Draft of
Student Friendly Quantum Field Theory
Volume 2
The Standard Model

(with solutions to problems)

by Robert D. Klauber

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Part One
Mathematical Preliminaries

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Name

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Group Theory

"I believe that ideas such as absolute certitude, absolute exactness, final truth, etc. are figments of the imagination which should not be admissible in any field of science... This loosening of thinking seems to me to be the greatest blessing which modern science has given to us."

Max Born

2.0 Introduction

In Part One of this book, we cover two topics that could be ignored for the canonical approach to QED (and were, in fact, ignored in Vol. 1), but play major roles elsewhere in QFT. These are

- group theory (this chapter), and
- Grassmann variables theory (Chapter 3).

The former is used throughout the theories of electroweak and strong interactions, while the latter plays an essential role in handling fermionic fields in the path integral approach to all standard model (SM) interactions. Hopefully, for most readers, a good part of the group theory presentation will be a review of course work already taken. For others, also hopefully, it will be sufficient for understanding, as treated in latter parts of this book, the structural underpinning that group theory provides for the SM of QFT.

As always, we will attempt to simplify, in the extreme, the presentations of these two theories herein, without sacrificing accuracy. And we will only present the essential parts of group theory needed for QFT. For additional applications of the theory in physics (such as angular momentum addition in QM), presented in a simplified manner, I recommend McKenzie[1] and Schwichtenberg[2]. In areas where McKenzie's article overlaps this one, I have borrowed (with approval from him and considerable gratitude from me) pedagogic presentation methodologies from that article.

2.1 Overview of Group Theory

Areas of study: groups, fields, vector spaces, and algebras

Group theory is, in one sense, the simplest of the theories about mathematical structures known as groups, fields, vector spaces, and algebras, but in another sense includes all of these, as the latter three can be considered groups endowed with additional operations and axioms. Wholeness Chart 2.1 summarizes the basic defining characteristics of each of these types of structures and provides a few simple examples. Hopefully, there is not too much new in there for most readers.

Note that for algebras, the first operation has all the characteristics of a vector space. The second operation, on the other hand, does not necessarily have to be associative, have an identity element or inverse elements in the set, or be commutative.

Algebras may or may not be associative, unital, or commutative

An algebra with (without) the associative property for the second operation is called an associative (non-associative) algebra. An algebra with (without) an identity element for the second operation is called a unital (non-unital) algebra or sometimes a unitary (non-unitary) algebra. We will avoid the second term as it uses the same word (unitary) we reserve for probability conserving operations. A unital algebra is considered to possess an inverse element under the 2nd operation (for the first operation, it already has one, by definition), for every element in the set.

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[1] McKenzie, D., An Elementary Introduction to Lie Algebras for Physicists, https://www.liealgebrasintro.com/

[2] Schwichtenberg, J., *Physics from Symmetry*, 2nd ed, (Springer 2018).

Wholeness Chart 2.1. Synopsis of Groups, Fields, Vector Spaces, and Algebras

Type of Entity	**Elements**	**Main Characteristics**	**Examples** **(A, B, C, D = elements in set)**	**Other Characteristics** **Shared**	 **Particular**
Group	Set of elements	1 (binary) operation "binary" = between two elements in the set	#1: Real numbers under addition. **A**+**B** = **C**, e.g., 2+3 = 5 #2: 2D rotations (can be matrices) under multiplication. **AB** = **C**	Closure; associative; identity; inverse;	May or may not be commutative
Field	As above	2 (binary) operations	#1: Real numbers under addition & multiplication. 2•3 + 4 = 10 #2: Complex numbers under addition & multiplication. $(1+2i)\,(1-2i) + (2+4i) = 7+4i$	As above	Commutative; distributive
Vector Space	Set of (vector) elements & 2nd set of scalars	1 (binary) operation & 1 scalar multiplication "scalar" = entity outside vec space set	#1: 3D vectors under vec addition & scalar multip. 3**A** + 2**B** = **C** #2: Vectors are matrices with matrix addition & scalar multip. 3**A** + 2**B** = **C** #3: Hilbert space in QM	As above	Commutative; distributive for scalar multip with vector operation; may have inner product, i.e., **A**•**B**=scalar
Algebra	As above	2 (binary) operations & 1 scalar multiplication	#1: 3D vectors under vec addition, vec cross product, scalar multip 3**A**X**B** + 2**D** = **C** #2: Matrices under matrix addition, matrix multip, scalar multip. 3**AB** + 2**D** = **C** #3: Matrices under matrix addition, matrix commut, scalar multip. 3[**A**,**B**+**D**] = **C**	1st operation (often addition): As above 2nd operation: Closure	Both operations distributive 1st operation: Commutative 2nd operation: Not required to be associative, have identity, have inverses, be commutative

Definitions (symbol ◦ represents a binary operation)		**Examples**
Closure	All operations on set elements yield an element in the set. **A**◦**B** = **C**, **C** in set for any and all set elements **A** and **B**.	All **C** in Examples column above are in original set of elements.
Associative	**A**◦(**B**◦**C**) = (**A**◦**B**)◦**C**	Real numbers, rotations, matrices, vectors, all under addition or multiplication.
Identity	There is a unique element of the set **I** with the property **A**◦**I**= **I**◦**A** for every element **A**	Real number addition, **I** = 0. Matrix multiplication, **I** = identity matrix.
Inverse	There is a unique element of the set $\mathbf{A}^{-1}$ with the property **A**◦ $\mathbf{A}^{-1}$ = $\mathbf{A}^{-1}$ ◦ **A** = **I** for every element **A**	Real number addition, $\mathbf{A}^{-1} = -\mathbf{A}$. Real number multiplication, $\mathbf{A}^{-1} = 1/\mathbf{A}$. Matrix multiplication, $\mathbf{A}^{-1}$ = matrix inverse of **A**
Commutative	**A**◦**B**= **B**◦**A** for all elements **A**,**B** in set.	Real number addition and multiplication. Vector addition. Non-commutative examples: 3D rotation, creation & destruction operators in QFT under multip.
Distributive	For two binary operations (✞ = 2nd binary operation) **A**◦(**B** ✞ **C**) = **A**◦**B** ✞ **A**◦**C**	Real numbers: ✞ as addition, ◦ as multiplication. Matrices: ✞ as addition, ◦ as multiplication

Doing **Problems 1 and 2** may help in understanding groups, fields, vector spaces, and algebras.

2.1.1 A Set of Transformations as a Group

Consider the set of rotation transformations in 3D, a typical element of which is symbolized by **A** herein. Such transformations can act as operations on a 3D vector, i.e., they rotate the 3D vector, which we designate by the symbol **v**. In the transformation, the vector **v** is rotated to a new position, designated **v′**. The transformations **A** comprise an abstract expression of rotation in physical space, i.e., **A** signifies rotation independent of any particular coordinate system. Any element **A** does the same thing to a given vector regardless of what coordinate system we choose to view the rotation operation from.

Rotation transformations = a set of elements

Now, if we select a given coordinate system, we can represent elements **A**, in one manner, as matrices, whose components depend on the coordinate system chosen. For practical applications, and for aid in teaching, we almost always have to express rotations as matrices.

One way to represent rotations is via matrices

$$\underbrace{\mathbf{A}\mathbf{v}=\mathbf{v}'}_{\substack{\text{Abstract}\\ \text{form}}} \xrightarrow[\text{and column vectors}]{\text{expressed as matrix}} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix}. \tag{2-1}$$

Note that **A** has the characteristics delineated in Wholeness Chart 2.1 for a group. In particular, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are two members of this set of transformations, then rotating the vector first via $\mathbf{A}_1$ and then via $\mathbf{A}_2$ is represented by (where $\mathbf{A}_2 \circ \mathbf{A}_1$ of Wholeness Chart 2-1 is represented by $\mathbf{A}_2\,\mathbf{A}_1$)

$$\mathbf{A}_2\mathbf{A}_1\mathbf{v}=\mathbf{v}''=\mathbf{C}\mathbf{v} \qquad \text{where } \mathbf{C}=\mathbf{A}_2\mathbf{A}_1 \text{ is a member of the set of 3D rotations}\,. \tag{2-2}$$

Therefore, the rotation transformation set has closure and the binary operation, when the transformation is expressed in matrix form, is matrix multiplication. Further, the operation on set members (we are talking operation between matrices here [in the matrix representation], not the operation of matrices on vectors) is associative, and inverses ($\mathbf{A}^{-1}$ for each **A**) and an identity (**I**) exist. Further, there is no other operation, such as addition, involved for the members of the set. (In the matrix representation, two successive transformations involve matrix multiplication, not matrix addition.) Hence, the transformations **A** form a group.

Set of rotation transformations satisfy criteria to be a group

Note, this rotation example is a non-commutative group, since

$$\text{in general, } \mathbf{A}_2\mathbf{A}_1 \neq \mathbf{A}_1\mathbf{A}_2\,. \tag{2-3}$$

You can prove this to yourself by rotating a book along its binder axis first, then along its lower edge second; then starting from the same original book position and reversing the order of the rotation operations.

A non-commutative group is denoted a non-Abelian group. Note that some pairs of elements in a non-Abelian group can still commute, just *not any and all* pairs. A group in which all elements commute is an Abelian group.

Set of 3D rotation transformations is non-Abelian (non-commutative)

2.1.2 Groups in QFT

As insight into where we are going with this, recall from Vol. 1 (see pg. 196, first row of eq. (7-49) that the *S* operator in QFT transforms an initial state $|i\rangle$ into a final state $|F\rangle$ (that's what happens during an interaction).

$$S\,|\,i\,\rangle=|\,F\,\rangle\,. \tag{2-4}$$

But that state could be further transformed (via another transformation) into another state $|\,F'\rangle$. So, for two such transformations S_1 and S_2, we would have

$$S_2S_1\,|\,i\,\rangle=S_3\,|\,i\,\rangle=|\,F'\rangle\,. \tag{2-5}$$

In a similar way, the set of S operator transformations on QFT states forms a group

Recall also from Vol. 1 (pg. 195, eq. (7-43)) that the *S* operator could be represented by a matrix (*S* matrix) and the initial and final states by column vectors. The parallels between (2-1) and (2-4), and between (2-2) and (2-5), should allow us to surmise directly that the set of all transformations (interactions) in QFT form a group. And so, the mathematics of groups should help us (and it does help us as we will eventually see) in doing QFT.

2.1.3 Quick Summary

1st Bottom line: A set of transformations on column vectors (or on QM states) can form a group. We can apply group theory to them (with or without considering the column vectors [or QM states]).

2nd Bottom line: The column vectors (or QM states) can form a vector space.

Do **Problem 3** to show this.

Group operation between group elements differs from what some groups have as operation of a group element on a vector

So, the group elements act as operators on the vectors (or QM states). Discern between *operations* (which are transformations) by group members *on vector space members* from the *group operation between group members* (matrix multiplication in our sample representation.)

