_1|\psi\rangle \\ \langle\psi_2|\psi\rangle \\ \vdots \\ \langle\psi_n|\psi\rangle \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} , \tag{2-97}$$

where in NRQM and RQM the vector space is Hilbert space.

A unitary operator, represented by the matrix P, operating on the state vector yields

$$P|\psi\rangle = |\psi'\rangle = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1n} \\ P_{21} & P_{22} & \cdots & P_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{nn} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = P_{ij} A_j = A'_i = \begin{bmatrix} A'_1 \\ A'_2 \\ \vdots \\ A'_n \end{bmatrix} , \tag{2-98}$$

where, because of unitarity [parallel to (2-92)],

Under unitary transformation, total probability (state vector "length") remains = 1

$$\langle\psi'|\psi'\rangle = \sum_i |A'_i|^2 = 1 = \sum_i |A_i|^2 = \langle\psi|\psi\rangle . \tag{2-99}$$

The absolute value of the state vector, its "length" (= total probability for the system to be found in *some* state), remains equal to one. This result generalizes to states in all branches of quantum mechanics, NRQM, RQM, and QFT.

Quantum Field Theory

Consider a unitary transformation, such as the S operator of QFT (Vol. 1, Sects. 7.4 to 7.5.2, pgs. 194-199), which describes a transition in Fock space (complex vector space with basis vectors being multiparticle eigenstates) from one state vector (components A_i) to another state vector (components $A_i{'}$). This parallels (2-95) to (2-99) for Fock space rather than Hilbert space. (See Vol. 1, Wholeness Chart 3-2, pg. 68, for a comparison of these two spaces.) In Vol. 1, Fig. 7-2, pg. 199, the S_{fi} there correspond to the $A_i{'}$ here. The total probability before and after the transformation remains unchanged, i.e., the total probability for the system to be in some state remains equal to one.

In QFT, under unitary S operator (interactions), total probability (state "length") remains = 1

In QFT, the Lagrangian (density) is an invariant scalar under external (global) transformations (Vol. 1, pg. 173), i.e., Poincaré (Lorentz plus 4D translation) transformations. It is called a world scalar, or a Lorentz scalar (Vol. 1, Sect. 2.5.1, point 11, pgs. 24-25). This particular invariance constitutes an external symmetry.

But the Lagrangian also has internal (local) symmetries, which leave it invariant under certain other transformations in other abstract spaces. These are also called gauge symmetries. (See Vol. 1, pgs. 178, 290-298.) In QED, we found the Lagrangian was symmetric (invariant) under the gauge transformations shown in the cited pages.

In QFT, scalar $\mathcal{L}$ invariant (symmetric) under external (Poincaré) and certain internal (gauge) transformations

As a simple example, consider the fermion mass term in $\mathcal{L}$, $m\bar{\psi}\psi$, under the gauge transformation

$$\psi \rightarrow \psi' = e^{-i\alpha}\psi , \tag{2-100}$$

$$m\bar{\psi}\psi \;\rightarrow\; m\bar{\psi}'\psi' = m\left(\bar{\psi}e^{i\alpha}\right)\left(e^{-i\alpha}\psi\right) = m\bar{\psi}\psi . \tag{2-101}$$

Example of an invariant term in $\mathcal{L}$ under a certain gauge transformation

That term in $\mathcal{L}$ is symmetric with respect to the transformation (2-100), and as we found in Chaps. 6 and 11 of Vol. 1, when we include the concomitant transformation on the photon field, so are all the other terms, collectively, in $\mathcal{L}$. ψ here is known as a gauge field.

With reference to (2-17) and Prob. 8, the transformation (2-100) is a $U(1)$ transformation. The QED Lagrangian is symmetric under (particular) $U(1)$ transformations. We have yet to discuss weak and strong interactions, which we will find are symmetric under particular $SU(2)$ and $SU(3)$ transformations, respectively, which is why the SM is commonly referred to by the symbology

$\mathcal{L}$ invariance under $U(1)$, $SU(2)$, and $SU(3)$ transformations underlies SM

$$SU(3)\times SU(2)\times U(1) \qquad [\text{the standard model}]. \tag{2-102}$$

There is some oversimplification here, as we will find the weak and QED interactions are intertwined in non-trivial ways, but that is the general idea.

Now consider certain terms in the free Lagrangian, where we note the subscripts e and ν_e refer to electron and electron neutrino fields, respectively, and the superscript L refers to something called left-handed chirality, the exact definition of which we leave to later chapters on weak interaction theory. For now, just consider it a label for a particular vector space we are interested in, which we will find later on in this book to be related to the weak interaction. For $\gamma^\mu \partial_\mu = \not{\partial}$,

$$\mathcal{L}_{\substack{two\\terms}} = \bar{\psi}^L_{\nu_e}\gamma^\mu\partial_\mu\psi^L_{\nu_e} + \bar{\psi}^L_e\gamma^\mu\partial_\mu\psi^L_e = \begin{bmatrix}\bar{\psi}^L_{\nu_e} & \bar{\psi}^L_e\end{bmatrix}\begin{bmatrix}\not{\partial}\psi^L_{\nu_e}\\ \not{\partial}\psi^L_e\end{bmatrix} = \begin{bmatrix}\bar{\psi}^L_{\nu_e} & \bar{\psi}^L_e\end{bmatrix}\not{\partial}\begin{bmatrix}\psi^L_{\nu_e}\\ \psi^L_e\end{bmatrix}, \tag{2-103}$$

We have cast the usual scalar terms just after the first equal sign into a two-component, complex, column vector (2D vector in complex space). In this case, it is composed of fields, not states.

Now, let's see how (2-103) transforms under a typical $SU(2)$ transformation, which we label M and recall that $M^\dagger = M^{-1}$. M here is a global transformation, i.e., not a function of spacetime x^μ.

Example of symmetry of two terms in $\mathcal{L}$, under a global $SU(2)$ transformation

$$\begin{aligned}\mathcal{L}'_{\substack{two\\terms}} &= \underbrace{\begin{bmatrix}\bar{\psi}'^L_{\nu_e} & \bar{\psi}'^L_e\end{bmatrix}\not{\partial}\begin{bmatrix}\psi'^L_{\nu_e}\\ \psi'^L_e\end{bmatrix}}_{\text{transformed functional form}} = \underbrace{\begin{bmatrix}\bar{\psi}^L_{\nu_e} & \bar{\psi}^L_e\end{bmatrix}M^\dagger}_{\text{transformed row vec}}\not{\partial}\;\underbrace{M\begin{bmatrix}\psi^L_{\nu_e}\\ \psi^L_e\end{bmatrix}}_{\text{transformed col vec}} \\ &= \begin{bmatrix}\bar{\psi}^L_{\nu_e} & \bar{\psi}^L_e\end{bmatrix}\not{\partial}\underbrace{M^\dagger M}_{M^{-1}}\begin{bmatrix}\psi^L_{\nu_e}\\ \psi^L_e\end{bmatrix} = \underbrace{\begin{bmatrix}\bar{\psi}^L_{\nu_e} & \bar{\psi}^L_e\end{bmatrix}\not{\partial}\begin{bmatrix}\psi^L_{\nu_e}\\ \psi^L_e\end{bmatrix}}_{\text{orignal functional form}} = \underbrace{\mathcal{L}_{\substack{two\\terms}}}_{\text{original value}}\end{aligned} \tag{2-104}$$

The transformed terms in (2-104) equal the original terms in (2-103), in both functional form and value. These terms are symmetric with respect to any global $SU(2)$ transformation.

We note that we have used a global transformation in order to make a point in the simplest possible way. The actual $SU(2)$ transformations in the SM are local (gauge) transformations, for which it is considerably more complicated to demonstrate invariance (but which we will do later in this book.)

One may wonder why we chose the particular two $\mathcal{L}$ terms of (2-103) to form our two component complex vector, instead of others, like perhaps an electron field and a muon field, or a right chirality (RC in this book, though many texts use RH) neutrino field and a left chirality (LC) electron field. The answer is simply that the form of (2-103) plays a fundamental role in weak interaction theory. Nature chose that particular two component vector as the one for which $SU(2)$ weak symmetry holds. But, we are getting ahead of ourselves. Much detail on this is yet to come in later chapters.

Nature has decided which particular terms form the 2-component vectors

These are LC electron field and its associated neutrino field

Note that quarks share the same $SU(2)$ symmetry with leptons, provided we form our two component quark weak field vector from the up and down quark fields. That is,

And LC up and down quarks

$$\mathcal{L}_{\substack{two\ other\\terms}} = \bar{\psi}^L_u\not{\partial}\psi^L_u + \bar{\psi}^L_d\not{\partial}\psi^L_d = \begin{bmatrix}\bar{\psi}^L_u & \bar{\psi}^L_d\end{bmatrix}\not{\partial}\begin{bmatrix}\psi^L_u\\ \psi^L_d\end{bmatrix}, \tag{2-105}$$

where (2-105) is invariant under the transformation M, in the same way we showed in (2-104).

Similar logic applies for $SU(3)$ transformations related to the strong (color) interaction. If quarks have three different eigenstates of color (red, green, and blue), we can represent terms for them in the Lagrangian as in (2-106). Note that up quarks can have any one of the three colors as eigenstates, and likewise, down quarks can have any one of the three. The same holds true for RC vs LC fields. So, in (2-106), we don't use the up/down subscripts for quark fields, nor the L superscript, as the results apply to both kinds of chirality and both components of the quark weak (2D) field vector in (2-105).

Similarly, 3-component color vectors exist for $SU(3)$

Consider then, $$\mathcal{L}_{\substack{three\ quark\\ terms}} = \bar{\psi}_r \not{\partial} \psi_r + \bar{\psi}_g \not{\partial} \psi_g + \bar{\psi}_b \not{\partial} \psi_b = \begin{bmatrix} \bar{\psi}_r & \bar{\psi}_g & \bar{\psi}_b \end{bmatrix} \not{\partial} \begin{bmatrix} \psi_r \\ \psi_g \\ \psi_b \end{bmatrix}. \qquad (2\text{-}106)$$

The three terms in (2-106) are symmetric under a global $SU(3)$ transformation.

Do **Problem 21** to prove it. Then do **Problem 22**.

Note that fermions differ in type [flavor] (electron, neutrino, up quark, down quark, etc.), chirality (LC vs RC), and color (R, G, or B). Each of these can be designated via a sub or superscript index. For example, an up, LC, green quark field could be written as (note use of capital letter Ψ)

$$\begin{gathered} \text{up, green, LC quark} \rightarrow \Psi^L_{ug} \qquad \text{generally} \rightarrow \Psi^h_{fa} \text{ where here } h = L;\, f = u;\, a = g \\ h = L,R \quad f = u,d \ (\text{for quarks}) \text{ or } \nu_e, e \ (\text{for leptons}) \quad a = r,g,b \ (\text{quarks});\ 0 (\text{leptons}) \end{gathered} \qquad (2\text{-}107)$$

Some symbols for LC vs RC, color

Leptons have no a component, as they are colorless (i.e., do not feel the strong force). And if, instead (which will often prove helpful), we want to use numbers for the two and three-component vectors to be acted upon by $SU(2)$ and $SU(3)$ matrices, we can take

$$f = 1,2 \ (\text{for quarks});\ 1,2 (\text{for leptons}) \qquad a = 1,2,3 \ (\text{quarks});\ \text{nothing} (\text{leptons}) \qquad (2\text{-}108)$$

Then, (2-107) is essentially an outer (tensor) product (Sect. 2.2.8, pg. 18) of a 2D vector with a 3D vector. An $SU(2)$ transformation on it would only affect the f indices and not the a indices. An $SU(3)$ transformation would only affect the a indices and not the f indices. That is, for quark fields, where M is an $SU(2)$ operation and N is an $SU(3)$ operation, we can write

Fields are outer products of 2D, 3D, spin, and spacetime elements

$$M_{mf} N_{na} \Psi^h_{fa} = \Psi'^{\,h}_{mn} \qquad f = 1,2 \quad a = 1,2,3 \quad m = 1,2 \quad n = 1,2,3\,. \qquad (2\text{-}109)$$

In the weak and strong interactions chapters, we will see how these $SU(2)$ and $SU(3)$ symmetries, via Noether's theorem (parallel to what we saw in QED for $U(1)$ symmetries) lead to weak and strong charge conservation (at least prior to symmetry breaking for weak charge). In addition, associated local transformation symmetries will lead to the correct interaction terms in the Lagrangian.

<u>Full Expressions of Typical Fields in QFT</u>

Finally, we note in (2-110) what typical fields might look like, if we expressed them in terms of column vectors, where the subscripts W and S refer to the weak and strong interactions, respectively. The subscripts u and e on the creation/destruction operators refer to up quark field and electron field, respectively. The down quark would have unity in the lower $SU(2)$ column vector component and zero in the upper. The neutrino field has unity in the top slot; the electron field, unity in the bottom one. The M matrix of (2-109) would operate on the two-component vector and nothing else; the N matrix on the three-component vector and nothing else. All of the operators we worked with in QED, such as electric charge, momentum, etc. would be formed from the ψ_u (or ψ_e) part, as in QED.

Examples of typical field outer products

$$\begin{gathered} \text{LC green up quark} \quad \Psi^L_{ug} = \Psi^L_{12} = \underbrace{\sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left(c_{ur}(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + d^\dagger_{ur}(\mathbf{p}) v_r(\mathbf{p}) e^{ipx} \right)}_{\text{general solution to Dirac equation for up quark, } \psi_u} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_W \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S = \psi_u \begin{bmatrix} 1 \\ 0 \end{bmatrix}_W \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S \\ \text{LC electron} \quad \Psi^L_e = \Psi^L_2 = \underbrace{\sum_{r,\mathbf{p}} \sqrt{\frac{m}{VE_{\mathbf{p}}}} \left(c_{er}(\mathbf{p}) u_r(\mathbf{p}) e^{-ipx} + d^\dagger_{er}(\mathbf{p}) v_r(\mathbf{p}) e^{ipx} \right)}_{\text{general solution for electron field, } \psi_e} \begin{bmatrix} 0 \\ 1 \end{bmatrix}_W = \psi_e \begin{bmatrix} 0 \\ 1 \end{bmatrix}_W \end{gathered} \qquad (2\text{-}110)$$

u_r and v_r are also column vectors (in 4D spinor space) not explicitly shown in (2-110). As shown later, $d^\dagger_{ur}$ and $d^\dagger_{er}$ here create antiparticles with “anti-color” (for quarks) *and* opposite (RC) chirality.

Note further that it will be easier in the future if we use slightly different notation (parallel to that in (2-103) to (2-105)), whereby the last terms in both rows of (2-110) are written as

$$\psi_e \begin{bmatrix} 0 \\ 1 \end{bmatrix}_W = \begin{bmatrix} 0 \\ \psi_e \end{bmatrix}_W = \begin{bmatrix} 0 \\ \psi^L_e \end{bmatrix} \qquad \psi_u \begin{bmatrix} 1 \\ 0 \end{bmatrix}_W \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S = \begin{bmatrix} \psi_u \\ 0 \end{bmatrix}_W \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S = \begin{bmatrix} \psi^L_u \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S . \qquad (2\text{-}111)$$

Alternative symbolism

Summary and Conclusions

Wholeness Chart 2-6 summarizes transformations and symmetries for various spaces.

Symmetries of the Lagrangian exist under $SU(n)$ transformations and these, as we will find, have profound implications for QFT.

Wholeness Chart 2.6. Symmetries in Various Spaces

Space	Operator	Example	Transformed	Sym?
Euclidean, 2D	$SO(2)$ rotation	vector length, $\vert\mathbf{v}\vert$	$\vert\mathbf{v}'\vert = \vert\mathbf{v}\vert$	Y
	"	function (circle) $x^2 + y^2 = r^2$	function $x'^2 + y'^2 = r^2$	Y
	"	function (ellipse) $x^2 + 3y^2 = r^2$	function $x'^2 + 3y'^2 \neq r^2$	N
Physical, 3D	$SO(3)$ rotation	vector length $\vert\mathbf{v}\vert$	$\vert\mathbf{v}'\vert = \vert\mathbf{v}\vert$	Y
	"	laws of nature (functional form)	laws in primed coordinates = laws in unprimed coordinates	Y
	"	vector component v^i	$v'^i \neq v^i$	N
	Non-orthogonal transform	vector length $\vert\mathbf{v}\vert$	$\vert\mathbf{v}'\vert \neq \vert\mathbf{v}\vert$	N
Minkowski, 4D	$SO(3,1)$ Lorentz transform	4D position vec. length = Δs	$\Delta s' = \Delta s$	Y
	"	mass m	$m^2 = p^\mu p_\mu = p'^\mu p'_\mu$	Y
	"	laws of nature (functional form)	Laws in primed coordinates = laws in unprimed coordinates	Y
	"	4-momentum component p^μ	$p'^\mu \neq p^\mu$	N
	Non-Lorentz (non 4D orthogonal) transform	4D position vec. length = Δs	$\Delta s' \neq \Delta s$	N
Hilbert, *n*D	$SU(n)$ ("rotation")	state vector length = total probability = 1	$\Sigma \vert A'_i\vert^2 = \Sigma \vert A_i\vert^2 = 1$ i.e., $\langle \psi' \vert \psi' \rangle = \langle \psi \vert \psi \rangle = 1$	Y
n "axes" = single particle eigenstates	"	Schrödinger eq (SE)	SE primed = SE unprimed	Y
	Non-unitary transform	state vector length	$\langle \psi' \vert \psi' \rangle = \Sigma \vert A'_i\vert^2 \neq \Sigma \vert A_i\vert^2 = \langle \psi \vert \psi \rangle$	N
Fock, *n*D	Complex "rotation", such as S operator	state vector length = total probability = 1	$\langle \psi' \vert \psi' \rangle = \Sigma \vert A'_i\vert^2 = \Sigma \vert A_i\vert^2 = \langle \psi \vert \psi \rangle$	Y
n "axes" = multiparticle eigenstates	"	K-G, Dirac, etc. field equations functs. of ϕ, ψ, etc	same field equations in terms of primed fields, ϕ', ψ', etc.	Y
	Non-unitary transform	state vector length	$\langle \psi' \vert \psi' \rangle = \Sigma \vert A'_i\vert^2 \neq \Sigma \vert A_i\vert^2 = \langle \psi \vert \psi \rangle$	N
fields operating on those states	$U(1)$ ("rotation") such as $\psi \to \psi' = \psi e^{-i\alpha}$	free fermion Lagrangian (density) $\mathcal{L}_0^{1/2}$	$\mathcal{L}'_0{}^{1/2}$ is same function of ψ' as $\mathcal{L}_0^{1/2}$ is of ψ	Y
Weak, 2D	$SU(2)$ ("rotation")	$\mathcal{L}_o$ functs. of ψ_e, ψ_ν, etc.	$\mathcal{L}_o'$ same functs. of ψ_e', ψ_ν', etc.	Y
2 "axes" = LC e^- & ν	"	field eqs. in ψ_e, ψ_ν, etc	same field eqs. in ψ_e', ψ_ν', etc.	Y
Color, 3D	$SU(3)$ ("rotation")	$\mathcal{L}_o$ functions of quark, lepton, boson fields	$\mathcal{L}_o'$ same functions of primed fields as $\mathcal{L}_o$ is of unprimed fields	Y
3 "axes" = RGB	"	field eqs. in unprimed fields	same field eqs. in primed fields.	Y

2.5 Singlets and Multiplets

2.5.1 Physical Space and Orthogonal Transformation Groups

In the ordinary 3D space of our experience, we know of two different types of entities, vectors and scalars. In an n dimensional real space, vectors have n components, but regardless of what n is, scalars always have one component. An orthogonal transformation on the vector will change the components of the vector. But an orthogonal transformation leaves the value of a scalar unchanged. It is as if the orthogonal operation acting on the scalar is simply the 1D identity "matrix", i.e., the number one.

For an $SO(n)$ group operating on n dimensional vectors, the matrix representation of the group is an nXn matrix. But for the group operating on scalars, the same group is represented by unity, a 1X1 matrix. The nXn matrix is one representation of the group (that acts on vectors). The 1X1 matrix is another representation of the same group (that acts on scalars).

It may seem confusing that we deal with 1X1 matrices for an $SO(n)$ group, instead of nXn matrices. To help, consider the $SO(2)$ group of (2-6), where **v** symbolizes a vector and $\hat{\mathbf{s}}$ symbolizes a scalar,

Rotations SO(2) change vector component values, but not scalar value

$$SO(2):\quad \hat{\mathbf{M}}(\theta)\mathbf{v} \xrightarrow[\text{rep}]{\text{matrix}} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} \qquad \hat{\mathbf{M}}(\theta)\hat{\mathbf{s}} \xrightarrow[\text{rep}]{\text{matrix}} [1]\hat{s} = \hat{s}' = \hat{s}\,. \tag{2-112}$$

We say the 2X2 matrix in (2-112) is a 2D representation of the group, and the unit "matrix" is the 1D representation of the same group. The former acts on vectors, the latter on scalars. In group theory lingo (as opposed to typical physics lingo), the vector is called a doublet, and the scalar a singlet.

2D vector = doublet, scalar = singlet, SO(2) operation has 2D & 1D reps

The matrix generators for each rep (we will start using "rep" sometimes as shorthand for "representation") are different, but they represent the same group. For the 2D rep, the generator is the $\hat{X}$ matrix of (2-42), which commutes with itself. Since the 1D rep is essentially the 1D identity matrix (the number one), and the group can be expressed as $e^{i\theta\hat{X}}$ (see Wholeness Chart 2-5, pg. 37), for any θ, the generator $\hat{X}$ in 1D is zero, since $1 = e^{i\theta 0}$. This commutes with itself, and thus, the commutation relations are the same, a criterion for different reps of the same group (with the same parametrization).

Same commutator relation for 2D & 1D reps

To express it slightly differently, consider the vector space of real numbers, which are 1X1 column vectors. Transform these vectors under an $SO(2)$ transformation, which operates on 1X1 vectors and so must be represented by 1X1 matrices. Such a transformation matrix must have determinant of one, so the only entry in the 1X1 matrix has to be one.

For SO(3), 3D vector = triplet, scalar = singlet, operation has 3D and 1D reps

In 3D space, the vector is called a triplet, and the scalar a singlet. The three generators $\hat{X}_i$ of (2-49) for the 3D rep obey the commutation relations (2-51). For the 1D rep, each of the $\hat{X}_i$ is zero. (See Wholeness Chart 2-5 where the $\hat{\theta}_i$ are arbitrary and the group operator must be unity.) Thus, in the 1D rep, the generators also obey (2-51) [rather trivially]. The commutation relation holds in both reps of the $SO(3)$ group.

Same commutators in both reps

The important point is that singlets are unchanged by $SO(n)$ group operations. In a nutshell, because $Det\,\hat{M} = 1$, and for 1D, $\hat{M}$ is 1X1, $\hat{M}$ must = 1. In practice, we simply remember that when an SO group operates on a scalar (a singlet), the scalar (singlet) remains unchanged.

Singlet unchanged under SO(n) action

2.5.2 Complex Space and Unitary Transformation Groups

As we've seen before, $SU(n)$ operations are the complex cousins of $SO(n)$ operations. Just as we had nD multiplets (vectors) and 1D singlets (scalars) upon which special orthogonal matrices operated in real n dimensional space, so we have nD multiplets and 1D singlets upon which special unitary groups operate in complex n dimensional space.

For SU(n), vector = multiplet, scalar = singlet, operation has nD and 1D reps

As an example, consider the $SU(2)$ group with 2D rep of (2-25), where **w** symbolizes a complex doublet and **s** symbolizes a complex singlet.

$$SU(2):\ \mathbf{Mw} \xrightarrow[\text{rep}]{\text{matrix}} \begin{bmatrix} \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_1' \\ w_2' \end{bmatrix} \quad \mathbf{Ms} \xrightarrow[\text{rep}]{\text{matrix}} [1]s = s' = s \tag{2-113}$$

For singlet, 1D "matrix" rep = 1

For the 1D case, **M** is the 1D identity matrix, and since $M = e^{i\beta_i X_i}$ (Wholeness Chart 2-5, pg. 37) for all β_i, all three $X_i = 0$. The commutation relations hold trivially. And the singlet is invariant.