2.1.4 Notation

__Bold__ = abstract element of group or vector space
Non-bold = matrix representation of element

We will generally use bold capital letters, such as **A**, for abstract group elements (which could characterize some operation in physical space, such as rotation); and non-bold capital letters, such as A, for matrix representations of abstract group elements. We will generally use bold lower-case letters, such as **v**, for abstract vector elements in a vector space; and non-bold lower-case letters, such as v, for column matrix (or row matrix) representations of those vectors. The binary operation on abstract elements $\mathbf{A} \circ \mathbf{B}$, for matrix multiplication in the matrix representation, will be expressed simply as AB. In our work we will focus, almost exclusively, on matrix representations.

2.2 Lie Groups

Lie group elements vary continuously and smoothly with a continuous, smooth parameter

A Lie group is a group whose elements are continuous, smooth functions of one or more variables (parameters) which vary continuously and smoothly.

2.2.1 A One Parameter Lie Group

A simple example of a Lie group is rotation in 2D, which can be represented by a matrix that operates on a 2D vector,

A simple example

$$M(\theta) = \begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}. \tag{2-6}$$

This group is characterized by a single parameter, the angle of rotation θ. As (2-6) is just a special case of the 3D rotations of (2-1), it forms a group. And because all of its elements can be generated continuously and smoothly by a smooth, continuous variation of a parameter, it is a Lie group.

Example of a non-Lie group: discrete elements

An example of a non-Lie group would be the set of 2D rotations through increments of 90°. The set of matrix elements would be (take θ = 0°, 90°, 180°, 270° in (2-6))

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad M_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{2-7}$$

Obviously M_1 is the identity. M_4 is the inverse of M_2, and M_1 and M_3 are each their own inverses. Additionally, (you can check if you like) any two of (2-7) multiplied together yield one of the other elements (closure). Further, matrix multiplication is always associative. So, (2-7) form a group under matrix multiplication. But since the elements are discrete and do not convert one into the other via a continuous variable parameter, it is not a Lie group.

Property of one parameter Lie groups

A property of Lie groups, for the one parameter case, is (for two values of θ of α and ϕ)

$$M(\alpha)M(\phi) = M(\alpha+\phi). \tag{2-8}$$

This should be evident from our general knowledge of 2D rotations. Rotating first through 30°, then second through 13° degrees is the same as rotating through 43° degrees. The parameter θ varies continuously and smoothly and so does $M(\theta)$.

As an aside, the 2D rotation group is commutative (Abelian), as rotating first by 13°, then second by 43° is the same as rotating first by 43° and second by 13°.

2.2.2 Orthogonal vs Special Orthogonal Groups

Both groups (2-6) and (2-7) are what are termed special orthogonal groups. "Orthogonal" means the elements of the group (represented by the matrices) are real and the transpose of the matrix is the same as its inverse, i.e. $M^{\mathrm{T}} = M^{-1}$. Recall from linear algebra that the magnitude of a vector remains unchanged under an orthogonal transformation (as in rotation). "Special" means the determinant of each matrix in the group is unity. Det$M = 1$.

Orthogonal, $O(n)$
$\rightarrow$ real, $M^{\mathrm{T}} = M^{-1}$
(magnitude of vector invariant under M)

Do **Problem 4** to find a 2D orthogonal group that is *not* a special orthogonal group and to understand the significance of special orthogonal transformations.

Special Orthogonal, $SO(n)$
$\rightarrow$ Det $M = 1$
(rotation without reflection of vector)

If you did the above problem, it should be relatively evident that multiplying -1 by our matrix that operates on a column vector does two things: 1) it reverses direction of (reflects) the final vector so it points in the opposite direction, and 2) it changes the sign of the matrix determinant. We can generalize. Changing the sign of any matrix operating on a vector will change the sign of the determinant and reflect the final vector.

The group dimension for n X n matrices is n, which in our example groups (2-6) and (2-7) above is 2. So, the shorthand notation for such a group is $SO(2)$.If the rotations were in 3D instead of 2D, as in (2-1), we denote it an $SO(3)$ group. For n dimensional space rotations, we would have $SO(n)$.

If the determinant were not constrained to be equal to positive unity, the group would be simply an orthogonal group, symbolized by $O(n)$.

Note that orthogonal and special orthogonal groups do not have to be Lie groups. For example, (2-7) is special orthogonal, but not a Lie group. We define these more general terms in this Section 2.2, which is specifically on Lie groups, because it is easiest to understand them in the context of the examples presented in this section.

$O(n)$ and $SO(n)$ groups can be Lie or non-Lie groups

2.2.3 Different Representations of the Same Group

Note that we can represent the $SO(2)$ rotation group in a different way as

$$M(x) = \begin{bmatrix} \sqrt{1-x^2} & -x \\ x & \sqrt{1-x^2} \end{bmatrix} \tag{2-9}$$

where $x = sin\theta$. We say that (2-9) is another representation of the same $SO(2)$ group of (2-6). A third representation is the transpose of (2-6),

Same group can have different representations

$$M(\theta') = \begin{bmatrix} cos\theta' & sin\theta' \\ -sin\theta' & cos\theta' \end{bmatrix} \tag{2-10}$$

where $\theta' = -\theta$.

Bottom line: A particular group is an abstract structure (in the above example, 2D rotations; in the example of (2-1), 3D rotations) that can be represented explicitly in different mathematical ways, called representations.

2.2.4 A Lie Group with More than One Parameter: SO(3)

Of course, there are many Lie groups with more than one parameter. As one example, let us express the $SO(3)$ group of 3D rotations (2-1) as a function of certain angles (successive rotations about different axes) θ_1, θ_2, and θ_3. Typically, for a solid object with three orthogonal axes visualized as attached to the object, the first rotation is about the x_3 axis; the second, about the x_2 axis; the third, about the x_1 axis.

Consider **A** in (2-1) as an abstract group element (characterized simply in that it performs rotations), and A as the particular mathematical matrix representation under consideration.

Example of Lie group with more than one parameter $\rightarrow$ 3D rotation

$$A(\theta_1,\theta_2,\theta_3)v = v' \quad \rightarrow \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} \qquad a_{ij} = a_{ij}(\theta_1,\theta_2,\theta_3) \tag{2-11}$$

Any A can be expressed using the building blocks

$$A_1(\theta_1)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_1 & -\sin\theta_1\\ 0 & \sin\theta_1 & \cos\theta_1\end{bmatrix}\quad A_2(\theta_2)=\begin{bmatrix}\cos\theta_2 & 0 & -\sin\theta_2\\ 0 & 1 & 0\\ \sin\theta_2 & 0 & \cos\theta_2\end{bmatrix}\quad A_3(\theta_3)=\begin{bmatrix}\cos\theta_3 & -\sin\theta_3 & 0\\ \sin\theta_3 & \cos\theta_3 & 0\\ 0 & 0 & 1\end{bmatrix}\qquad (2\text{-}12)$$

One example of building blocks of 3D rotation group

There are a number of ways the matrices (2-12) can be combined to form a representation of $SO(3)$, but to be consistent with our above noted order of rotations, we will use [note the operations proceed from the right side to the left side in (2-13)]

One of many ways to represent that group using building blocks

$$A(\theta_1,\theta_2,\theta_3)=A_1(\theta_1)A_2(\theta_2)A_3(\theta_3)\,. \qquad (2\text{-}13)$$

The matrix A varies continuously and smoothly with continuous, smooth variation in the three parameters θ_1, θ_2, and θ_3. We could, of course, define the order of operation on the RHS of (2-13) differently and have a different representation of the same $SO(3)$ rotation group. Similarly, we could define our building blocks with different parameters (similar to the x in (2-9) for $SO(2)$ rotations) and have yet other, different representations. Again, we see that a group itself is an abstract structure (3D rotations in this case) that can have a number of different representations.

Key point:

Note the A_i of (2-12) are not bases in the vector space sense of spanning the space of all possible 3X3 matrices. We would need 9 of them, all independent, in that case. However, all possible 3D rotations, expressed as matrices, can be obtained from the three A_i, so they are the foundation of the group.

Don't confuse building blocks of groups with bases of vector spaces. They are generally different.

The 3D rotation group matrices form a subset of all 3X3 real matrices, and the reader should be able to verify this by doing the problem referenced below. Further, any group element can be formed, in this representation, by matrix multiplication of three group building blocks. But in a typical vector space, any element is formed from a linear combination (adding, not multiplying) of basis vectors (which are matrices here).

Similarly, in 2D rotations, we only had one matrix, such as (2-6), which is a function of one parameter (θ in the referenced representation). For a basis in 2D for matrices, we would need four independent matrices. Don't confuse the building blocks of a Lie group with the basis vectors of a linear vector space.

Do **Problem 5** to help illustrate the difference between matrices as vector space elements and matrices as group elements.

2.2.5 Lorentz Transformations Form a Lie Group

The Lorentz transformation, with $c = 1$ and boost velocity v in the direction of the x^1 coordinate, is

$$\Lambda^{\alpha}{}_{\beta}(v)\,dx^{\beta}=\begin{bmatrix}\frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} & & \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & & \\ & & 1 & \\ & & & 1\end{bmatrix}\begin{bmatrix}dx^0\\ dx^1\\ dx^2\\ dx^3\end{bmatrix}=\begin{bmatrix}dx'^0\\ dx'^1\\ dx'^2\\ dx'^3\end{bmatrix}=dx'^{\alpha}\,. \qquad (2\text{-}14)$$

The set of transformations $\Lambda^{\alpha}{}_{\beta}$ satisfies our criteria for a group. The elements $\Lambda^{\alpha}{}_{\beta}$ are subject to a single operation (matrix multiplication) under which the set has an identity member (when $v = 0$), obeys closure, is associative, and possesses an inverse for every member. The Lorentz transformations constitute a group.

Lorentz transformations form a group

Due to the Minkowski metric in special relativity's 4D space, things get a little tricky comparing the Lorentz transformation to matrices in Euclidean space. So, even though the magnitude of the four-vector $|dx^{\beta}|$ remains invariant under the transformation (Section 2.2.2 above), the inverse of $\Lambda^{\alpha}{}_{\beta}(v)$ is not its transpose, but $\Lambda^{\alpha}{}_{\beta}(-v)$. (You can check this or save time by just taking my word for it, as this material is a bit peripheral.) In a special relativity sense, therefore, the Lorentz transformation is considered orthogonal. As the determinant of $\Lambda^{\alpha}{}_{\beta}$ is unity (you can check using (2-14) and the rules for calculating 4D matrix determinants), it is special. To discern the special relativistic nature of this particular kind of orthogonality, the Lorentz group is denoted by $SO(3,1)$ [for 3 dimensions of space, and one of time.]

Note that the 4D transformation matrix is a continuous function of v, the velocity between frames, whose possible values vary continuously. So, the Lorentz group is a Lie group. The addition property (2-8) holds for relativistic velocity addition, i.e.,

$$\Lambda^{\alpha}{}_{\beta}(v)\Lambda^{\beta}{}_{\delta}(v') = \Lambda^{\alpha}{}_{\delta}(v+v') \quad \text{for relativistic velocity addition.} \tag{2-15}$$

In particular, a special orthogonal Lie group, SO (3,1)

Extending the form of (2-14) to include the more general cases of 3D coordinate axes rotation plus boosts in any direction leads to the same conclusions. Lorentz transformations comprise a Lie group $SO(3,1)$.