Same commutators

Similar logic holds for $SU(3)$ with the 3D complex vectors as triplets and complex scalars are singlets. In the 1D rep, the group takes the form of a 1D identity matrix, the generators are all zero, and a singlet is unchanged under the group action. These conclusions hold for any $SU(n)$.

For 1D rep in $SU(n)$, generators all = 0; singlet unchanged under action of $SU(n)$

As a foreshadowing of things to come, we will see later on that right-chiral fermions are weak [$SU(2)$] singlets and left-chiral ones form weak doublets. Similarly, each lepton is a color [$SU(3)$] singlet and quarks form color triplets. But don't think too much more about this right now.

Some fields in QFT are singlets, some are multiplet components

2.5.3 Other Multiplets

There are yet other dimensional representations for any given group. For example, 3D and 4D representations of $SO(2)$ are

$$\hat{\underline{M}} = \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{bmatrix} \qquad \hat{\underset{\sim}{M}} = \begin{bmatrix} \cos\theta & -\sin\theta & & \\ \sin\theta & \cos\theta & & \\ & & \cos\theta & -\sin\theta \\ & & \sin\theta & \cos\theta \end{bmatrix}. \tag{2-114}$$

Reps can have dimensions other than n and 1

These share similarities with (2-6) and, respectively, would act on an $SO(2)$ triplet and an $SO(2)$ quadruplet.

The generators for (2-114) are, respectively,

$$\hat{\underline{X}} = \begin{bmatrix} 0 & i & \\ -i & 0 & \\ & & 0 \end{bmatrix} \qquad \hat{\underset{\sim}{X}} = \begin{bmatrix} 0 & i & & \\ -i & 0 & & \\ & & 0 & i \\ & & -i & 0 \end{bmatrix}, \tag{2-115}$$

which share similarities with (2-42). The commutation relation for the generator of each of the matrices in (2-114) is the same as for the generators of (2-6). (In this trivial case of only one generator, the generator commutes with itself.)

Similarly, there can be different dimension representations for any given group. For the same parametrization, the commutation relations for every such representation will be the same.

Reps of different dimensions have same commutators for generators

In the simplest view, for a given choice of parameters, the representation comprising an nXn matrix is called the <u>fundamental (or standard) representation</u>. There are more sophisticated ways, steeped in mathematical jargon, to define the fundamental rep, but this definition, though perhaps not fully precise, should suffice for our purposes. The rep associated with the singlet is called the <u>trivial representation</u>.

Fundamental rep, simple definition = nXn matrix rep

We will not do much, if anything, in this book with representations of dimensions other than n and 1 (acting on n-plet and singlet) for any group of degree n.

2.5.4 General Points for Multiplets and Associated Reps

In both real and complex spaces, we know that components of a multiplet change when acted upon by the ($SO(n)$ or $SU(n)$) group, but the singlet is invariant. We can also recognize that a matrix representation of a group does not have to be of dimension n. For the same group, matrix reps of different dimensions having the same parametrizations all have the same structure constants (same commutation relations).

Singlets invariant under $SU(n)$ transformations

Commonly used names for fundamental reps of certain groups are shown in Table 2-3. Note we show the formal mathematical symbols (see Table 2-1, pg. 23) for the vector spaces in parentheses.

Common names for fundamental reps

Table 2-3. Common Names of Some Fundamental Representations

Group	Vector Space	Name
$SU(2)$	2D complex ($\mathbb{C}^2$)	spinor
$SO(3)$	3D real ($\mathbb{R}^3$)	vector
$SO(3,1)$	4D relativistic, real ($\mathbb{R}^4$)	four-vector

2.5.5 Singlets and Multiplets in QFT

As you may have guessed, what we called 2-component and 3-component vectors in (2-103) to (2-111) are more properly called doublets and triplets. In the referenced equations they were $SU(2)$ LC (weak interaction) field doublets and $SU(3)$ color (strong interaction) field triplets.

In QFT, as we know, fields create and destroy states. So, we will find that the individual fields, which are the components in doublets and triplets, create and destroy particles in (generally) multiparticle states. In the SM, we will deal with both fields and states, just as we have in all the QFT we have studied to date.

In QFT, components of multiplet fields create and destroy multiparticle states

2.6 Matrix Operators on States vs Fields

In Section 2.4.3, we began to apply what we had learned of group theory to quantum mechanics. We reviewed how the action of a unitary operator (represented by a matrix) on a state left the magnitude of that state unchanged, even though the component parts of the state changed.

We then discussed the action of unitary operators on *fields* (as opposed to states) in QFT. We showed examples of this for $U(1)$ in (2-101), $SU(2)$ in (2-104), and $SU(3)$ in Problem 21. The question can then arise as to whether, in QFT, we take our vectors (multiplets upon which group operators operate), to be fields or states. The answer is a little subtle.

In QFT, do group operators act on field multiplets or state multiplets?

2.6.1 The Spin Operator on States and Fields

We start by referencing Vol. 1, Sect. 4.1.10, pg. 93. There we show the RQM spin operator, where for simplicity we only discuss the z direction component of the total spin operator, acting on a particular RQM (single particle) spinor state at rest with spin in the z direction. See Vol. 1, (4-40).

To help answer: spin operator example from Vol. 1

$$\text{For RQM} \qquad \Sigma_3 \left| \psi_{\substack{spin\,up \\ \mathbf{p}=0}} \right\rangle = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-ipx} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-ipx} = \frac{1}{2}\left| \psi_{\substack{spin\,up \\ \mathbf{p}=0}} \right\rangle \tag{2-116}$$

However, in QFT, as we discussed in detail in Vol. 1, Sect. 4.9, pgs. 113-115, we need to take into account that a state may be multiparticle. We found that by defining our spin operator (which acts on states) to be the LHS of (2-117) below, we got the RHS. See Vol. 1, (4-110) and (4-119).

$$_{\text{QFT}}\Sigma_3 = \int_V \psi^\dagger \Sigma_3 \psi \, d^3x \quad \rightarrow \quad _{\text{QFT}}\Sigma_3 = \sum_{r,\mathbf{p}} \frac{m}{E_\mathbf{p}} \left(u_r^\dagger(\mathbf{p}) \Sigma_3 u_r(\mathbf{p}) \, N_r(\mathbf{p}) + v_r^\dagger(\mathbf{p}) \Sigma_3 v_r(\mathbf{p}) \, \bar{N}_r(\mathbf{p}) \right) \tag{2-117}$$

QFT spin operator on states includes number operators and spinors

So, for a state with three at rest spin up electrons, we get

$$\begin{aligned} _{\text{QFT}}\Sigma_3 \left| 3\psi_{\substack{spin\,up \\ \mathbf{p}=0}} \right\rangle &= \sum_{r,\mathbf{p}} \frac{m}{E_\mathbf{p}} \left(u_r^\dagger(\mathbf{p}) \Sigma_3 u_r(\mathbf{p}) \, N_r(\mathbf{p}) + v_r^\dagger(\mathbf{p}) \Sigma_3 v_r(\mathbf{p}) \, \bar{N}_r(\mathbf{p}) \right) \left| 3\psi_{\substack{spin\,up \\ \mathbf{p}=0}} \right\rangle \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} 3 \left| 3\psi_{\substack{spin\,up \\ \mathbf{p}=0}} \right\rangle = \frac{3}{2}\left| 3\psi_{\substack{spin\,up \\ \mathbf{p}=0}} \right\rangle . \end{aligned} \tag{2-118}$$

The eigenvalue, representing total spin, is three times that of a single spin up electron, so it all works. In QFT, the (multiparticle) state is just represented by a symbol, the ket symbol like that of (2-118), and we generally don't think of it as having structure, such as spinor column vectors. The definition (2-117) provides the spinor column vectors in the operator itself. In any operation on a ket of given spin, these column vectors in the QFT spin operator give us the needed structure that leads to the correct final result. We don't have to worry about spinor structure in the kets. It is included in the spin operator.

If we had used a single particle in (2-118), we would have found the spin eigenvalue of ½. Note, as in (2-119) below, that if we had used the original operator of (2-116) and operated with it on the QFT field (instead of the state) we would also get an eigenvalue of ½.

$$\text{On a quantum field } \Sigma_3 \psi_{\substack{spin\,up\\ \mathbf{p}=0}} = \frac{1}{2}\begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix}\left(c_r(\mathbf{p})\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-ipx} + d_r^{\dagger}(\mathbf{p})\begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{ipx} \right)$$
$$= \frac{1}{2}\left(c_r(\mathbf{p})\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-ipx} + d_r^{\dagger}(\mathbf{p})\begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{ipx} \right) = \frac{1}{2}\psi_{\substack{spin\,up\\ \mathbf{p}=0}} . \tag{2-119}$$

Σ_3 operating on spin up field yields same eigenvalue as $\int \psi^{\dagger}\Sigma_3 \psi \, d^3x$ on one particle spin up state

Conclusion: Σ_3 acting on a quantum field of given spin yields the same eigenvalue as ${}_{\text{QFT}}\Sigma_3 = \int \psi^{\dagger}\Sigma_3 \psi \, d^3x$ operating on a single particle state of the same spin.

2.6.2 Other Operators on States vs Fields

Similar for other QFT operators

Somewhat similar logic works for other operators in QFT. Commonly, our operators, like M in (2-104), operate on fields directly, as we show in (2-104). If we want the corresponding operation to apply to (multiparticle) states, we parallel what we did for spin in going from (2-116) to (2-117). That is,

$$\text{Operation on quantum fields } M\Psi^L = M\begin{bmatrix}\psi_{\nu_e}^L \\ \psi_e^L\end{bmatrix} = M\psi_{\nu_e}^L\begin{bmatrix}1\\0\end{bmatrix} + M\psi_e^L\begin{bmatrix}0\\1\end{bmatrix}$$
$$\text{Corresponding operation on state } \left(\int_V \Psi^{L\dagger} M \Psi^L \, d^3x\right)\left| n_{\nu_e}\psi_{\nu_e}^L, n_e \psi_e^L \right\rangle \tag{2-120}$$

Generally, SU(n) group operators in QFT act on field multiplets, but we can find associated operators for states

where the ket here could have any number n_{ν_e} of neutrinos and any number n_e of electrons, where the neutrinos and electrons could have any spin and momentum values, and where M only operates on the two-component column vector shown and not the spacetime or 4D spinor column vector factors in the fields.

Let's consider the particular form for M of

$$M = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{2-121}$$

Hopefully, in parallel with (2-117), and recalling (2-110), 2nd row, we can intuitively deduce that

$$\int_V \Psi^{L\dagger} M \Psi^L \, d^3x = \sum_{r,\mathbf{p}}\left(\begin{matrix} (1\;\;0)M\begin{pmatrix}1\\0\end{pmatrix}N_{\nu_e,r}^L(\mathbf{p}) + (0\;\;1)M\begin{pmatrix}0\\1\end{pmatrix}N_{e,r}^L(\mathbf{p}) \\ -(1\;\;0)M\begin{pmatrix}1\\0\end{pmatrix}\overline{N}_{\nu_e,r}^R(\mathbf{p}) - (0\;\;1)M\begin{pmatrix}0\\1\end{pmatrix}\overline{N}_{e,r}^R(\mathbf{p}) \end{matrix}\right) \tag{2-122}$$

Do **Problem 23** to prove (2-122). Or take less time by just looking at the solution booklet answer.

Now let's consider an example where we have a single electron ket ($n_{\nu_e} = 0$, $n_e = 1$ in (2-120)).

$$\left(\int_V \Psi^{L\dagger} M \Psi^L \, d^3x\right)\left|\psi_e^L\right\rangle = \sum_{r,\mathbf{p}}(0\;\;1)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}0\\1\end{pmatrix}N_{e,r}(\mathbf{p})\left|\psi_e^L\right\rangle = -\left|\psi_e^L\right\rangle . \tag{2-123}$$

For M acting directly on the electron field, we get

$$M\Psi_e^L = M\begin{bmatrix}0\\ \psi_e^L\end{bmatrix} = M\psi_e^L\begin{bmatrix}0\\1\end{bmatrix} = \psi_e^L\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = -\psi_e^L\begin{bmatrix}0\\1\end{bmatrix} = -\begin{bmatrix}0\\ \psi_e^L\end{bmatrix} = -\Psi_e^L . \tag{2-124}$$

The eigenvalues in (2-123) and (2-124) are the same, i.e., – 1. We generalize this result below.