2.3 More on Groups

2.3.1 Complex Groups: Unitary vs Special Unitary

So far, we have looked exclusively at groups represented by real matrices. But since QM is replete with complex numbers, we need to expand our treatment to include representations of groups using complex matrices. See Vol.1, Box 2-3, pg. 27, to review some differences and similarities between real, orthogonal transformations and complex, unitary transformations. Unitary groups (symbol $U(n)$ for dimension n) are effectively the complex number incarnation of (real number) orthogonal groups.

As a simple case of a unitary group representation, consider the set of matrices for continuous θ

$$U = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}. \tag{2-16}$$

Do **Problem 6** to show that U of (2-16) forms a group. Then do **Problem 7**.

Unitary, $U(n)$ $\rightarrow$ complex, $U^{\dagger} = U^{-1}$ (magnitude of vector invariant under U)

U in (2-16) is unitary because $U^{\dagger}U = I$. (Compare to orthogonal matrices, which are real, and for which $M^{\mathrm{T}}M = I$.) It is special because Det $U = 1$. (Note Problem 7 gives an example of a unitary group that is not special.) So (2-16) represents a special unitary group, designated $SU(2)$, where the 2 represents the dimension of the matrix. For dimension n, one uses the symbolism $SU(n)$.

Special Unitary, $SU(n)$ $\rightarrow$ Det $U = 1$

So, U of (2-16) would be better expressed as $SU(2)$ (here of a single parameter). Also, similar to an orthogonal transformation on a real vector, a unitary transformation on a complex vector leaves the magnitude of the vector unchanged.

Do **Problem 8** to prove the last statement.

Unitary transformations operating on quantum state vectors leave probability unchanged

If the vector happens to be a normalized QM state, this means that the total probability (to find the quantum system in *some* quantum state) remains unity under the action of the transformation U.

Additionally, since the set elements of (2-16) vary continuously and smoothly with the continuous, smooth variation of the parameter θ, (2-16) comprises a Lie group.

Given that QFT teems with complex numbers (of which operators and states are composed) and the theory is inherently unitary (conservation of total probability = 1 under transformations), we can expect to be focusing herein on special unitary groups.

Problem 9 can help in understanding how some physical world phenomena can be described by different kinds of groups.

2.3.2 Direct Sums, Sub-groups and Reducibility

Consider a group **C** represented by the 5-dimensional matrices C of the form exhibited in (2-17) (where the matrix components could be real or complex and could be a function of one or more parameters). Components left blank signify zero values. Note the meaning of the ⊕ sign, which implies what is called a direct sum.

$$C=\begin{bmatrix} a_{11} & a_{12} & a_{13} & & \\ a_{21} & a_{22} & a_{23} & & \\ a_{31} & a_{32} & a_{33} & & \\ & & & b_{11} & b_{12} \\ & & & b_{21} & b_{22} \end{bmatrix} = A\oplus B \quad \text{where} \quad A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad (2\text{-}17)$$

Direct sum of matrices nXn and mXm yields (n+m)X(n+m) matrix

When the matrix C operates on a five-component vector v, the A submatrix only acts on the top three components and has no effect on the bottom two. Similarly, the B submatrix only acts on the bottom two components and does nothing to the top three.

$$\mathbf{Cv}=\mathbf{v}' \xrightarrow[\text{coordinate system}]{\text{in some specific}} Cv=\begin{bmatrix} a_{11} & a_{12} & a_{13} & & \\ a_{21} & a_{22} & a_{23} & & \\ a_{31} & a_{32} & a_{33} & & \\ & & & b_{11} & b_{12} \\ & & & b_{21} & b_{22} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}=\begin{bmatrix} v_1' \\ v_2' \\ v_3' \\ v_4' \\ v_5' \end{bmatrix}=v'. \qquad (2\text{-}18)$$

nXn and mXm sub-matrices act independently and represent independent subgroups

The A and B matrices act independently on different components of a vector and can be considered independent groups acting on independent vector spaces (of dimensions 3 and 2, respectively.) A and B are called subgroups of C. The form of the matrix C in (2-17) and (2-18) is said to be block diagonal, for what hopefully is a fairly obvious reason, i.e., non-zero submatrix blocks along the diagonal and zeroes everywhere else.

The dimension of the new group C is 5 which equals the dimension of A (=3) plus the dimension of B (=2). More generally, in direct summing, the resulting group dimension equals the sum of the subgroup dimensions.

Direct sum group dimension = sum of subgroup dimensions

Now consider a transformation T, which acts on the matrix operator C, that fills up at least some of the original zero value matrix components of C.

$$TCT^{-1}=\begin{bmatrix} \underset{\sim}{c}_{11} & \underset{\sim}{c}_{12} & \underset{\sim}{c}_{13} & \underset{\sim}{c}_{14} & \underset{\sim}{c}_{15} \\ \underset{\sim}{c}_{21} & \underset{\sim}{c}_{22} & \underset{\sim}{c}_{23} & \underset{\sim}{c}_{24} & \underset{\sim}{c}_{25} \\ \underset{\sim}{c}_{31} & \underset{\sim}{c}_{32} & \underset{\sim}{c}_{33} & \underset{\sim}{c}_{34} & \underset{\sim}{c}_{35} \\ \underset{\sim}{c}_{41} & \underset{\sim}{c}_{42} & \underset{\sim}{c}_{43} & \underset{\sim}{c}_{44} & \underset{\sim}{c}_{45} \\ \underset{\sim}{c}_{51} & \underset{\sim}{c}_{52} & \underset{\sim}{c}_{53} & \underset{\sim}{c}_{54} & \underset{\sim}{c}_{55} \end{bmatrix}=\underset{\sim}{C} \qquad \begin{array}{c}\text{Same abstract operation } \mathbf{C}\text{, expressed} \\ \text{in different coordinate system.}\end{array} \qquad (2\text{-}19)$$

In other coordinate systems, the subgroup independence may not be obvious

We can think of the $\underset{\sim}{C}$ matrix as representing the same group **C**, just expressed in different form. That is, T has essentially changed our coordinate system (a passive transformation). Matrices and vectors in the new coordinate system are denoted with "squiggles" underneath; those in the old system, as plain letters. So, for vectors,

$$\underset{\sim}{v}=Tv \qquad \text{Same abstract vector } \mathbf{v}\text{, expressed in different coordinate system} \qquad (2\text{-}20)$$

$$\text{We still have} \quad \mathbf{Cv}=\mathbf{v}' \xrightarrow[\text{coordinate system}]{\text{but in this new}} \underset{\sim}{C}\,\underset{\sim}{v}=\underset{\sim}{v}' \quad \left(\text{where } \underset{\sim}{v}'=Tv'\right). \qquad (2\text{-}21)$$

The T transformation gives us different components for the matrices and the column vectors, even though the abstract operation **C** carries out the same operation on the same abstract vector **v**.

For example, the operation carried out by the C group could be rotation of a 5D vector in the 5D space. Physically (imagining a 5D space) the vector would be an independent physical quantity like position rotated through a particular angle. The angle and the vector length remain the same physically regardless of which coordinate system we prefer to view them in. But the components of the matrix representing the rotation, and the components of the 5D vector, will be different in different coordinate systems. The right-hand sides of (2-18) and (2-21) represent that same rotation as observed in different coordinate systems. The T transformation changes the coordinate system.

The subgroup independence is still there in the coordinate independence sense

Of course, the operation represented by the matrix C could be any number of things, not just rotation. But similar logic applies, regardless of the particular operation **C** carries out.

Note that if we had started with (2-19), instead of (2-17), we would not be immediately aware that there were two subgroups in the group. They would still be there, but it would not be obvious. By operating on $\underset{\sim}{C}$ with T^{-1}, i.e., $T^{-1}\underset{\sim}{C}T$ we would get C, and then it would be obvious.

In going from $\underset{\sim}{C}$ to C, we have *reduced* the matrix to two sub-matrices, i.e., reduced the group representation to two (independent) sub-group representations. One could then imagine a scenario where either A or B matrices could be further reduced to subgroups within them, such as

$$\underset{\sim}{A} = \hat{T}A\hat{T}^{-1} = \begin{bmatrix} \underset{\sim}{a}_{11} & \underset{\sim}{a}_{12} & \\ \underset{\sim}{a}_{21} & \underset{\sim}{a}_{22} & \\ 0 & 0 & \underset{\sim}{a}_{33} \end{bmatrix}. \tag{2-22}$$

A reducible group contains subgroups

An irreducible group does not

However, if we cannot further reduce A or B, in a manner such as that of (2-22), then we say the representation C of (2-17) is irreducible. An irreducible group has no sub-groups into which it could be reduced. A reducible group does have such sub-groups.

For matrices, a reducible group is transformable to block diagonal

For groups expressed as matrices, we can define a reducible group as one for which a similarity transformation (a transformation such as T in (2-19) or $\hat{T}$ in (2-22)) can result in block diagonal form (such as in (2-18) or (2-22)). An irreducible group could not be transformed to such form. Note that a direct sum, as in (2-17), always gives us a reducible group.

D and TDT^{-1} are equivalent representations

Two representations of a given group, such as C in (2-18) and $\underset{\sim}{C}$ in (2-19), are said to be equivalent representations if they are related by a similarity transformation, such as T in (2-19). Note the group may or may not be reducible, yet still have different, equivalent representations. To be equivalent only means one can be transformed into the other. Equivalence of D and D' only means there is some transformation T such that $D' = TDT^{-1}$, nothing more (i.e., no block diagonal necessarily implied.)

Invariant subspace = vector space outside of which group action does not take a vector

The set of all possible vector components v_1, v_2, v_3 in (2-18) comprise what is termed an invariant subspace (of vectors in a vector space) under C because any action of C on the five component vector v_i will not involve any of the three components v_1, v_2, v_3 in determining v'_4 or v'_5. Likewise, the set of all v_4, v_5 components will not, under C, play any role in determining v'_1, v'_2, v'_3, so it is also an invariant subspace. Two invariant subspaces behave independently under C.

By way of our prior example of 5D rotation, any vector initially in a 2D plane spanned by the basis vectors along the 4th and 5th axes in (2-18) would be rotated by **C** inside that plane. **C** could never rotate such a vector outside the plane. Even if we change coordinate axes via the transformation T of (2-19), that same physical vector would not be rotated by **C** outside of that original 2D plane. It may be rotated out of a coordinate plane defined by the x'_4, x'_5 axes in the new primed coordinate system, and have non-zero values in any or all of the five vector components in the new primed coordinates, but it would not be rotated out of the original 2D plane formed by the x_4, x_5 axes. That lack of ability of **C** to move the original vector out of the original plane indicates **C** has a sub-group that has its own independent action on vectors in the vector space.

Parallel logic applies, of course, to the other invariant subspace, the 3D volume formed by the x_1, x_2, x_3 axes outside of which **C** would not rotate any vector originally inside that volume.