Conclusion #1: Eigenvalues from any unitary operator M operating on field multiplets Ψ are the same as those from the associated operator $\int \Psi^{\dagger} M \, \Psi \, d^3x$ operating on single particle states.

Eigenvalues for an operator M acting on field multiplets same as $\int \Psi^{\dagger} M \, \Psi \, d^3x$ on one particle states.

Conclusion #2: In group theory applied to QFT, elements of the group (such as M in our example above) act on multiplets composed of fields. The group theory vectors comprise fields, not states.

We will see this material again, in a different, more rigorous way later in the book, when we get to four-currents and conservation laws in weak and strong interaction theories.

SU(n) group elements in QFT act on field multiplets, not states

2.7 Cartan Subalgebras and Eigenvectors

2.7.1 SU(2) Cartan Subalgebra

Note that one of the $SU(2)$ generators (2-63) is diagonal (and like M of (2-121) in the example above).

$$X_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} . \tag{2-125}$$

Observe the operation of this generator matrix on particular $SU(2)$ doublets.

$$X_3 \begin{bmatrix} C_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ 0 \end{bmatrix} = \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \qquad X_3 \begin{bmatrix} 0 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ C_2 \end{bmatrix} = -\begin{bmatrix} 0 \\ C_2 \end{bmatrix} . \tag{2-126}$$

Diagonal generators operating on single component multiplets yield eigenvalues

---

Do **Problem 24** to show the effect of X_3 on a more general $SU(2)$ doublet.

The first doublet in (2-126) has an eigenvalue, under the transformation X_3, of +1; the second, of –1. We can use these eigenvalues to label each type of doublet. If d symbolizes a given doublet, one with $C_1 = 1$ and $C_2 = 0$, it would be d_{+1}; for $C_1 = 0$ and $C_2 = 1$, d_{-1}. A general doublet, with any values for the constants C_i, would be $d_i = C_1 \, d_{+1} + C_2 \, d_{-1}$. As another foreshadowing of things to come, consider the particular QFT weak interaction field doublets of (2-127), where a constant factor of $g/2$ (where the constant g is discussed more below) is used by convention. Note this can be considered as simply taking $X_3 = ½\sigma_i$ instead of σ_i, with a constant factor of g.

$$\begin{aligned} \frac{g}{2} X_3 \Psi_e^L &= \frac{g}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \psi_e^L \end{bmatrix} = -\frac{g}{2} \begin{bmatrix} 0 \\ \psi_e^L \end{bmatrix} = -\frac{g}{2} \Psi_e^L \\ \frac{g}{2} X_3 \Psi_{\nu_e}^L &= \frac{g}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \psi_{\nu_e}^L \\ 0 \end{bmatrix} = \frac{g}{2} \begin{bmatrix} \psi_{\nu_e}^L \\ 0 \end{bmatrix} = \frac{g}{2} \Psi_{\nu_e}^L \end{aligned} \tag{2-127}$$

Can be used to assign component eigenvalues (quantum numbers) to weak doublet states

If the doublet is a LC electron, we get an eigenvalue for the $\frac{g}{2} X_3$ operation of $-\frac{g}{2}$. If it is a LC neutrino, we get $+\frac{g}{2}$. From what we learned in Section 2.6 (eigenvalues for a group operator on a field are the same as those for the associated operator on a single particle state), we can use these eigenvalues to label the associated particles.

Note that for singlets, in the 1D rep, the generator X_3 is zero (as are the other generators). So,

$$\frac{g}{2} X_3 \psi_e^R = \frac{g}{2} [0] \psi_e^R = 0 \qquad \frac{g}{2} X_3 \psi_{\nu_e}^R = \frac{g}{2} [0] \psi_{\nu_e}^R = 0 . \tag{2-128}$$

For singlets, eigenvalues = 0

It will turn out, when we get to weak interactions, that the operator $\frac{g}{2} X_3$ corresponds to the weak charge operator. Weak doublets and singlets are eigenstates of that operator, with eigenvalues that correspond to the weak charge of the respective particles.

Recall that RC fermions do not feel the weak force, and from (2-128), we see that RC fermions have zero weak charge. From (2-127), LC electrons have weak charge $-\frac{g}{2}$; LC neutrinos, $+\frac{g}{2}$.

Weak operator eigenvalues = weak charges

Due to the negative signs in (2-122) on the antiparticle number operators, an antiparticle ket would have the opposite sign for weak charge (as antiparticles have for any type charge in the SM). The constant g reflects the inherent strength of the weak interaction, as greater value for it means a higher weak charge. It is called the weak coupling constant, though it is actually on the order of the e/m coupling constant e. The weak interaction is weak for another reason we will delve into later.

In practice, when discussing weak charge, the coupling constant g is commonly ignored. That is, one generally says the weak charge of the LC electron is – ½; of the LC neutrino, + ½. We just need to keep in mind that it is actually g times +½ or – ½.

Weak charge includes weak coupling g, but g commonly omitted in weak charge designations

Note that diagonal matrices commute. Note also that the diagonal matrix X_3, by itself (in either the 2D or 1D rep), forms a Lie algebra. It conforms to all the properties we list for an algebra in Wholeness Chart 2-1, pg. 9, with the second operation as commutation, closure, and the structure constant of the parent algebra (or really any structure constant since all elements in the algebra commute). Hence, X_3 comprises a subalgebra (within the entire algebra for $SU(2)$), and is called a Cartan subalgebra, after its discoverer Élie Cartan. A Cartan sub-algebra, in the general case, comprises a sub-algebra of a parent Lie algebra, wherein the sub-algebra comprises all commuting elements of the parent algebra (for matrices, all diagonal matrices).

Subset of diagonal matrices called Cartan subalgebra

The Cartan subalgebra provides us with a means for labeling the vectors (upon which the parent algebra acts) with vector eigenvalues under operation of the Cartan generators. And the eigenvectors, in turn, provide a basis with which we can construct any general (non-eigenstate) vector.

Wholeness Chart 2-7a lists the weak charge eigenvalues for the first generation (first family) of fermions. As one might expect, the other two known lepton generations (muon/muon neutrino and tau/tau neutrino) are directly parallel. That is, they form the same multiplets, with the same weak charges, just as they had parallel e/m charges in electrodynamics.

Quarks form weak doublets and singlets as well, and they parallel those for leptons. As might be expected, the LC up quark typically occupies the upper slot in the doublet; and the LC down quark, the lower slot. Just as with leptons, there are second and third generations of quarks, which, like their lepton cousins, play a far smaller role in creation than their first generation counterparts. The second of these is comprised of the charmed and strange quarks; the third, of the top and bottom quarks. As a mnemonic, remember that the more positive "quality" (up, charmed, top) for quarks gets the upper slot (positive weak charge) in the doublet, while the more negative one (down, strange, bottom), the lower (negative weak charge) slot.

You may wish to go back and compare what we said about charges in Section 2.2.8 (pg. 18) subheading A Hypothetical Example with this present section.

Aside on Group Theory Lingo

In formal group theory, the X_3 eigenvalues of the LC electron neutrino (up quark) and the LC electron (down quark) doublets are called weights. Weight I is + ½, and weight II is – ½. We will not be using this terminology in this book.

Wholeness Chart 2-7a. *SU*(2) Cartan Subalgebra Generator Eigenvalues for 1st Generation Fermions

Leptons

Particle	Multiplet	Weak Charge
RC electron neutrino	singlet $\psi^R_{\nu_e}$	0
RC electron	singlet ψ^R_e	0
LC electron neutrino	doublet $\begin{bmatrix}\psi^L_{\nu_e}\\0\end{bmatrix}=\Psi^L_{\nu_e}$	$+\frac{1}{2}$
LC electron	doublet $\begin{bmatrix}0\\\psi^L_e\end{bmatrix}=\Psi^L_e$	$-\frac{1}{2}$

Quarks

Particle	Multiplet	Weak Charge
RC up quark	singlet ψ^R_u	0
RC down quark	singlet ψ^R_d	0
LC up quark	doublet $\begin{bmatrix}\psi^L_u\\0\end{bmatrix}=\Psi^L_u$	$+\frac{1}{2}$
LC down quark	doublet $\begin{bmatrix}0\\\psi^L_d\end{bmatrix}=\Psi^L_d$	$-\frac{1}{2}$

2.7.2 SU(3) *Cartan Subalgebra*

The $SU(3)$ generators (2-80) have two diagonal matrices,

$$\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}. \tag{2-129}$$

SU(3) *Cartan subalgebra*

Consider their operations on a typical strong (color) triplet state, such as that created and destroyed by the green quark field of (2-111), where g_S is the strong coupling constant. As with $SU(2)$, it is conventional to divide g_S by 2 (or equivalently, simply take $X_i = ½\lambda_i$ instead of λ_i, with constant g_S) for finding eigenvalues,

$$\begin{gathered} \frac{g_S}{2}\lambda_3 \begin{bmatrix} \psi_u^L \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S = \frac{g_S}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi_u^L \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S = \begin{bmatrix} \psi_u^L \\ 0 \end{bmatrix} \frac{g_S}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S = -\frac{g_S}{2} \begin{bmatrix} \psi_u^L \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S \\ \frac{g_S}{2}\lambda_8 \begin{bmatrix} \psi_u^L \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S = \frac{g_S}{2\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} \psi_u^L \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S = \frac{g_S}{2\sqrt{3}} \begin{bmatrix} \psi_u^L \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_S . \end{gathered} \tag{2-130}$$

Finding the two eigenvalues of an SU(3) *color triplet state*

The up green quark has two eigenvalues from the two diagonal matrices. We will label them

$$\varepsilon_3 = -\frac{g_S}{2} \qquad \varepsilon_8 = +\frac{g_S}{2\sqrt{3}}. \tag{2-131}$$

SU (3) *eigenvalues of green quark*

Any quantum field with those two eigenvalues will be a green quark field. The shortcut way, rather than using two different eigenvalues to label a strong interaction triplet eigenvector, is to conglomerate them into one, i.e., color (R, G, or B).

Do **Problem 25** to gain practice with finding different quantum numbers (eigenvalues) for quark and lepton states other than the green quark.

With the results of Problem 25, we can build Wholeness Chart 2-7b. Antiparticle states have eigenvalues of opposite signs from particle states. As with weak charges, the strong interaction eigenvalues are usually expressed without the coupling constant factor of g_S. And as before, the value of that constant reflects the inherent strength of the strong interaction. And it is, as one might expect, significantly greater than the weak or electromagnetic coupling constants.

SU (3) *eigenvalues commonly expressed without g_S factor*

In parallel with $SU(2)$, $SU(3)$ singlets have zero color charge. These are the leptons, which do not feel the strong force. With regard to color, there is no difference between RC and LC quarks or leptons.

SU (3) *eigenvalues for singlets* = 0

Wholeness Chart 2-7b. *SU*(3) Cartan Subalgebra Generator Eigenvalues for Fermions

Color	Multiplet	ε_3	ε_8
R quark	triplet $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$	$\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$
G quark	triplet $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$-\frac{1}{2}$	$\frac{1}{2\sqrt{3}}$
B quark	triplet $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	0	$-\frac{1}{\sqrt{3}}$
Colorless (all leptons)	singlet	0	0

What About a Single Color (Strong Charge) Operator?

One could ask why we have two separate operators ½λ_3 and ½λ_8, with two separate eigenvalues, for the single variable of color. Why not a single 3X3 matrix operator, as we implied (for pedagogic reasons) on pg. 19? For example, the operator

$$\frac{g_s}{2}\left(\lambda_3+\sqrt{3}\lambda_8\right)=\frac{g_s}{2}\left(\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}+\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}\right)=g_s\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix} \tag{2-132}$$

would have a red eigenvalue (ignoring the g_S factor) of +1; a green eigenvalue of 0, and a blue eigenvalue of –1.