Things to note

The 3D rotation group $SO(3)$ (2-11) is irreducible because any vector in the 3D space can be rotated into any other vector. Nothing in (2-11) constricts any 3D vector to any particular 2D plane, i.e., any particular subspace. That group has no subgroups, and the 3D vector space it acts on is an invariant subspace.

However, we could imagine another $SO(3)$ rotation group that would constrict rotation to a 2D plane, and such a group would be reducible to a 2D subgroup and a 1D subgroup. The 3D vector space would then not be an invariant subspace under the group.

Don't confuse eigenvalue matrix diagonalization with subgrouping

Also, one must be careful not to confuse eigenvector analysis with subgroups. Eigenvalues can be found by diagonalizing a matrix, via a suitable transformation (which is also applied to the vector). In that transformed state, the new basis vectors are eigenvectors. One might think we then have subgroups of the matrix, each subgroup of one dimension. But this is not the case. The diagonalized matrix is relevant for only the eigenvectors, *not all* vectors in the space. That is, only eigenvectors acted on by the matrix remain in the 1D space they started in. Other vectors are rotated to different alignments. None of these other vectors could be considered to be in one of the supposed invariant subspaces on which the supposed 1D subgroups act.

All of the subgroups must, collectively, act on all the vectors in the space and keep those in their respective invariant subspaces within those subspaces. But in the eigenvalue diagonalized matrix situation, vectors that are not eigenvectors do not stay in a given subspace under the action of the group.

With diagonalized matrix, non-eigenvectors change subspace

When we see the symbol ⊕ for direct summing two groups together to form a larger group (as in (2-17)), we should recognize that this new larger group has two subgroups, the ones direct summed together via the ⊕ symbol. This is *not* a binary operation, as defined in Wholeness Chart 2-1, pg. 3. A binary operation occurs between members of a set. The symbol ⊕ means we are adding (direct summing) two different sets (sub-groups, really). It is an operation that combines groups, not an operation between elements of a given group.

Symbol ⊕ means two subgroups in resulting group

This is not a group binary operation, but a combination of groups

After all this simplification with vectors, the real meaning of sub-group

We have visualized the action of sub-groups on vectors, as represented in (2-18) and (2-22), as a non-mixing of certain vector components. However, a group does not have to operate on a vector to be a group. Yet, it could still have sub-groups into which it is reducible.

Consider a particular group with two sub-groups. In (2-23), we show the (binary) group operation of two group members as one group member represented by a 4X4 matrix D matrix multiplied with another group member represented by another 4X4 matrix H. The sub-groups are represented as 2X2 matrices, F and G for group element D and R and S for group element H.

$$\underbrace{\boldsymbol{DH}}_{\substack{\text{elements}\\ \text{of group}}} \xrightarrow[\text{representation}]{\text{matrix}} \underbrace{\begin{bmatrix} F & \\ & G \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} R & \\ & S \end{bmatrix}}_{H} = \begin{bmatrix} f_{11} & f_{12} & & \\ f_{21} & f_{22} & & \\ & & g_{11} & g_{12} \\ & & g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} r_{11} & r_{12} & & \\ r_{21} & r_{22} & & \\ & & s_{11} & s_{12} \\ & & s_{21} & s_{22} \end{bmatrix}$$

$$= \begin{bmatrix} f_{11}r_{11}+f_{12}r_{21} & f_{11}r_{12}+f_{12}r_{22} & & \\ f_{21}r_{11}+f_{22}r_{21} & f_{21}r_{12}+f_{22}r_{22} & & \\ & & g_{11}s_{11}+g_{12}s_{21} & g_{11}s_{12}+g_{12}s_{22} \\ & & g_{21}s_{11}+g_{22}s_{21} & g_{21}s_{12}+g_{22}s_{22} \end{bmatrix} = \underbrace{\begin{bmatrix} FR & \\ & GS \end{bmatrix}}_{DH}. \tag{2-23}$$

Note that all the resulting elements in the upper left 2X2 block matrix in the last line of (2-23) are from the matrix multiplication of the F and R sub-matrices, which are members of one of the subgroups. Similarly, all the resulting elements in the lower right 2X2 block matrix are from the matrix multiplication of the G and S matrices, which are members of the other subgroup.

So, we see that just as the action of a subgroup on a vector does not mix certain components of the vector, so the group operation between elements of a group does not mix the subgroups together. They stay separate. In a binary group operation, no subgroup changes any part of any other subgroup. It only affects other members of its own subgroup.

Real meaning of subgroups = under group operation, no mixing of subgroups (as opposed to vector subspaces)

We introduced this concept originally with vectors in hopes of making it easier to understand. In these last few paragraphs, we have made it more precise, and more in the true spirit of group theory, which is defined in terms of group (binary) operations, not operations on vectors.

2.3.3 Direct (Tensor, Outer) Products

In group theory applications, one commonly runs into a type of multiplication that is called by three equivalent names: direct product, tensor product, or outer product.

Consider an example where elements of two groups are represented by matrices, and although they could have the same dimension, in our case the first matrix A has dimension 3, and the second matrix B, dimension 2. Note the symbol ⊗ represents direct (tensor, outer) product. Often times in the literature you will see just X used instead, however.

$$\mathbf{A}\otimes\mathbf{B} \xrightarrow[\text{by matrices}]{\text{represented}} A\otimes B=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\otimes\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \tag{2-24}$$

In index notation,

Direct product definition

$$A_{ij}\otimes B_{kl}=A_{ij}B_{kl}=C_{ijkl}\,. \tag{2-25}$$

The <u>order</u> of a matrix is its number of indices, so the resulting matrix (2-25) is 4th order (four different indices = 2 indices for A_{ij} plus 2 indices for B_{kl}), with different dimensions for each order (index). The i index dimension is 3; the j dimension is 3; the k dimension is 2; and the l dimension is 2. The total number of components is 3 X 3 X 2 X 2 = 36. Said another way, it is the total number of components of A times the total number of components of B, i.e., 9 X 4 = 36.

Direct product group order = sum of orders of constituent groups

Number of components = product of number of components of constituent groups

We have already used this concept in Vol. 1 (see (4-123), pg. 115, reproduced below) for the outer (tensor) product of fermion fields. It is traditional to not use the ⊗ symbol in that case, however, but to imply the outer product by having the adjoint (complex conjugate transpose) factor on the RHS. (The inner product is symbolized by having it on the LHS.)

Direct product example from prior work

$$\underbrace{\psi\bar{\psi}}_{\substack{\text{not writing}\\ \text{out spinor}\\ \text{indices}}}=\underbrace{\psi_\alpha\bar{\psi}_\beta}_{\substack{\text{with spinor}\\ \text{indices}\\ \text{written}}}=\psi_\alpha\psi^\dagger_\delta\gamma^0_{\delta\beta}=X_{\alpha\beta}=\text{ a matrix quantity in spinor space}\,. \tag{2-26}$$

In (2-26), we are outer product multiplying two vectors (column and row matrices), whereas in (2-24) it is two square matrices. But the general principle is the same. We get a separate component in the result for each possible multiplication of one component in the first matrix times one component in the second.

Direct product of components of two vectors = a matrix

Now consider **A** and **B** being operators that operate on vectors **v** in a vector space. Imagine the quantities represented by vectors have two characteristics. They are colored, and they are charged. The colors are red, green, blue and the charges are +1 and –1. We represent a given vector as a direct product of a 3D vector **w** representing color (r, g, b being the amount of each color in the particular vector) and a 2D vector **y** representing charge (p being the amount of positive charge; n, the amount of negative charge).

A hypothetical example for illustrative purposes

$$\mathbf{v}=\mathbf{w}\otimes\mathbf{y} \xrightarrow[\text{and row matrices}]{\text{represented by column}} v=\begin{bmatrix} r \\ g \\ b \end{bmatrix}\begin{bmatrix} p & n \end{bmatrix}=\begin{bmatrix} rp & rn \\ gp & gn \\ bp & bn \end{bmatrix} \quad\text{or}\quad v_{jl}=w_j y_l\,. \tag{2-27}$$

In this example, the **A** operator is related to color and thus acts only on the **w** part of **v**. It is blind to the **y** part. The **B** operator is related to charge and acts only on the **y** part. So, given (2-24) and (2-25),

A particular operator commonly acts on only one of the constituent groups of a direct product group

$$\mathbf{Cv}=(\mathbf{A}\otimes\mathbf{B})\,\mathbf{v}=\mathbf{v}' \qquad C_{ijkl}v_{jl}=A_{ij}B_{kl}v_{jl}=v'_{ik}\,. \tag{2-28}$$

Do **Problem 10** to show (2-28) in terms of matrices.

You are probably already considering the color part of the vector **v** above in terms of the strong force, and that, in fact, is why I choose color as a characteristic for this example. When we get to the strong interaction, we will see that operators like **A** will act on a 3-component field (such as a quark). For example, a quark field having a given color charge of r (red) would be in a color eigenstate, and a particular **A** operator, a color operator, acting on that would yield r as the eigenvalue.

Similarly, the same quark field represented symbolically by **v** may have a particular weak interaction charge. A particular **B** operator, a weak charge operator, acting on it would yield an eigenvalue equal to that weak charge. This is all pretty rough around the edges, and somewhat inaccurate, since we are trying to convey the general idea as simply as possible. Many details and modifications will be needed (in future chapters) to make it more complete and accurate.

In fact, the famous $SU(3)$ X $SU(2)$ X $U(1)$ relation (where X is used in place of ⊗) represents the action of three groups (strong/color interaction operators in $SU(3)$, weak interaction operators in

SU(2), and QED operators in U(1)) whose operators act on fields (vectors in the sense here). Each field has a separate part in it for each of the three interaction types. And each such part of the field is acted upon only by operators associated with that particular interaction.

An aside

An aside on spin and group theory

The following will not be relevant for our work, but I mention it in case you run across it in other places (such as the reference in the footnote on pg. 2). Do not spend too much time scratching your head over this, but save it to come back to if and when you run into this elsewhere.

In some applications, the two parts of the state vector, such as **w** and **y** in (2-27), respond to the same operator(s). For example, a spin 1 particle interacting with a spin ½ particle in NRQM are both operated on by a spin operator. When two such particles interact to form a bound system, that total system has six possible spin eigenstates (four states with J_{tot} = 3/2, J_z = 3/2, ½, - ½ , – 3/2, and two with J_{tot} = 1/2, J_z = ½, - ½). The state vector for spin of the system is the outer product of the spin 1 state multiplied by the spin ½ state. Both parts of the system state vector relate to spin and both parts are acted on by a spin operator.

The two parts of the system state vector have, respectively, 3 spin components (spin 1 particle has eigenstates J_z = 1, 0, 1) and 2 spin components (spin ½ particle as eigenstates J_z = ½, - ½),. The spin operator acts in the 3D space of the spin 1 particle and also in the 2D space of the spin ½ particle.

In that case, instead of a 3X2 state vector matrix with six components representing the system, one can formulate the math using a six-component column vector for the system. And then the spin operator for the system becomes a 6X6 matrix, instead of a 3X3 matrix outer multiplied by a 2X2 matrix.

This is commonly done and can be confusing when one considers outer products defined as in (2-24) (equivalently, (2-25)), as we do here.