But that implies the green charge is zero, so there would be no attraction, or repulsion, of it by a red or blue quark, and we know that is not the case. There is, in fact, no choice of a single diagonal matrix for which we would have equal magnitude eigenvalues of *r,g,b* quarks, but different signs for the three (as there are only two signs.). There are ample reasons, besides not being the accepted convention, why using a single diagonal matrix as a color operator would not be advantageous.

Another Aside on Group Theory Lingo

As in $SU(2)$, mathematicians call the sets of eigenvalues in Wholeness Chart 2-7b "weights". For $SU(3)$, there are three weights. Weight I = $\left(\frac{1}{2},\frac{1}{2\sqrt{3}}\right)$, weight II = $\left(-\frac{1}{2},\frac{1}{2\sqrt{3}}\right)$, weight III = $\left(0,\frac{-1}{\sqrt{3}}\right)$. But again, we will not be using this terminology in this book. We mention it because you will run into it in other texts.

2.7.3 Cartan Subalgebras and Observables

Almost all observables are built out of Cartan subalgebra elements of certain Lie algebras. Recall that the eigenvalues for particular operators are observables. We see with our examples from $SU(2)$ and $SU(3)$ above, and $U(1)$ from Vol. 1, that we identify (and thus distinguish between) particles by their e/m, weak and strong operator eigenvalues. These eigenvalues are the charges associated with the respective interactions (electric, weak, and strong charges). For the weak and strong interactions, these are the eigenvalues of the Cartan subalgebra operator(s). In the strong interaction case, we streamline by labeling certain eigenvalue pairs as particular colors.

Cartan subalgebra elements are operators corresponding to observables

2.7.4 SU(n) Cartan Subalgebras

We will not delve into special unitary groups which act on vectors (multiplets) in spaces of dimension greater than 3. However, such spaces play a key role in many advanced theories, so we sum up the general principles we have uncovered as applied to special unitary operations in complex spaces of any dimension.

For an $SU(n)$ Lie group in matrix representation, we will have $n^2 - 1$ generators in the associated Lie algebra. One can choose a basis where $n - 1$ of these are simultaneously diagonal. These diagonal matrices commute and form a subalgebra called the Cartan subalgebra. Each vector (multiplet) that is an eigenvector of all of these Cartan (diagonal) generators has $n - 1$ eigenvalues associated with those generators. These eigenvalues can be used to label the n independent vectors (multiplets).

$SU(n)$ has $n - 1$ generators in Cartan subalgebra

2.8 Group Theory Odds and Ends

2.8.1 Graphic Analogy to Lie Groups and Algebras

We noted earlier that the vector space of a Lie algebra is commonly known as the tangent space to its associated Lie group. This is because, essentially, the basis vectors (the generators, which are matrices for us) of the vector space, are first derivatives (with respect to particular parameters), and a first derivative is tangent to its underlying function.

Fig. 2-1 illustrates this in a graphic, and heuristic, way, where the Lie group is represented as the surface of a sphere, and the Lie algebra, as a tangent plane to that sphere at the identity of the group. One can imagine more extended analogies, in higher dimensional spaces, wherein there are more than two generators X_i.

In the figure, in order for $e^{i\alpha_i X_i}$ to represent an element of the group, we need to restrict the parameters so $|\alpha_i| << 1$, i.e., we have to be very close to the identity. Each set of α_i corresponds to a different point (group element) on the sphere surface. As the α_i change, one moves along that surface. Group elements not infinitesimally close to the identity can, in principle (as summarized in Wholeness Chart 2-5, pg. 37), be represented by $e^{i\beta_j(\alpha_i)X_j}$, where the β_j are dependent on the α_i.

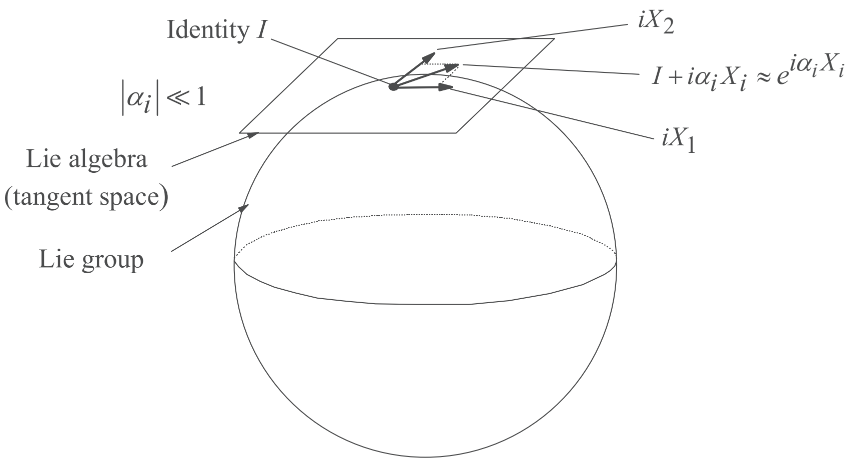

Figure 2-1. Schematic Analogy for Lie Groups and Algebras

2.8.2 Hermitian Generators and Unitarity

Note from the summary in Wholeness Chart 2-8 that all the generators we have looked at are Hermitian. This is a general rule for $SO(n)$ and $SU(n)$ groups.

Wholeness Chart 2-8. Lie Groups and Their Lie Algebra Generators

Lie Group	Matrix Rep	Generators for Fundamental Representation	Exponentiated Form
$SO(2)$	$\hat{M}$ (2-6)	$\hat{X} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$	$e^{i\theta\hat{X}}$
$SO(3)$	$\hat{N}(=A)$ (2-11)	$\hat{X}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{bmatrix}$ $\hat{X}_2 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$ $\hat{X}_3 = \begin{bmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$e^{i\hat{\theta}_i\hat{X}_i}$
$SO(n)$	$\hat{P}$	$\hat{Y}_i$ (imaginary, Hermitian, and traceless)	$e^{i\hat{\theta}_i\hat{Y}_i}$
$SU(2)$	M (2-25)	$X_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $X_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ $X_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	$e^{i\beta_i X_i}$
$SU(3)$	N (2-26)	$\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $\lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}$ $\lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}$ $\lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$	$e^{i\beta_i\lambda_i}$
$SU(n)$	P	Y_i (complex, Hermitian, and traceless)	$e^{i\beta_i Y_i}$

Do **Problem 26** to prove that all generators of $SO(n)$ and $SU(n)$ groups are Hermitian.

If you did the suggested problem, you saw that in order to generate a unitary group matrix (or orthogonal group matrix in the case of real matrices) via exponentiation of a Lie algebra element, the generators must be Hermitian, i.e., $Y_i^\dagger = Y_i$.

Note also that the four cases we studied all have traceless generators. This too is a general rule for generators of all special orthogonal and special unitary groups.

$SU(n)$ groups have Hermitian, traceless generators

Do **Problem 27** to prove it.

2.8.3 Terminology: Vector Spaces, Dimensions, Dual Spaces, and More

Wholeness Chart 2-9, pg. 53, summarizes all of this section and more.

Use of Terms "Vector Space" and "Dimension"

Don't confuse the use of the term "vector space" when applied to matrix operation of a group on a vector versus the use of the same term when applied to Lie algebras. In the former case, the vector space is the home of the vector (or multiplet in group theory language) upon which the matrix acts. In the latter case, the vector space is the space of matrices (which are vectors in this sense) whose basis matrices (basis vectors) are the generators.

Vector space on which group matrices operate different from Lie algebra vector space

The dimension of a given representation equals the dimension of the vector space in which it acts. The $SU(2)$ representation with generators X_i shown in Wholeness Chart 2-8 operates on 2D vectors (doublets), i.e., it comprises 2x2 matrices and has dimension 2. However if an $SU(2)$ representation operates on a singlet, a 1D entity, such as in (2-128), it has dimension 1.

Dimension of a rep = dimension of vectors operated on

The dimension of a Lie algebra, on the other hand, equals the number of generators for the associated Lie group, since each generator is a basis vector (a matrix in this case) for the vector space of the Lie algebra. An $SU(n)$ Lie algebra has $n^2 - 1$ generators, so its dimension is $n^2 - 1$.

Dimension of a Lie algebra = number of generators

Further, and making it even more confusing, the dimension of a Lie group is commonly taken to be the dimension of its Lie algebra. So, an $SU(3)$ Lie group would have dimension 8, but its matrix representation in the fundamental rep (acting on three component vectors) would have dimension 3.

Dimension of a Lie group same as dimension of its algebra

So, be careful not to confuse the two uses of the term "vector space" and the two uses of the term "dimension" in Lie group theory.

Dual Vectors

Dual vector is a mathematical term for what we, in certain applications, have called the complex conjugate transpose of a vector. More generally, it is the entity with which a vector forms an inner product to generate a real scalar, whose value equals the square of the absolute value ("length" or magnitude) of the vector.

Dual vector inner product with associated vector = square of vector magnitude

Examples include the dual $\mathbf{r}^\dagger$ of the position vector $\mathbf{r}$ in any dimension, the bra $\langle\psi|$ as the dual of the ket in QM, the covariant 4D spacetime position vector as the dual of the contravariant 4D position vector, and the complex conjugate transpose as the dual of the weak doublet.

$$r_i = \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \underbrace{r_i^\dagger = r_i^{\mathrm{T}} = \begin{bmatrix} x & y \end{bmatrix}}_{\text{dual vector}} \quad r_i^\dagger r_i = |\mathbf{r}|^2 \qquad |\phi\rangle \rightarrow \underbrace{\langle\phi|}_{\text{dual vector}} \quad \langle\phi|\phi\rangle = \begin{array}{c}\text{NRQM probability,} \\ \text{QFT norm (real scalar)}\end{array}$$

$$x^\mu = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \rightarrow \underbrace{x_\mu = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \end{bmatrix}}_{\text{dual vector}} \qquad x_\mu x^\mu = \text{ spacetime interval (real scalar)} \qquad (2\text{-}133)$$

$$\begin{bmatrix} \psi_{\nu_e}^L \\ \psi_e^L \end{bmatrix} \rightarrow \underbrace{\begin{bmatrix} \bar{\psi}_{\nu_e}^L & \bar{\psi}_e^L \end{bmatrix}}_{\text{dual vector}} \qquad \begin{bmatrix} \bar{\psi}_{\nu_e}^L & \bar{\psi}_e^L \end{bmatrix} \begin{bmatrix} \psi_{\nu_e}^L \\ \psi_e^L \end{bmatrix} = \bar{\psi}_{\nu_e}^L \psi_{\nu_e}^L + \bar{\psi}_e^L \psi_e^L \quad \text{(real scalar)}$$

To be precise, the components of the dual vector for the weak doublet are adjoints, i.e., they are complex conjugate transposes in spinor space post multiplied by the γ^0 matrix. (See Vol. 1, Sect. 4.1.6, pg. 91.)

See Jeevanjee (footnote on pg. 8) for more on dual vectors.

Other Terminology

The order of a group is the number of elements in the group. The order of a matrix (even a matrix representing a group), on the other hand, is mXn, where m is the number of rows and n is the number of columns. For a square nXn matrix, the order is simply stated as n, and is often used interchangeably with the term dimension (of the matrix).