End of aside

Things to Note

In NRQM, we have already used the concept of separate operators acting independently on state vectors, where we have free particle states like

NRQM example of different operators operating on different parts of state vector (i.e., on a direct product entity)

$$\text{NRQM spin up state} \qquad \psi_{state} = Ae^{-i(Et-\mathbf{k}\cdot\mathbf{x})}\begin{bmatrix}1\\0\end{bmatrix}. \tag{2-29}$$

The Hamiltonian operator $H = i\frac{\partial}{\partial t}$ acts on the $e^{-i(Et-\mathbf{k}\cdot\mathbf{x})}$ part of the wave function and does nothing to the 2-component spinor part. The spin operator S_z operates on the spinor part, but not the $e^{-i(Et-\mathbf{k}\cdot\mathbf{x})}$ part.

We commonly write the outer (direct, tensor) products of two vectors as a column vector on the LHS times a row vector on the RHS, as in (2-27). An inner product, conversely, is represented as a row vector on the LHS with a column vector on the RHS, as most readers have been doing for a long time in other areas.

Often write inner product as row on left, column on right; outer product as column on left, row on right

$$\mathbf{w}_1\bullet\mathbf{w}_2 \xrightarrow[\text{row on left and column on right}]{\text{inner product represented by}} =\begin{bmatrix}r_1 & g_1 & b_1\end{bmatrix}\begin{bmatrix}r_2\\g_2\\b_2\end{bmatrix} = r_1r_2 + g_1g_2 + b_1b_2 \tag{2-30}$$

However, as noted in the solution book, in the answer to Problem 10, the row vs column vector methodology can become cumbersome, and even a bit confusing, at times. Of all the choices, the most foolproof notation is the index notation, as on the RHS of (2-27) and (2-28).

But index notation is most foolproof

2.3.4 Summary of Types of Operations Involving Groups

Wholeness Chart 2-2 lists the types of operations associated with groups that we have covered.

Wholeness Chart 2-2. Types of Operations Involving Groups

Operation	What	Type	Relevance	In Matrix Representation
$\mathbf{A} \circ \mathbf{B}$	Group operation (binary)	Between 2 elements **A** and **B** of the group	Defining characteristic of the group	Matrix multiplication, AB
Av	Group action on a vector space	A group element **A** operates on a vector **v**	Some, but not all groups, may do this.	Matrix multiplication with column vector, Av
$\mathbf{A} \oplus \mathbf{B}$	Direct sum	Combining groups (**A** & **B** here symbolize entire groups **A** & **B**.)	Larger group formed from smaller ones	2 sub-groups form higher dimension reducible group. Dim = Dim A + Dim B
$\mathbf{A} \otimes \mathbf{B}$	Direct (tensor, outer) product	Combining groups (**A** & **B** here symbolize entire groups **A** & **B**.)	Larger group formed from smaller ones	2 groups form higher order group* Indices = Indices A + Indices B

* The direct product, in some applications (see "An Aside" section on pg. 13), can be re-expressed as the same order as each of **A** and **B**, but of the same dimension as the direct sum of group **A** and group **B**.

2.3.5 Overview of Types of Groups

Infinite vs finite groups

The types of groups we have encountered are summarized in Wholeness Chart 2-3, along with one (first row) we have yet to mention, infinite groups, which simply have an infinite number of group elements. One example is all real numbers with group operation addition. Another is continuous 2D rotations (see (2-6)), which is infinite because there are an infinite number of angles θ through which we can rotate (even when θ is constrained to $0 > \theta \geq 2\pi$, since θ is continuous). A finite group has a finite number of elements. One example is shown in (2-7), which has only four group members.

Note the various types of groups are not mutually exclusive. For example, we could have an $SO(n)$ reducible, direct product, Abelian, Lie group. Or many other different combinations of group types.

Wholeness Chart 2-3. Overview of Types of Groups

Type of Group	Characteristic	Symbols	Matrix Representation
Infinite (vs. finite)	Group has an infinite number of elements.		Example: 2D rotation matrices as function of θ
Abelian (vs Non-abelian)	All elements commute	**AB = BA**	Some groups of matrices Abelian, but generally no.
Lie (vs Non-Lie)	Elements continuous smooth functions of continuous, smooth variable(s) θ_i	$\mathbf{A}=\mathbf{A}(\theta_i)$	Example: rotation matrices as function of rotation angle(s)
Orthogonal, $O(n)$	Under **A**, magnitude of vector unchanged. All group elements real.	$\lvert\mathbf{Av}\rvert = \lvert\mathbf{v}\rvert$	$A^{-1} = A^{T}$ $\lvert\mathrm{Det}\,A\rvert = 1$
Special Orthogonal, $SO(n)$	As above	As above	As above, but Det $A = 1$
Unitary, $U(n)$	Under **A**, magnitude of vector unchanged. At least some group elements complex.	$\lvert\mathbf{Av}\rvert = \lvert\mathbf{v}\rvert$	$A^{-1} = A^{\dagger}$ $\lvert\mathrm{Det}\,A\rvert = 1$
Special Unitary, $SU(n)$	As above	As above	As above, but Det $A = 1$
Reducible (vs Irreducible)	Group **C** is reducible into sub-groups	$\mathbf{C} = \mathbf{A} \oplus \mathbf{B}$	$C = \begin{bmatrix} A & \\ & B \end{bmatrix}$
Direct product	Group **C** is formed by tensor (outer) product of two groups	$\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$	$C_{ijkl} = A_{ij}B_{kl}$

2.3.6 Same Physical Phenomenon Characterized by Different Groups

It is interesting that certain natural phenomena can be characterized by different groups. For example, consider 2D rotation. Mathematically, we can characterize rotation by the *SO*(2) group, one representation of which is (2-6). (Some other representations are shown in (2-9) and (2-10)). This group rotates a vector, such as the position vector (*x*,*y*), through an angle θ.

Different mathematical groups can characterize the same physical phenomenon

But we can also characterize 2D rotation via

$$U(1) = e^{i\theta} , \qquad (2\text{-}31)$$

One example

where the unitary group (2-31) rotates a complex number $x + iy$ though an angle θ. (See Problem 9.)

Note, the *SO*(2) group and the *U*(1) groups above are different groups (here characterizing the same real world phenomenon), and *not* different representations of the same group.

These are different groups, not different representations of the same group

That is why we prefer to say a particular group is a *characterization* of a given natural phenomenon and not a "representation" of the phenomenon.

In a similar way, which we will look at very briefly later on, both *SO*(3) and *SU*(2) can characterize 3D rotation. In fact, *SU*(2) is a preferred way of handling spin (which is a 3D angular momentum vector) for spin ½ particles in NRQM, as seen from different orientations (*z* axis up, *x* axis up, *y* axis up, or other orientations). Many QM textbooks show this.[1]

2.3.7 Most Texts Treat Group Theory More Formally Than This One

We have purposefully not used formal mathematical language in our development of group theory, in keeping with the pedagogic principles delineated in the preface of Vol. 1. In short, I think that, for most of us, it is easier to learn a theory introduced via concrete examples than via more abstract presentations, as in some other texts.

But, for those who may consult other books, the following table shows some symbols you will run into in more formal treatments, and what they mean in terms of what we have done here.

Brief look at more formal treatment of group theory

Table of Some Symbols Used in Formal Group Theory

Symbol	Use	Meaning
$\in$	$\mathbf{A} \in G$	**A** is a member of group (or set) *G*
$\notin$	$\mathbf{A} \notin G$	**A** is not a member of group (or set) *G*
$\subseteq$	$\mathbf{A} \subseteq S$	**A** is a subset of set *S*
$\leq$	$\mathbf{A} \leq G$	**A** is a subgroup of group *G*
$\forall$	$\forall\ \mathbf{A}$	For any and all elements **A**
$\mathbb{R}$		Set of real numbers
$\mathbb{C}$		Set of complex numbers
$\mathbb{R}^3$		3D space of real numbers

For example, where we said that the closure property of a group means the result of the group operation between any two elements in the group is also an element of the group, the more formal statement of this would be

$$\forall \mathbf{A}, \mathbf{B} \in G, \quad \mathbf{A} \circ \mathbf{B} \in G \ . \qquad (2\text{-}32)$$

2.4 Lie Algebras

In parallel with our definition for Lie groups, a Lie algebra contains elements that vary continuously and smoothly with one or more parameters, as those parameters vary continuously and smoothly. But, different from a group, a Lie algebra has *two* binary operations (in a matrix representation, matrix addition plus a matrix multiplication type operation) between set elements and also a scalar operation (which will be scalar multiplication by matrix elements in our applications).

Lie algebra definition

In our work, the group element multiplication operation that will turn out to be most beneficial is not simple matrix multiplication such as *AB*, but a matrix commutation operation. That is,

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[1] For one, see Merzbacher, E. *Quantum Mechanics*, 2nd ed, (Wiley, 1970), pg. 271 and Chap. 16.

common 2^{nd} operation in physics $\rightarrow$ $\left[\mathbf{A},\mathbf{B}\right]=\mathbf{AB}-\mathbf{BA}=\mathbf{C}$ where $\mathbf{C}$ an element of the set. (2-33)

In many physics applications, 2nd binary operation is commutation

It turns out, as we will see and as is extremely important for physics applications, that any given Lie group can be related to a Lie algebra. Our goal now is to derive this relationship, and then show how it applies to certain areas of physics.

2.4.1 Relating a Lie Group to a Lie Algebra: Simple Example of SO(2)

Consider the $SO(2)$ 2D rotation group of (2-6), reproduced below for convenience.

$$M(\theta)=\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{2-6}$$

One parameter, real Lie group (2D rotation)

We can express $M(\theta)$ as a Taylor expansion around $\theta = 0$.

$$M(\theta)=M(0)+\theta M'(0)+\frac{\theta^2}{2!}M''(0)+\frac{\theta^3}{3!}M''(0)+\dots\ , \tag{2-34}$$

which for small θ (<< 1) becomes (where the factor of i is inserted at the end because it will make things easier in the future)

$$\begin{aligned} M(\theta\ll 1)\approx M(0)+\theta M'(0) &= \begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}+\theta\begin{bmatrix} -\sin 0 & -\cos 0 \\ \cos 0 & -\sin 0 \end{bmatrix}, \\ &= I+\theta\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=I+i\theta X \qquad \theta\ll 1\,. \end{aligned} \tag{2-35}$$

Taylor expansion

where[1]

$$X=-iM'(0)=-i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \tag{2-36}$$

The single matrix X and the continuous scalar field θ represent a Lie algebra, where the elements are all of the θX, the first operation is matrix addition, and the second operation is matrix commutation.

Lie algebra for SO(2) has one matrix X, and elements θX

Do **Problem 11** to show this and gain a valuable learning experience. If you have some trouble, it will help to continue reading to the end of Sect. 2.4.3, and then come back to this.

A Lie algebra such as this one (i.e., θX) is often called the <u>tangent space</u> of the group (here, $M(\theta)$) because the derivative of a function, when plotted, is tangent to that function at any given point. X in the present case is the first derivative of M at $\theta = 0$.