Order of group = num of elements

Order of square matrix same as its dimension = num of rows (or columns)

Wholeness Chart 2-9. Various Terms in Group, Matrix, and Tensor Theory

Term	Used with	Meaning	Examples
Degree	groups	n in $SO(n)$, $SU(n)$, $U(n)$, etc.	$SO(3)$ has degree 3 $SU(2)$ has degree 2
Vector space	matrix (group representation)	space of vectors upon which matrix representation acts	$SO(2)$: $M\begin{bmatrix} x \\ y \end{bmatrix}$ 2D space of x, y axes
	Lie algebra	space of matrix generators (= basis vectors in the space)	$SU(2)$: space spanned by X_1, X_2, X_3 generators
Dimension	Lie group	same as Lie algebra below	see Lie algebra below
	matrix (can be a group rep)	number of components in a vector upon which square matrix acts	$M\begin{bmatrix} x \\ y \end{bmatrix}$ M has dimension 2
	Lie algebra	number of generators (= number of independent parameters)	$SU(2)$ has dimension 3 $SU(n)$ has dimension $n^2 - 1$
Dual vector space	vector space	separate space of vecs: each inner product with original vec = vec magnitude squared	$\mathbf{r}^\dagger$, $\langle\psi\|$, x_μ
Order	group	number of elements in underlying set	any Lie group = ∞
	matrix	mXn where m = rows, n = columns (for square matrix, same as dimension)	2X3 matrix has order 2X3 3X3 matrix has order 3
	tensor	same as rank of tensor below	see rank of tensor below
Sub	group	subset of elements in a parent group that by itself satisfies properties of a group (including closure within the subgroup)	2D rotations is a subgroup of 3D rotations
	matrix	submatrix obtained by removing rows and/or columns from parent matrix	$\begin{bmatrix} a & b \\ d & e \end{bmatrix}$ submatrix of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$
	algebra	subset of elements in a parent algebra that by itself satisfies properties of an algebra (including closure within the subalgebra)	Cartan subalgebra (diagonal matrices) for the Lie algebra of any $SU(n)$ group
Rank	matrix	number of independent vectors (columns)	identity matrix in 2D: rank = 2
	Lie algebra	number of generators in Cartan sub-algebra	$SU(3)$ has rank 2; $U(1)$, rank 1 $SU(n)$ has rank $n - 1$
	tensor	number of indices	rank of tensor T_{ijk} is 3

2.8.4 Spinors in QFT and the Lorentz Group

In Vol. 1, pg. 171, we noted that when our reference frame undergoes a Lorentz transformation, the 4D spinors (spinor space vectors) transform too. The manner in which they transform was signified there by the symbol D, which is a particular 4D matrix group (spinor space operator). D is the spinor space *representation* of the Lorentz group. It is a representation of that group in spinor space, as opposed to the usual way we see the Lorentz group represented, as the Lorentz transformation in 4D spacetime. With considerable time and effort, which we will not take here (but which is shown in the citations of the footnote on the aforenoted page in Vol. 1), one can deduce the precise form of the spinor space representation from the spacetime representation.

Lorentz transformation has a rep in spinor space

2.8.5 Casimir Operators

An operator for a Lie algebra that commutes with all generators is called a Casimir operator. The identity operator multiplied by any constant is a Casimir operator, for example. However, it must be (aside from the arbitrary constant) constructed from the generators. An example from $SU(2)$ is

Casimir operator commutes with all generators

$$X_1X_1 + X_2X_2 + X_3X_3 = I + I + I = 3I \tag{2-134}$$

We will not be doing anything further with Casimir operators herein, but mention them because they are usually part of other developments of group theory you will run into.

2.8.6 Jacobi Identity

We note in passing that many texts define a Lie algebra using what is called the Jacobi identity,

Jacobi identity is formal, most general way of expressing 2nd Lie algebra operation

$$\left[[X,Y],Z\right] + \left[[Y,Z],X\right] + \left[[Z,X],Y\right] = 0 \ , \tag{2-135}$$

to define the second binary operation in the algebra, rather than the more straightforward way (at least for our special case) of simply defining that operation as a commutator. Each bracket in (2-135) is considered a Lie bracket, and it constitutes the second operation. Formally, the Lie brackets do not have to be commutators, they just have to satisfy (2-135), which commutators do.

Do **Problem 28** if you wish to show that for the brackets in the Jacobi identity signifying commutation, then the identity is satisfied. (This will not be relevant for our work, so it is not critical to do this problem.)

We will not do anything more with the Jacobi identity. We mention it only because you will no doubt run into it in other texts. The bottom line is that the Jacobi identity is simply the formal way of defining the second operation for a Lie algebra. For our purposes, this is commutation.

2.8.7 Reducible and Irreducible Group Representations

Most presentations of group theory discuss what are known as reducible (or irreducible) representations of groups. We are going to postpone treatment of that topic until Part 4 of this book, however, as it will not be really relevant before we get to strong interactions.

2.8.8 Abelian (Commuting) vs Non-Abelian Operators

Recognize that if a matrix is diagonal, then exponentiation of it results in a diagonal matrix as well. So, an element of the Cartan subalgebra exponentiated will yield a diagonal element of the associated Lie group. Diagonal matrices commute.

If all elements of a particular Lie group commute, then all of its Lie algebra generators will commute as well. As we have mentioned, a commuting group is called an Abelian group; a non-commuting group, non-Abelian. The $SU(2)$ and $SU(3)$ groups are non-Abelian and their Lie algebras are non-Abelian, as well. The $U(1)$ group of (2-17) is Abelian. Cartan subalgebras are Abelian.

$SU(2)$, $SU(3)$ Lie algebras non-Abelian; $U(1)$ and Cartan subalgebras Abelian

2.8.9 Raising and Lowering Operators vs Eigenvalue Operators

As we have mentioned before (pg. 34), some operators change components of a multiplet and some can leave a multiplet unchanged, but typically multiplied by a constant.

In the latter case the constants are eigenvalues and the multiplet is an eigenvector of the operator. Examples of such operators include (see Wholeness Chart 2-8) X_3 of $SU(2)$, as well as λ_3 and λ_8 of

2 kinds of matrices: diag → eigenvalues; non-diag → raise or lower multiplet components

$SU(3)$, acting on multiplets with only one non-zero component. See (2-127) and (2-130). Such operators are generally diagonal and members of the Cartan subalgebra.

<u>$SU(2)$</u>

Other, non-diagonal operators like the $SU(2)$ generators X_1 and X_2 of (2-63) raise and lower components. For examples,

$$X_1 v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_2 \\ 0 \end{bmatrix} \quad X_1 \underset{\sim}{v} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} C_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ C_1 \end{bmatrix} \quad X_2 v = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ C_2 \end{bmatrix} = -i\begin{bmatrix} C_2 \\ 0 \end{bmatrix} . \tag{2-136}$$

However, since, for example, X_1 can either raise or lower components of a doublet, it is generally preferred to call entities such as

$SU(2)$ raising and lowering operators

$$\frac{X_1 + iX_2}{2} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i\frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \frac{X_1 - iX_2}{2} = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - i\frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{2-137}$$

the <u>raising operator</u> (LHS of (2-137)) and <u>lowering operator</u> (RHS of (2-137)).

<u>$SU(3)$</u>

All of the $SU(3)$ generators other than λ_3 and λ_8 will similarly raise and lower components of triplets in 3D complex space.

Fig. 2-2 is a plot of the $SU(3)$ eigenvalues listed in Wholeness Chart 2-7b, pg. 49. Bars over color symbols indicate anti-particles (with opposite color charges, and thus opposite eigenvalues). Note that the tips of the vectors signifying the quarks lie on a circle and all leptons sit at the origin, as they are $SU(3)$ singlets, and thus have zero for each $SU(3)$ eigenvalue.

Plot of fermion $SU(3)$ eigenvalues

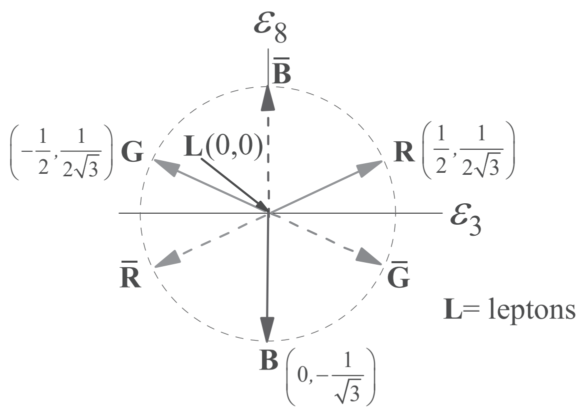

Figure 2-2. Quark and Lepton *SU*(3) Eigenvalue Plot

Note that the raising operator (see (2-80))

$$\frac{\lambda_1}{2} + i\frac{\lambda_2}{2} = \frac{1}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + i\frac{1}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2-138}$$

operating on a green quark triplet

$$\left(\frac{\lambda_1}{2} + i\frac{\lambda_2}{2}\right)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{2-139}$$

$G \rightarrow R$ raising operator

raises the triplet component from the second slot to the first, i.e., it turns it into a red quark. A comparable lowering operator does the reverse, turning a red quark into a green one.

$$\left(\frac{\lambda_1}{2} - i\frac{\lambda_2}{2}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{2-140}$$

$R \rightarrow G$ lowering operator

In terms of Fig. 2-2, the raising operator of (2-138) rotates the green quark vector of (2-139) clockwise 120°. The lowering operator of (2-140) rotates the red quark in the opposite direction by the same amount.

SU(3) raising and lowering operators rotate color states in Fig. 2-2

Do **Problem 29** to find other SU(3) raising and lowering operators.

All of the non-diagonal SU(3) generators play a role in raising and lowering components in quark triplets. The diagonal SU(3) generators determine the eigenvalues for quark triplet eigenstates.

<u>Bottom line:</u> There are two types of operators. One raises and/or lowers multiplet components. The other merely multiplies eigen-multiplets by an eigenvalue. Similar reasoning applies to operators in $SU(n)$ for any n, as well as any $SO(n)$. We could, of course, have operators which are linear combinations of those two types.

2.8.10 Colorless Composite States

Consider a quark combination, such as one would find in any baryon, of RGB. The eigenvalues of the three parts of the composite add to zero. Graphically, vector addition in Fig. 2-2 of the three color eigenstates sums to the origin. The composite is colorless. It has zero for both eigenvalues.

Hadrons are colorless (vectors in Fig. 2-2 add to zero)

Similarly, a meson is comprised of a quark and an antiquark of the same color (actually anti-color). The eigenvalues sum to zero, and the two vectors in Fig. 2-2 vector sum to the origin.

One can figuratively think of the strong force as a tendency for points on the outer circle of Fig. 2-2 to attract points on the opposite side of the circle. We will, of course, speak more about this when we get to a more formal treatment of strong interactions.

and have total eigenvalues for $\lambda_3/2$ and $\lambda_8/2 = 0$

2.8.11 Group Action of S Operator versus SU(n) Operators

Note that the S operator (see (2-4)) acts on *states*, whereas the $SU(n)$ operators act on *field* multiplets. The S operator is really a group of operators and the vector space on which it operates is composed of quantum states. The U(1), SU(2), and SU(3) groups we have been looking at, on the other hand, act on vectors (multiplets) composed of quantum fields.

S operator acts on states; $SU(n)$ operators act on fields

2.8.12 Unitary vs Special Unitary

The most general complex square matrix P of dimension n has n^2 different complex number components. Each of these has two real numbers, one of which is multiplied by i, so, there are $2n^2$ real numbers, call them real variables, in all. P equals a sum of $2n^2$ independent matrices (for which half of them could be real and half imaginary), each such matrix multiplied by an independent real variable.

Progression from general matrix to unitary to special summarized in Wholeness Chart 2-10

$$P=\begin{bmatrix} p_{Re11}+ip_{Im11} & p_{Re12}+ip_{Im12} & \cdots & p_{Re1n}+ip_{Im1n} \\ p_{Re21}+ip_{Im21} & p_{Re21}+ip_{Im21} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ p_{Ren1}+ip_{Imn1} & \cdots & \cdots & p_{Renn}+ip_{Imnn} \end{bmatrix}$$
$$= p_{Re11}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + p_{Im11}\begin{bmatrix} i & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + p_{Re12}\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix} + \ldots.. + p_{Imnn}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & i \end{bmatrix}. \qquad (2\text{-}141)$$

General complex matrix as sum of $2n^2$ basis (matrix) vectors with $2n^2$ indep real variables

Consider what happens when we impose the restriction that P is unitary.

$$\begin{gathered} P \text{ a general } n\mathrm{X}n \text{ matrix with } 2n^2 \text{ independent real variables} \\ P^{-1}P = P^{\dagger}P = I \;\rightarrow\; n^2 \text{ constraint equations} \;\rightarrow\; 2n^2 - n^2 = n^2 \text{ independent real variables} \end{gathered} \qquad (2\text{-}142)$$

Unitary $\rightarrow n^2$ constraint eqs on real variables $\rightarrow n^2$ independent real variables

The imposition of unitarity yields n^2 separate scalar equations, one for each component of I, in terms of the $2n^2$ real variables. These constrain the variables so only n^2 of them are independent. P is then equal to a sum of the same $2n^2$ independent matrices, for which half of them could be real and half imaginary, but half of the variables in front of the matrices are dependent, and half independent.

We could, instead, combine the real and imaginary matrices to give us half as many complex matrices. We also have the freedom to choose whatever variables we wish as independent, as long as the remaining variables satisfy the unitary constraint equations. In effect, by doing these things, we are just changing our basis vectors (matrices here) in the n dimensional complex space P lives in.

$\rightarrow n^2$ complex basis (matrix) vectors

The matrices can then be combined in such a way that we can construct P from a sum of these n^2 generally complex matrices, each multiplied by an independent real variable.

Now consider that P is also special.

$$\text{Det}\, P = 1 \quad \rightarrow \quad \text{one scalar constraint equation} \quad \rightarrow \quad \text{one variable becomes dependent}\;, \qquad (2\text{-}143)$$

Making group special imposes one more constraint

so, we have $n^2 - 1$ independent variables with the same number of independent generators (basis vectors in the matrix space of the algebra).

Wholeness Chart 2-10 summarizes these results. Compare them with Wholeness Charts 2-5 and 2-8 on pgs. 37 and 51, respectively.

Wholeness Chart 2-10. Imposition of Unitarity and Special-ness on a General Matrix

Matrix P of Order n	**Scalar Constraint Equations**	**Independent Real Variables**	**Number of Matrices as Basis for P**	**P as Sum of Matrices (infinitesimal)**
Most general P	0	$2n^2$	$2n^2$ complex matrices Z_i	$P = p_i Z_i$ $i = 1,.., 2n^2$
Unitary $P^{-1} = P^\dagger$	n^2	n^2	n^2 generally complex	$P = I + i\alpha_i Y_i$ $i = 1,.., n^2$
Also, special Det $P = 1$	1	$n^2 - 1$	$n^2 - 1$ plus identity matrix	$P = I + i\alpha_i Y_i$ $i = 1,.., n^2 - 1$

2.8.13 A Final Note on Groups vs Tensors

I hope this section does not confuse you, as up to here, we have carefully discriminated between groups and tensors.

Tensors represent a lot of things in our physical world. Examples (of rank 2) include the stress and strain tensors (continuum mechanics), the vector rotation and moment of inertia tensors (classical mechanics), the electromagnetic tensor $F^{\mu}{}_{\nu}$ (see Vol. 1, pg. 138), and the stress-energy tensor of general relativity $T^{\mu}{}_{\nu}$. All of these examples are grounded in the physical world of space and time, and when expressed as matrices, all have real components. They act on real vectors and yield other (related) real vectors.

Some examples of tensors

However, in a more general sense, tensor components could be complex, and are in fact found in applications in QM, which, as we know, is a theory replete with complex numbers and complex state vectors.

A common definition of tensors delineates how they transform under a change in coordinate system. Hopefully, you have seen that before, as we cannot digress to consider it in depth here. (See Jeevanjee (footnote on pg. 8) for more details, if needed.)

Tensor action same in physical world, different components in different coordinate systems

(2-6), for fixed θ, is an expression of a tensor in an orthonormal coordinate system. This tensor rotates a 2D physical space vector through an angle θ. If we change our coordinate system (transform it so the x axis is no longer horizontal), we will transform the components of (2-6) and the components of the vector it operates on. But, in physical space, the same physical vector **v** will rotate from its original physical position by the same angle θ. Only the coordinate values are different. The actual physical operation is the same. For (2-6) to be a tensor, that same physical operation has to be performed regardless of what coordinate system one has transformed (2-6) to (what component values one has for the transformed matrix).

(2-6), which rotates a vector, is a tensor (for a fixed value of θ). So is each of (2-7). So is each of (2-13) for any given fixed set of values for θ_i. A set of such tensors, such as the elements of (2-7), or (2-6) with $0 \leq \theta < 2\pi$, or (2-13) with $0 \leq \theta_i < 2\pi$, can form a group.

Here is the point. The group $SO(2)$ comprises a set of rotation tensors in 2D. Each element in the group is a tensor. The same goes for $SO(3)$. And, if we extrapolate to complex spaces, the same thing can be considered true of any $SU(n)$. The group elements can be thought of as complex tensors[1].

Group members of $SO(n)$ and $SU(n)$ are tensors

Recall that groups can have many different types of members, from ordinary numbers to vectors to matrices, etc. In this chapter, we have examined groups made up of tensor elements.

BUT, the language can then get cumbersome. Earlier (Wholeness Chart 2-2, pg. 21), we distinguished between the direct product of group elements and the tensor product of vectors. But, in our cases, the direct product of group elements is, technically speaking, a tensor product of tensors. Confusing? Yes, certainly.

To circumvent this confusion as best we can, in this text, we will henceforth avoid calling our group elements tensors. We will treat group elements as represented by matrices, and call the vectors they operate on multiplets, vectors, or occasionally, rank 1 tensors. We only mention the connection of rank 2 tensors to group elements in this section. I do this because I, the author, was once confused by the similarity between groups and tensors, and I believe others must have similar confusion, as well.

But we will refer to them as "matrices", and avoid the term "tensors"

2.8.14 More Advanced Aspects of Group Theory

We note that we have presented group theory in the simplest possible way, and have done more "showing" than "proving". In particular, strict mathematical derivations of abstract Lie algebras are intimately involved with the concept of differentiation on manifolds (which we won't define here) and manifold tangent spaces. Matrix Lie algebras, as opposed to abstract ones, can be derived and defined using parametrizations and exponentiation. For matrix groups, the two approaches (abstract and matrix) are equivalent, though the latter is generally considered easier to grok (English vernacular for "understand"). Though we have focused in this chapter on matrix Lie groups and the matrix approach, we have related that to differentiation of group matrices and thus indirectly to differentiable manifolds. (See Fig. 2-1, pg. 51, and recall the Lie algebra is the tangent space of the Lie group.)

Additionally, you may have heard of other types of groups associated, for example, with string (or M) theory, such as E_8 and $O(32)$ groups. We will not be delving into any of these in this book. In fact, $U(1)$ and special unitary groups $SU(n)$ are all we will be concerned with from here on out, and only those in the latter group with $n = 2$ or 3.

There is much more to group theory, but we've developed what we need for the SM

Finally, as you may be aware, there are many more aspects and levels to group theory than we have covered here. These include additional terminology, advanced theory, and numerous applications[2]. We have developed what we need for the task at hand, i.e., understanding the SM.

2.9 Chapter Summary

There are several ways to summarize this chapter. The first is the graphic overview in Section 2.9.1. Second is the verbal summary of Section 2.9.2. Third is in chart form and found in Section 2.9.3. To get a more detailed summary simply review the other wholeness charts in the chapter. Being structured, graphic, and concise yet extensive, they paint a good overview of group theory as it applies to QFT.

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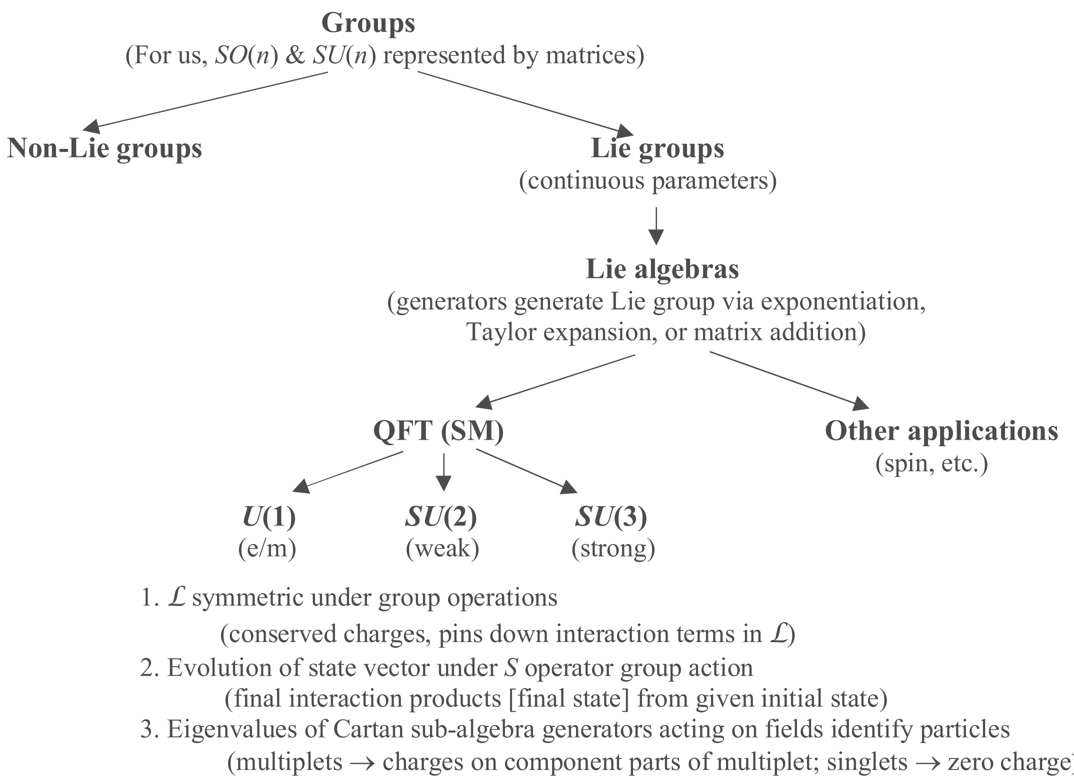
[1] For veterans of tensor analysis only: To be precise, a group element in this context is a mixed tensor, as it transforms a vector to another vector in the same vector space (not to a vector in the dual vector space.) For real vectors and orthonormal bases, there is no need to bring up this distinction, as there is no difference between the original vector space and its dual space.

[2] See the references on pg. 8. Also, for a thorough, pedagogic treatment of even more advanced group theory, such as that needed for study of grand unified theories, see Marina von Steinkirch's *Introduction to Group Theory for Physicists* (2011). She wrote it as a grad student, for grad students, as an elaboration on H. Georgi's far more terse *Lie Algebras in Particle Physics* (1999). Free legal download available at www.astro.sunysb.edu/steinkirch/books/group.pdf.

2.9.1 Pictorial Overview

Wholeness Chart 2-11. is a schematic "big picture" view of this chapter, and needs little in the way of explanation.

Wholeness Chart 2-11. Schematic Overview of Chapter 2



2.9.2 Verbal Summary

A group can be represented as a set of matrices and a binary operation between the matrices that satisfy the properties of a group, such as closure, being invertible, etc. The group may act on vectors in a vector space, and in our applications will virtually always do so.

There are many classifications of groups (orthogonal, unitary, special, Abelian, Lie, direct product, finite, and more) and a particular group could belong to one or several of these classifications. For unitary (orthogonal) groups, the complex conjugate transpose (transpose) of any element equals the inverse of that element. For special groups, the determinant of every element equals unity. Special orthogonal groups [*SO*(n)] rotate vectors (active transformation) or coordinate systems (passive transformation) in real spaces. Special unitary groups [*SU*(n)] can be conceptualized as comparable "rotations" in complex spaces.

An algebra can be represented as a set of matrices with scalar multiplication and two binary operations that satisfy certain properties such as closure.

Lie groups have all elements that vary as continuous, smooth functions of one or more continuous, smooth variables (parameters). Lie algebras have elements that satisfy the properties of an algebra, generate an associated Lie group, and, as a result, are continuous, smooth functions of one or more continuous, smooth variables (parameters).

The basis elements of a Lie algebra can generate the Lie group

1) globally, and locally, as factors in the terms of a Taylor expansion about the identity,
2) globally and locally, via exponentiation (which is simplified locally), or
3) locally (generally, but in some particular cases globally, as well), via matrix addition of the identity plus the Lie algebra basis elements times the independent parameters.

Such Lie algebra elements are called the generators of the group. The generators can generate the group via exponentiation, though in the global case, finding the form of the associated real variables used in the exponentiation can be very complicated for all but the simplest matrix groups. For infinitesimal values of the parameters, however, the exponentiation simplifies, as we only deal with first order terms in the expansion of the exponential.