<u>Generating SO(2) Lie Group from Its Lie Algebra</u>

Given (2-35), and knowing X, we can actually reproduce the group M from the Lie algebra θX. We show this using (2-34) with (2-6) and its derivatives, along with (2-36). That is,

From X, can generate SO(2) group via expansion

$$M(\theta)=\underbrace{\begin{bmatrix} \cos 0 & -\sin 0 \\ \sin 0 & \cos 0 \end{bmatrix}}_{I}+\theta\underbrace{\begin{bmatrix} -\sin 0 & -\cos 0 \\ \cos 0 & -\sin 0 \end{bmatrix}}_{iX}+\frac{\theta^2}{2!}\underbrace{\begin{bmatrix} -\cos 0 & \sin 0 \\ -\sin 0 & -\cos 0 \end{bmatrix}}_{-I}+\frac{\theta^3}{3!}\underbrace{\begin{bmatrix} \sin 0 & \cos 0 \\ -\cos 0 & \sin 0 \end{bmatrix}}_{-iX}+\,.. \tag{2-37}$$

Realizing that every even derivative factor in (2-34) is the identity matrix (times either 1 or – 1), and every odd derivative factor is iX (times 1 or – 1), we can obtain the group element for any θ directly from the Lie algebra matrix X.

Note that (2-37) can be re-expressed as

---

[1] Some authors define X as simply $M'(0)$ without a factor of i.

$$M(\theta)=\begin{bmatrix} 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}.. & -\theta+\frac{\theta^3}{3!}-\frac{\theta^5}{5!}+.. \\ \theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}.. & 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+.. \end{bmatrix}=\begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}. \tag{2-38}$$

<u>Exponentiation of Lie Algebra to Generate *SO*(2) Lie Group</u>

More directly, as shown below, we can simply exponentiate θX to get M. From (2-34) and (2-37), where we note $XX = I$,

$$\begin{aligned}\begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix} &= M(\theta)=M(0)+\theta M'(0)+\frac{\theta^2}{2!}M''(0)+\frac{\theta^3}{3!}M''(0)+... \\ &= I+\theta(iX)+\frac{\theta^2}{2!}(-I)+\frac{\theta^3}{3!}(-iX\,I)+... \qquad . \\ &= I+(i\theta X)+\frac{(i\theta X)^2}{2!}+\frac{(i\theta X)^3}{3!}+... \;= e^{i\theta X}\;. \end{aligned} \tag{2-39}$$

or via $SO(2) = e^{i\theta X}$

In essence, X (along with I, which is sort of a "given") can generate M. The matrix X is called the <u>generator</u> of the group M (or of the Lie algebra). It is a basis vector in the Lie algebra vector space. Actually, it is *the* basis vector in this case, as there is only one basis matrix, and that is X.

X called the generator of the Lie group/algebra

As an aside, inserting the i in the last step of (2-35) to define X as in (2-36) led to (2-39).

<u>Key point</u>

Knowing the generator of the Lie algebra, we can construct (generate) the associated Lie group simply by exponentiation of the generator (times the scalar field). Knowing the associated Lie algebra is the same as knowing the group. They each, ultimately, contain the same information.

Knowing generator = knowing the group

2.4.2 Starting with a Different Representation of SO(2)

Let's repeat the process of the prior section, but start with a different representation of the same *SO*(2) group, i.e., (2-9), which we repeat below for convenience.

$$M(x)=\begin{bmatrix} \sqrt{1-x^2} & -x \\ x & \sqrt{1-x^2} \end{bmatrix} \tag{2-9}$$

Different rep of SO(2)

We calculate the generator in similar fashion as before.

$$X=-iM'(x)_{x=0}=-i\begin{bmatrix} \frac{-2x}{2\sqrt{1-x^2}} & -1 \\ 1 & \frac{-2x}{2\sqrt{1-x^2}} \end{bmatrix}_{x=0}=-i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \tag{2-40}$$

Has same generator as other rep

(2-40) is the same as (2-36). Different representations of the same group have the same generator and thus, the same associated Lie algebra.

Different reps of group have same Lie algebra

Interestingly, exponentiation using (2-40), as in the last part of (2-39), gives us the original group representation, as in the first part of (2-39), rather than the other group representation (2-9).

2.4.3 Lie Algebra for a Three Parameter Lie Group: SO(3) Example

The 2D rotation example above was a one parameter (i.e., θ) Lie group. Consider the 3D rotation group matrix representation A of (2-11) to (2-13) with the three parameters θ_1, θ_2, and θ_3. The multivariable Taylor expansion, parallel to (2-39), is

3D rotation = SO(3) group

3 parameters

$$A(\theta_1,\theta_2,\theta_3)=A_1(\theta_1)A_2(\theta_2)A_3(\theta_3)=e^{i\theta_1X_1}e^{i\theta_2X_2}e^{i\theta_3X_3}=$$
$$\left(M_1\big|_{\theta_1=0}+\theta_1 M_1'\big|_{\theta_1=0}+\tfrac{1}{2!}\theta_1^2 M_1''\big|_{\theta_1=0}+\dots\right)\left(M_2\big|_{\theta_2=0}+\theta_2 M_2'\big|_{\theta_2=0}+\tfrac{1}{2!}\theta_2^2 M_2''\big|_{\theta_2=0}+\dots\right)\times$$
$$\left(M_3\big|_{\theta_3=0}+\theta_3 M_3'\big|_{\theta_3=0}+\tfrac{1}{2!}\theta_3^2 M_3''\big|_{\theta_3=0}+\dots\right)= \qquad (2\text{-}41)$$
$$\left(I+(i\theta_1X_1)+\frac{(i\theta_1X_1)^2}{2!}+\dots\right)\left(I+(i\theta_2X_2)+\frac{(i\theta_2X_2)^2}{2!}+\dots\right)\left(I+(i\theta_3X_3)+\frac{(i\theta_3X_3)^2}{2!}+\dots\right).$$

Taylor expansion in 3 parameters

Parallel to what we did for the one parameter case, we can find three generators

$$X_i=-i\frac{\partial A_i}{\partial\theta_i}\bigg|_{\theta_i=0} \qquad i=1,2,3 \text{ (no sum)}, \qquad (2\text{-}42)$$

which, more explicitly expressed (taking first derivatives of components in (2-12)), are

$$X_1=-i\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix} \qquad X_2=-i\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix} \qquad X_3=-i\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}. \qquad (2\text{-}43)$$

3 generators X_i

Note, for future reference, the commutation relations between the X_i. For example,

$$\begin{aligned}[X_1,X_2]&=i^2\left(\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}-\begin{bmatrix}0&0&-1\\0&0&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix}\right)\\&=-\begin{bmatrix}0&0&0\\-1&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}0&-1&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix}=iX_3\ .\end{aligned} \qquad (2\text{-}44)$$

In general, where ε_{ijk} is the Levi-Civita symbol, and as you can prove to yourself by calculating the other two commutations involved (or just take my word for it),

$$\left[X_i,X_j\right]=i\varepsilon_{ijk}X_k \qquad i,j,k \text{ each take on value } 1,2, \text{ or } 3\ . \qquad (2\text{-}45)$$

Commutation relations for generators

As we are about to show, scalar field multiplication of the θ_i by their respective X_i, under matrix addition and matrix commutation, comprise a Lie algebra.

<u>Proof that (2-43) with θ_i comprises a Lie algebra</u>

X_i and θ_i comprise a Lie algebra

An algebra has two binary operations (matrix addition and commutation of two matrices here) and one scalar multiplication operation. We need to check that these operations satisfy the requirements for an algebra, as given in Wholeness Chart 2-1, pg. 3.

Proof

First, look at the 1st binary operation of matrix addition with scalar field multiplication.

1st binary operation satisfies group properties

Closure: Consider the X_i to comprise a basis for a vector space. Every possible matrix addition between elements in the set results in an element (matrix) in the same set. So, we have closure under the first operation.

Note: Subscripts *a, b, c* below imply (typically different) elements in the set. That is, θ_aX_a, θ_bX_b, and θ_cX_c are each elements, where repeated indices indicate summation over values 1,2, and 3, e.g., $\theta_aX_a = \theta_{a=1}X_1 + \theta_{a=2}X_2 + \theta_{a=3}X_3$

Associative: $\theta_aX_a + (\theta_bX_b + \theta_cX_c) = (\theta_aX_a + \theta_bX_b) + \theta_cX_c$. Matrix addition is associative.

Identity: For $\theta_c' = 0$ with c values = 1,2,3, $\theta_bX_b + \theta_c'X_c = \theta_bX_b$. So, $\theta_c'X_c = [0]_{3X3}$ is the identity element for matrix addition.

Inverse: Each element θ_aX_a of the set has an inverse $(-\theta_aX_a)$, since $\theta_aX_a + (-\theta_aX_a) = [0]_{3X3}$ (the identity element).

Commutation: $\theta_aX_a + \theta_bX_b = \theta_bX_b + \theta_aX_a$. Matrix addition is commutative.

1st binary operation commutative, so we have a vector space (vectors are the matrices)

Thus, under addition and scalar multiplication, the set of all elements $\theta_i X_i$ comprises a vector space and satisfies the requirements for one of the operations of an algebra.

Second, look at the 2nd binary operation of matrix commutation with scalar field multiplication.

Closure: $[\theta_a X_a,\ \theta_b X_b] = \theta_a \theta_b\ [X_a,\ X_b]$, which from (2-45) $= i\theta_a \theta_b \varepsilon_{abc} X_c = i\theta_c X_c$. apart from the factor of i, this is an element in the set. But with the factor of i, it is not. So, we need to define our 2nd operation a bit differently, to get true closure, as follows.

2nd binary operation satisfies closure requirement of an algebra

$$\text{2nd binary operation definition:} \quad -i[\theta_a X_a,\ \theta_b X_b] \tag{2-46}$$

Given (2-45), we find (2-46) is $-i[\theta_a X_a,\ \theta_b X_b] = -i\theta_a \theta_b\ [X_a,\ X_b] = \theta_a \theta_b \varepsilon_{abc} X_c = \theta_c X_c$, which is in the set. Under the 2nd operation of (2-46), there is closure.

Third, look at both binary operations together.

Distributive: From Wholeness Chart 2-1, $\mathbf{A}\circ(\mathbf{B} ✞ \mathbf{C}) = \mathbf{A}\circ\mathbf{B} ✞ \mathbf{A}\circ\mathbf{C}$ for us, if distributive, need $-i[A,B + C] = -i[A,B] + (-i)[A,C]$ or simply $[A,B + C] = [A,B] + [A,C]$.

Both binary operations satisfy distributive requirement of an algebra

$$[A,B + C] = [\theta_a X_a,\ \theta_b X_b + \theta_c X_c] = \theta_a X_a\,(\theta_b X_b + \theta_c X_c) - (\theta_b X_b + \theta_c X_c)\ \theta_a X_a$$

$$= \theta_a X_a \theta_b X_b + \theta_a X_a \theta_c X_c - \theta_b X_b \theta_a X_a - \theta_c X_c \theta_a X_a = [\theta_a X_a, \theta_b X_b] + [\theta_a X_a, \theta_c X_c]$$

$= [A,B] + [A,C]$. The operations are distributive.

<u>Conclusion:</u> The set $\theta_i X_i$ under matrix addition, the matrix commutation operation of (2-46), and scalar field multiplication is an algebra. It is a Lie algebra because every element in the set is a smooth, continuous function of the smooth, continuous variables θ_i.

So, we have an algebra, a Lie algebra, since θ_i are continuous, smooth

Further, regarding the 2nd binary operation, one sees from the analysis below that this particular algebra is non-associative, non-unital, and non-Abelian.

Associative: General relation $\mathbf{A}\circ(\mathbf{B}\circ\mathbf{C}) = (\mathbf{A}\circ\mathbf{B})\circ\mathbf{C}$**.** For us, if associative, need $-i\,[A,[B,C]] = -i\,[[A,B],C]$ or simply $[A,[B,C]] = [[A,B],C]$.

$$[A,[B,C]] = [\theta_a X_a,[\theta_b X_b,\theta_c X_c]] = \theta_a\theta_b\theta_c[X_a,[X_b,X_c]] = \theta_a\theta_b\theta_c[X_a,(X_b X_c - X_c X_b)] =$$
$$\theta_a\theta_b\theta_c(X_a X_b X_c - X_a X_c X_b - X_b X_c X_a + X_c X_b X_a)$$

$$[[A,B],C] = [[\theta_a X_a,\theta_b X_b],\theta_c X_c] = \theta_a\theta_b\theta_c[[X_a,X_b],X_c] = \theta_a\theta_b\theta_c[(X_a X_b - X_b X_a),X_c] =$$
$$\theta_a\theta_b\theta_c(X_a X_b X_c - X_b X_a X_c - X_c X_a X_b + X_c X_b X_a)$$

These relations are not equal. The middle terms differ because the X_i do not commute. So, the second binary operation is non-associative and we say that this Lie algebra is non-associative.

This algebra is non-associative

Identity: An element I would be the identity element under commutation relation (2-46), if and only if, $-i\ [I,\ X'] = X'$ where X' could be any element in the set. As shown by doing Problem 12, there is no such I. Since no identity element exists, this algebra is non-unital.

and non-unital

Inverse: If there is no identity element, there is no meaning for an inverse.

Commutative: General relation $\mathbf{A}\circ\mathbf{B} = \mathbf{B}\circ\mathbf{A}$ needed for all elements, for the binary operation $\circ$. For us, $\circ$ is commutation, so we need commutation of the commutation operation. That is, we need, in general, $[A,B] = [B,A]$. Thus, as one example, $-i[X_1, X_2] = -i[X_2, X_1]$. But from (2-45), or simply from general knowledge of commutation, this is not true (we are off by a minus sign), so there are elements in the set that do not commute under the 2nd binary operation (2-46) (which is itself a commutation relation). This 2nd binary operation is non-Abelian, and thus, so is the algebra.

and non-Abelian

<u>End of proof</u>

Do **Problem 12** to show there is no identity element for the 2nd operation (2-46) in the $SO(3)$ Lie algebra.

Do **Problem 13** to see why we took matrix commutation as our second binary operation for the Lie group, rather than the simpler alternative of matrix multiplication.

The commutation relations structure the Lie algebra and Lie group and tell us almost everything one needs to know about both the algebra and the group. Because of this "structuring", the ε_{ijk} of (2-45) are often called the structure constants of $SO(3)$. We will see that other groups have their own particular (different) structure constants, but in every case, they tell us the properties of the algebra and associated group.

Constants in the generator commutation relations called "structure constants". These structure the group (contain the key info about the group)

Quick intermediate summary for $SO(3)$

For $SO(3)$,

$$\text{the generators obey } \left[X_i, X_j\right] = i\varepsilon_{ijk} X_k \qquad i,j,k \text{ each take on a value } 1, 2, \text{ or } 3\,, \tag{2-45}$$

and the Lie algebra operations are addition and

$$\text{the 2}^{\text{nd}} \text{ binary operation } \; -i[\theta_a X_a,\, \theta_b X_b] \tag{2-46}$$

Summary of $SO(3)$:
1) 3 generators
2) 3 commutation rels
3) binary operations: addition & commutation

2.4.4 Generating SO(3) from Its Lie Algebra

As with the $SO(2)$ Lie group, one can generate the $SO(3)$ group from its generators, via the expansion in the last line of (2-41). We won't go through all the algebra involved, as the steps for each factor parallel those for $SO(2)$, and the actual doing of it is fairly straightforward.

2.4.5 Exponentiation Relating SO(3) Lie Group to Its Lie Algebra

General Case is Tricky

For a one parameter Lie group such as (2-6) in θ, the relationship between it and the associated Lie algebra θX [see (2-36)] was simple exponentiation (2-39). One can generate the group via $M(\theta) = e^{i\theta X}$. For a Lie group of more than one parameter, however, things get a little trickier, due to the Baker-Campbell-Hausdorf relationship for exponentiation of operators,

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}\left(X,[X,Y]+\left[Y,[Y,X]\right]\right)+\dots}\,, \tag{2-47}$$

where we imply the infinite series of nested commutators after the second commutator relation. If X and Y commute (as numbers do), we get the familiar addition of exponents relation. When they don't, such as with many operators, things get more complicated.

In our example of $SO(3)$ (2-41), one might naively expect to obtain the Lie group from the Lie algebra using the exponentiation relation on the RHS of (2-48), but due to (2-47) one cannot.

Exponential addition law for operators makes exponentiation of generators to get $SO(3)$ Lie group not simple

$$A(\theta_1,\theta_2,\theta_3) = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3) = e^{i\theta_1X_1}e^{i\theta_2X_2}e^{i\theta_3X_3} \neq e^{i(\theta_1X_1+\theta_2X_2+\theta_3X_3)} = e^{i\theta_iX_i} \tag{2-48}$$

So, if you have a particular Lie algebra element $\theta_i X_i$ (some sum of the generators), you do not use the RHS of (2-48) to generate the Lie group (2-41). You have to use the relationship in the middle of (2-48). Conversely, if you have a Lie group element in terms of three θ_i, such as A on the LHS of (2-48), you cannot assume the associated Lie algebra element to be exponentiated is $\theta_i X_i$.

To get the Lie algebra element, call it $\theta_i' X_i$, associated with a given Lie group element in terms of θ_i, we need to use (2-47). That is, we need to find the θ_i' values in

$$A(\theta_1,\theta_2,\theta_3) = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3) = e^{i\theta_i'X_i} \qquad \theta_i' = \theta_i'(\theta_i)\,. \tag{2-49}$$

As a simple example, consider the case where $\theta_3 = 0$, so the total group action amounts to a rotation through θ_1 followed by a rotation through θ_2. Using (2-47), we find

$$\begin{aligned} A(\theta_1,\theta_2,\theta_3=0) &= A_1(\theta_1)A_2(\theta_2) = e^{i\theta_1X_1}e^{i\theta_2X_2} \\ &= e^{i\theta_1X_1+i\theta_2X_2+\frac{1}{2}[i\theta_1X_1,i\theta_2X_2]+\frac{1}{12}\left(\left[i\theta_1X_1,[i\theta_1X_1,i\theta_2X_2]\right]+\left[i\theta_2X_2,[i\theta_2X_2,i\theta_1X_1]\right]\right)+\dots} \\ &= e^{i\theta_i'X_i}\,. \end{aligned} \tag{2-50}$$

So, using the defining commutation relation of the Lie algebra (2-45), we find

$$\begin{aligned} i\theta_i' X_i &= i\theta_1 X_1 + i\theta_2 X_2 + \tfrac{1}{2}\left[i\theta_1 X_1, i\theta_2 X_2\right] + \tfrac{1}{12}\left[i\theta_1 X_1,\left[i\theta_1 X_1, i\theta_2 X_2\right]\right] + \tfrac{1}{12}\left[i\theta_2 X_2,\left[i\theta_2 X_2, i\theta_1 X_1\right]\right].. \\ &= i\theta_1 X_1 + i\theta_2 X_2 - \tfrac{1}{2}\theta_1\theta_2\left(iX_3\right) - i\tfrac{1}{12}\theta_1^2\theta_2\left[X_1, iX_3\right] - i\tfrac{1}{12}\theta_1\theta_2^2\left[X_2, -iX_3\right] + ... \\ &= i\theta_1 X_1 + i\theta_2 X_2 - i\tfrac{1}{2}\theta_1\theta_2 X_3 - i\tfrac{1}{12}\theta_1^2\theta_2 X_2 - i\tfrac{1}{12}\theta_1\theta_2^2 X_1 + .. \end{aligned} \quad (2\text{-}51)$$

But we can still generate the group from X_i using the generator commutation relations

At second order, $\theta_i' X_i \approx \theta_1 X_1 + \theta_2 X_2 - \frac{1}{2}\theta_1\theta_2 X_3$, so $\theta_1' \approx \theta_1$, $\theta_2' \approx \theta_2$, $\theta_3' \approx -\frac{1}{2}\theta_1\theta_2$. In principle, we can find the θ_i' at any order by using all terms in (2-51) up to that order. And for cases where $\theta_3 \neq 0$, one just repeats the process one more time using the results of (2-51) with (2-47) and the third operator in the exponent $\theta_3 X_3$.

Do **Problem 14** to obtain the third order θ_i' values for our example.

A key thing to notice is that any two group elements $A(\theta_{A1}, \theta_{A2}, \theta_{A3})$ and $B(\theta_{B1}, \theta_{B2}, \theta_{B3})$ of form like (2-49), when multiplied together via the group operation of matrix multiplication, are also in the group, i.e., $AB = C$, where C is in the group. That is, due to the commutation relations (2-45) used in (2-47) we will always get a result equal to the exponentiation of $\theta_i'' X_i$, i.e., $C = e^{i\theta_i'' X_i}$, where the θ_i'' can be determined. That is, every group operation on group elements yields a group element, and that group element has an associated Lie algebra element $\theta_i'' X_i$. All of this is only because each of the commutation relations (2-45) used in (2-47) [and thus, (2-51)] yields one of the Lie algebra basis matrices X_i.