Lie algebras we will deal with comprise matrices and the operations of matrix addition and commutation. Structure constants relate such commutation to the particular matrices (within the Lie algebra) the commutation results in.

A symmetry under any group operation (matrix multiplication for us) means some value, as well as the functional form of that value, remains unchanged under that operation.

For a given group $SU(n)$, there will generally be multiplets (vectors) of n dimensions and singlets (scalars) of dimension one, upon which the group elements operate. Diagonal Lie algebra elements of the associated Lie group commute, and when operating on multiplets with one non-zero component, yield associated eigenvalues. The set of such diagonal elements is called the Cartan subalgebra. Operation of such elements on singlets yields eigenvalues of zero. The dimension of a representation of the group equals the number of components in the multiplet (or singlet) upon which it acts. The same group, when acting on different dimension vectors, is represented by different matrices, i.e., it has different representations.

In the standard model (SM), the weak force is embodied by the $SU(2)$ group; the strong force, the $SU(3)$ group; and the electromagnetic force, the $U(1)$ group. Cartan subalgebra elements in these spaces acting on quantum field multiplets yield eigenvalues that are the same as those of associated operators acting on single particle states. For $SU(2)$ and $SU(3)$, these eigenvalues, alone or in sets, correspond to weak and strong (color) charges. Certain symmetries in each of the $U(1)$, $SU(2)$, and $SU(3)$ groups dictate the nature of the interactions for each of the various forces. Much of the rest of this book is devoted to exploring these symmetries and their associated interactions.

2.9.3 Summary in Chart Form

Wholeness Chart 2-12. Table Overview of Chapter 2

Entity	Definition/Application	Comment
Group	Elements satisfy criteria for a group	
Lie Group	Elements = continuous smooth function(s) of continuous smooth parameter(s) α_i	
$SU(n)$ Lie Group	Det $P = 1$ $\quad P^\dagger P = I$	$P(n)$ = matrix rep of $SU(n)$
Generators = Bases of Lie Algebra	$Y_i = -i\dfrac{\partial P}{\partial \alpha_i} \qquad i = 1, \ldots, n^2 - 1$	
Commutation relations	$\left[Y_i, Y_j\right] = i\,c_{ijk} Y_k$	Usually different structure constants c_{ijk} for different parameters α_i.
Lie Algebra, 2nd Operation	$i\,[Y_i, Y_j]$	Elements satisfy algebra criteria. 1st binary operation is addition. Both operations have closure.
Group from Algebra For $\lvert\alpha_i\rvert << 1$ $\rightarrow$	$P = e^{i\beta_i(\alpha_j)Y_i}$ $P \approx e^{i\alpha_i Y_i} \approx I + i\alpha_i Y_i \quad \delta P \approx i\alpha_i Y_i$	Can also construct P from Taylor expansion with Y_i used therein.
Cartan subalgebra	Diagonal generators of Y_i (symbol Y_i^{Cart} here) Acting on vector space $\rightarrow$ eigenvectors & eigenvalues	All elements commute.
QFT	Y_i^{Cart} eigenvalues of field multiplets (vectors in a vector space) correspond to weak & strong (color) charges.	Later chapters to show these do, in fact, correspond to such charges.
	P & Y_i act on vector space of quantum fields. S operator acts on Fock (vector) space of (multiparticle) states. $\int \Psi^\dagger P \Psi d^3x$ & $\int \Psi^\dagger Y_i \Psi d^3x$ act on Fock (vector) space of states.	Y_i acting on field multiplet yields same eigenvalue(s) as $\int \Psi^\dagger Y_i \Psi d^3x$ acting on associated single particle state.
	Symmetries of $\mathcal{L}$ for quantum field multiplet transformations under $SU(n)$ operators yield 1) charge conservation (via Noether's theorem) and 2) the form of the interaction $\mathcal{L}$.	To be shown in later chapters.

2.10 Appendix: Proof of Particular Determinant Relationship

In Problem 27 one is asked to start with relation (2-144) from matrix theory, where A and B are matrices.

$$\text{If } B = e^{A}, \text{ then } Det\, B = e^{Trace\, A} \tag{2-144}$$

Here we prove that relation.

We can diagonalize A in the first relation of (2-144) via a similarity transformation, and B will be diagonalized at the same time.

$$B_{diag} = e^{A_{diag}} \tag{2-145}$$

For any component of the matrix in (2-145) with row and column number k ,

$$B_{diag}^{kk} = e^{A_{diag}^{kk}} \tag{2-146}$$

Then, taking determinants, we have

$$Det\, B_{diag} = B_{diag}^{11} B_{diag}^{22} B_{diag}^{33} \ldots = e^{A_{diag}^{11}} e^{A_{diag}^{22}} e^{A_{diag}^{33}} \ldots = e^{\left(A_{diag}^{11} + A_{diag}^{22} + A_{diag}^{33} \cdots\right)} = e^{Trace\, A_{diag}} \; . \tag{2-147}$$

The Baker-Campbell-Hausdorf relation is not relevant for the next to last equal sign because all exponents are mere numbers. Since both the trace and the determinant are invariant under a similarity transformation, we have, therefore,

$$Det\, B = e^{Trace\, A} \; . \tag{2-148}$$

2.11 Problems

1. Give your own examples of a group, field, vector space, and algebra.

2. Is a commutative group plus a second binary operation a field?
   Is a commutative group plus a scalar operation with a scalar field a vector space?
   Is a vector space plus a second binary operation an algebra?
   Does a field plus a scalar operation with a scalar field comprise an algebra? Is this algebra unital?

3. Show that 3D spatial vectors, under addition, form a vector space. Then show that QM states do as well. What do we call the space of QM states?

4. Show that the matrices $\hat{N}$ below are an orthogonal group $O(2)$ that is not special orthogonal, i.e., not $SO(2)$.

   $$\hat{N}(\theta) = \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   Show the operation of $\hat{N}$ on the vector [1,0] graphically. Graph the operation on the same vector for $\theta = 0$. Do these graphs help in understanding why special orthogonal transformations are more appropriate to represent the kinds of phenomena we see in nature? Explain your answer. For the operation $\mathbf{Av} = \mathbf{v'}$, from matrix theory, we know $|\mathbf{v'}| = |\text{Det}\, A\,||\mathbf{v}|$. Does this latter relation make sense for your graphical depiction? Explain.

5. Write down a 2D matrix that cannot be expressed using (2-6) and a 3D matrix than cannot be expressed using (2-12) and (2-13).

6. Show that U of (2-17) forms a group under matrix multiplication.

7. Show that any unitary operation U (not just (2-17)) operating on a vector (which could be a quantum mechanical state) leaves the magnitude of the vector unchanged.

8. Does $e^{i\theta}$ [which is a representation of the unitary group $U(1)$] acting on a complex number characterize the same thing as the $SO(2)$ group representation (2-6) acting on a 2D real vector? Explain your answer mathematically. (Hint: Express the components of a 2D real vector as the real and imaginary parts of a complex scalar. Then, compare the effect of $e^{i\theta}$ on that complex scalar to the effect of (2-6) on the 2D real vector.) Note that $U(1)$ and $SO(2)$ are *different groups*. We do *not* say that $e^{i\theta}$ here is a representation of $SO(2)$. $U(1)$ and $SO(2)$ can describe the same physical world phenomenon, but they are *not* different representations (of a particular group), as the term "representation" is employed in group theory.

9. Show that M of (2-20) obeys the group closure property under the group operation of matrix multiplication.

10. Show (2-35) in terms of matrices.

11. Show that $\hat{X} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ along with a scalar field multiplier θ comprise an algebra. Use Wholeness Chart 2-1 as an aid. Note that every $\theta\hat{X}$ is in the set of elements comprising the algebra, and the operations are matrix addition and matrix commutation. This is considered a trivial Lie algebra. Why do you think it is considered such?

12. Show there is no identity element for the 2nd operation (2-52) in the $SO(3)$ Lie algebra. (Hint: The identity element has to work for every element in the set, so you only have to show there is no identity for a single element of your choice.)

13. Why did we take a matrix commutation relation as our second binary operation for the algebra for our $SO(3)$ Lie group, rather than the simpler alternative of matrix multiplication? (Hint: Examine closure.)

14. Find the generators in $SO(3)$ for the parametrization $\theta'_i = -\theta_i$ in (2-12). Then, find the commutation relations for those generators.

15. Obtain the $\hat{\theta}_i$ values for (2-58) up to third order.

16. Show that $X_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, and $X_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ generate the three-parameter $SU(2)$ group. (Hint: Use (2-61), along with (2-20), (2-23), and the derivative of (2-23) to get M, and prove that all elements shown in the text in that expansion can be obtained with the generators and the identity matrix. Then, presume that all other elements not shown can be deduced in a similar way, with similar results.) Then sum up the second order terms in the expansion to see if it gives you, to second order, the group matrix (2-20).

17. For $SU(2)$ with $|\alpha_1|, |\alpha_2|, |\alpha_3| << 1$, show $e^{i(\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3)} \approx I + i\alpha_1 X_1 + i\alpha_2 X_2 + i\alpha_3 X_3 \approx M$.

18. Show that $M(\theta_1, \theta_2, \theta_3) = \begin{bmatrix} \cos\theta_1 e^{i\theta_2} & \sin\theta_1 e^{i\theta_3} \\ -\sin\theta_1 e^{-i\theta_3} & \cos\theta_1 e^{-i\theta_2} \end{bmatrix}$ is an $SU(2)$ Lie group. That is, verify that the group is both unitary and special.

19. Although this is already explicit in (2-81), confirm that the λ_i, in (2-80), are the generators of the matrix N of (2-28). (Hint: Expand N in terms of α_i. Comparing with (2-61) to (2-63) and (2-79) may help.)

20. Show that $[\lambda_1, \lambda_2] = i\, 2\lambda_3$ and that $[\lambda_6, \lambda_7] = i\sqrt{3}\lambda_8 - i\lambda_3$.

21. Show that the three terms $\mathcal{L}_{\substack{three\ quark \\ terms}} = \bar{\psi}_r \not{\partial} \psi_r + \bar{\psi}_g \not{\partial} \psi_g + \bar{\psi}_b \not{\partial} \psi_b = \begin{bmatrix} \bar{\psi}_r & \bar{\psi}_g & \bar{\psi}_b \end{bmatrix} \not{\partial} \begin{bmatrix} \psi_r \\ \psi_g \\ \psi_b \end{bmatrix}$ are symmetric under an $SU(3)$ transformation that is independent of space and time coordinates x^μ.

22. Show that for a non-unitary group acting on any arbitrary vector, the vector magnitude is generally not invariant.

23. Prove (2-122). This is time consuming, so you might just want to look directly at the solution in the solutions booklet.

24. Show the effect of the diagonal $SU(2)$ generator on a doublet where neither component is zero. What do you conclude from the result?

25. Find the strong interaction quantum numbers for the down red quark, the up blue quark, the LC electron, and the RC electron neutrino states.

26. For any unitary matrix P, $P^\dagger P = I$. If $P = e^{i\beta_i Y_i}$, show that all Y_i are Hermitian. For any orthogonal matrix $\hat{P}$, $\hat{P}^{\mathrm{T}}\hat{P} = I$. If $\hat{P} = e^{i\hat{\theta}_i \hat{Y}_i}$, show that all $\hat{Y}_i$ are purely imaginary and Hermitian.

27. Given what was stated in Problem 26, and using the relation from matrix theory below, show that all special orthogonal and special unitary generators must be traceless. The relation below is derived in the appendix of this chapter.

    From matrix theory: For $B = e^A$, Det $B = e^{Trace\ A}$.

28. Show that if the brackets in the Jacobi identity $\left[[X,Y],Z\right] + \left[[Y,Z],X\right] + \left[[Z,X],Y\right] = 0$ are commutators, then the identity is satisfied automatically.

29. Find the $SU(3)$ raising operator that turns a blue quark into a green one. Then find the lowering operator that does the reverse, i.e., changes a green quark into a blue one. In terms of Fig. 2-2, what does the action of each of these operators do. What does the action of the ½($\lambda_1 + i\lambda_2$) operator do to a blue quark?