Group property of AB=C, with A,B,C in group still holds

Infinitesimal Scalars θ_i Case is Simpler

For small θ_i in (2-51), at lowest order $\theta_i' X_i \approx \theta_1 X_1 + \theta_2 X_2$, so $\theta_1' \approx \theta_1$, $\theta_2' \approx \theta_2$, $\theta_3' \approx \theta_3 = 0$. It is common to simply consider the group and the algebra to be local (small values of θ_i), so orders higher than the lowest are negligible, and one can simply identify $\theta_i' \approx \theta_i$. Then, we find (2-50) becomes

$$A\left(\theta_1,\theta_2,\theta_3 = 0\right) = A_1\left(\theta_1\right)A_2\left(\theta_2\right) = e^{i\theta_1 X_1}e^{i\theta_2 X_2} \approx e^{i\theta_1 X_1 + i\theta_2 X_2} \qquad \theta_1,\theta_2 \ll 1 \;, \quad (2\text{-}52)$$

and for the more general case,

$$A\left(\theta_1,\theta_2,\theta_3\right) = A_1\left(\theta_1\right)A_2\left(\theta_2\right)A_3\left(\theta_3\right) = e^{i\theta_1 X_1}e^{i\theta_2 X_2}e^{i\theta_3 X_3} \approx e^{i\theta_i X_i} \approx I + \theta_i X_i \qquad \theta_i \ll 1 \quad (2\text{-}53)$$

Exponentiation to get SO(3) group is simpler in infinitesimal case

In principle, we can generate the global (finite) Lie group by taking $\theta_i \rightarrow d\theta_i$ in (2-53) and carrying out step-wise integration. And of course, we can always generate the finite group with the first part of (2-48), $A\left(\theta_1,\theta_2,\theta_3\right) = e^{i\theta_1 X_1}e^{i\theta_2 X_2}e^{i\theta_3 X_3}$.

2.5 The SU(2) Lie Group

The $SU(2)$ Lie group is a very important group in QFT, as it is intimately involved in theoretical descriptions of the weak force. We will examine the group as represented by matrices that are 2X2, have complex components, and may operate on two component vectors in a complex vector space.

The SU(2) Lie group, important in physics

2.5.1 The Matrix Form of the SU(2) Lie Group

A general two-dimensional complex matrix M has form

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \qquad m_{ij} \text{ complex, in general}. \quad (2\text{-}54)$$

2X2 complex matrices

But since the group we will examine is special unitary, it must satisfy

$$M^\dagger M = I \qquad Det M = 1 \;. \quad (2\text{-}55)$$

satisfying special unitary group requirements

I submit the following satisfies (2-55), where a and b are complex.

$$M = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} = \begin{bmatrix} a_{Re} + ia_{Im} & b_{Re} + ib_{Im} \\ -b_{Re} + ib_{Im} & a_{Re} - ia_{Im} \end{bmatrix} \qquad \text{with} \quad aa^* + bb^* = 1 \quad (2\text{-}56)$$

One representation that does satisfy them

The RHS (last part) of (2-55) is obvious from the constraint imposed at the end of (2-56). To save you the time and tedium, I show the LHS (first part) of (2-55) below.

$$M^{\dagger}M=\begin{bmatrix} a^* & -b \\ b^* & a \end{bmatrix}\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}=\begin{bmatrix} a^*a+b^*b & a^*b-a^*b \\ ab^*-ab^* & b^*b+a^*a \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2-57}$$

The Det M = 1 requirement means 1 parameter dependent on other 3 independent

Now consider M as a Lie group, where the a and b are continuous and smoothly varying. Since a and b are complex numbers, there are four real number variables $a_{Re}, a_{Im}, b_{Re}, b_{Im}$, which vary continuously and smoothly. From the constraint at the end of (2-56), only three of these are independent. They are related by

$$a_{Re}^2+a_{Im}^2+b_{Re}^2+b_{Im}^2=1, \tag{2-58}$$

and we choose a_{Im}, b_{Re}, b_{Im} to be independent, and $a_{Re}=a_{Re}\left(a_{Im},b_{Re},b_{Im}\right)$. For future reference, we find the partial derivative of a_{Re} with respect to each of the independent variables via (2-58).

Evaluating derivatives of dependent parameter

$$\begin{aligned}\frac{\partial a_{Re}}{\partial a_{Im}}&=\frac{\partial}{\partial a_{Im}}\left(1-a_{Im}^2-b_{Re}^2-b_{Im}^2\right)^{\frac{1}{2}}=\frac{1}{2}\left(1-a_{Im}^2-b_{Re}^2-b_{Im}^2\right)^{-\frac{1}{2}}\left(-2a_{Im}\right)\\ &=-\frac{a_{Im}}{\sqrt{1-a_{Im}^2-b_{Re}^2-b_{Im}^2}}=-\frac{a_{Im}}{a_{Re}} \quad \text{and} \quad \frac{\partial a_{Re}}{\partial b_{Re}}=-\frac{b_{Re}}{a_{Re}} \quad \frac{\partial a_{Re}}{\partial b_{Im}}=-\frac{b_{Im}}{a_{Re}}\end{aligned} \tag{2-59}$$

Note that when $a_{Im}=b_{Re}=b_{Im}=0$, $a_{Re}=1$, and the partial derivatives in the last line of (2-59) all equal zero.

For the Lie group, when the continuously variable independent parameters are all zero, nothing has changed, so we must have the identity element, and with (2-58), this is true for (2-56), i.e.,

$$M\left(a_{Im}=b_{Re}=b_{Im}=0\right)=I. \tag{2-60}$$

For all independent parameters = 0, M = I

Thus, we have shown that M of (2-56) represents the $SU(2)$ Lie group of three real parameters.

Do **Problem 15** to show that M obeys the group closure property.

Note that the $SU(2)$ group of (2-16) is a special case of (2-56) where $b=0$.

2.5.2 The SU(2) Associated Lie Algebra

We find the Lie algebra generators for the $SU(2)$ Lie group (2-56) in similar fashion to what we did for $SO(2)$ and $SO(3)$. That is, from the multivariable Taylor expansion

Expanding SU(2) group of 3 independent parameters to get generators

$$\begin{aligned}M\left(a_{Im},b_{Re},b_{Im}\right)=\underbrace{M(0,0,0)}_{I}&+a_{Im}\left.\frac{\partial M}{\partial a_{Im}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}+b_{Re}\left.\frac{\partial M}{\partial b_{Re}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}+b_{Im}\left.\frac{\partial M}{\partial b_{Im}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}\\ &+\frac{\left(a_{Im}\right)^2}{2!}\left.\frac{\partial^2 M}{\partial a_{Im}^2}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}+\frac{a_{Im}b_{Re}}{2!}\left.\frac{\partial^2 M}{\partial a_{Im}\partial b_{Re}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}+\ldots,\end{aligned} \tag{2-61}$$

the generators, where we choose the numbering with an eye towards the future result, are

$$X_1=-i\left.\frac{\partial M}{\partial b_{Im}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} \qquad X_2=-i\left.\frac{\partial M}{\partial b_{Re}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} \qquad X_3=-i\left.\frac{\partial M}{\partial a_{Im}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}. \tag{2-62}$$

Evaluating (2-62) for (2-56), we find, with (2-59),

The 3 generators X_i

$$X_1 = -i\frac{\partial M}{\partial b_{Im}}\bigg|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = -i\left(\frac{\partial M}{\partial b_{Im}}\begin{bmatrix} a_{Re}+ia_{Im} & b_{Re}+ib_{Im} \\ -b_{Re}+ib_{Im} & a_{Re}-ia_{Im}\end{bmatrix}\right)\bigg|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = -i\begin{bmatrix} 0 & i \\ i & 0\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}$$

$$X_2 = -i\frac{\partial M}{\partial b_{Re}}\bigg|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = \begin{bmatrix} 0 & -i \\ i & 0\end{bmatrix} \qquad X_3 = -i\frac{\partial M}{\partial a_{Im}}\bigg|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = \begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix}, \tag{2-63}$$

The 3 generators X_i are the Paul matrices

which are the Pauli matrices, and which have the commutation relations

$$\left[X_i, X_j\right] = i\varepsilon_{ijk}X_k \xrightarrow[\text{symbols}]{\text{more common}} \left[\sigma_i, \sigma_j\right] = i\varepsilon_{ijk}\sigma_k\,. \tag{2-64}$$

with the Pauli matrices commutation relations

We will not take all the time to show that the X_i along with the three scalar field multipliers comprise a Lie algebra under the binary operations of addition and commutation. We have done that twice before for other algebras, and we should be able to accept it to be the case here. I assure you it is indeed an algebra.

The Lie algebra for the $SU(2)$ group has the same number of generators with the same commutation relations as the $SO(3)$ group (see (2-45)). That is, it has the same structure constants ε_{ijk}. The two different groups have the same structure and are similar in many ways. For one, which we won't get into in depth here[1], as it doesn't play much role in the standard model of QFT, the 3-dimensional (pseudo) vector of angular momentum can be treated under either the 3D rotation group $SO(3)$ or the 2D $SU(2)$ group. As you may have run into in other studies, spin angular momentum is often analyzed using a 2D complex column vector with the top component representing spin up (along z axis) and the lower component representing spin down. The Pauli matrices, via their operations on the column vector, play a key role in all of that.

which happen to be the same as for $SO(3)$

$SU(2)$ and $SO(3)$ different groups, but can both characterize 3D rotation

Take caution that $SO(3)$ and $SU(2)$ are *not* different representations of the same group. They are different groups, even though they share the same structure (same associated Lie algebra) and can characterize the same physical phenomenon. This is similar to the relationship between 2D rotation group $SO(2)$ and the $U(1)$ group we looked at in Problem 9.

They share the same Lie algebra

2.5.3 Generating the SU(2) Group from the SU(2) Lie Algebra

Prove to yourself that the X_i above generate the $SU(2)$ group by doing **Problem 16.**

From the results of Problem 16, we see that (2-61) can be expressed as

$$M\left(a_{Im}, b_{Re}, b_{Im}\right) = I + ia_{Im}X_3 + ib_{Re}X_2 + ib_{Im}X_1 - \frac{a_{Im}^2}{2!}I - \frac{b_{Re}^2}{2!}I - \frac{b_{Im}^2}{2!}I + \frac{a_{Im}b_{Re}}{2!}[0] + \ldots \tag{2-65}$$

Expressing $SU(2)$ elements in terms of generators and independent parameters

2.5.4 Exponentiation of the SU(2) Lie Algebra

General Case

One can obtain the Lie group from the Lie algebra via the expansion (2-61) along with (2-62), expressed in (2-65). One can also obtain it a second, related, way, which involves exponentiation, in a manner similar to what we saw earlier with $SO(2)$ and $SO(3)$. However, we would find doing so to be a mathematical morass, so we will simply draw parallels to what we saw with earlier groups.

Exponentiating Lie algebra to get $SU(2)$ Lie group is a nightmare

Consider a general Lie algebra element

We illustrate how done in principle

$$X = b_{Im}X_1 + b_{Re}X_2 + a_{Im}X_3\,, \tag{2-66}$$

where one could exponentiate it as

$$e^{iX} = e^{i\left(b_{Im}X_1 + b_{Re}X_2 + a_{Im}X_3\right)}\,. \tag{2-67}$$

and where we note, in passing, that (see (2-47) and (2-48))

$$e^{i\left(b_{Im}X_1 + b_{Re}X_2 + a_{Im}X_3\right)} \neq e^{ib_{Im}X_1}e^{ib_{Re}X_2}e^{ia_{Im}X_3}\,. \tag{2-68}$$

---

[1] See footnote references on pg. 2 or almost any text on group theory.

We would like to explore whether (2-67) equals (2-56) [equivalently, (2-65)],

$$\begin{aligned} e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)} &\overset{?}{=} M\left(a_{Im},b_{Re},b_{Im}\right) \\ &= I-\frac{a_{Im}^2}{2!}I-\frac{b_{Re}^2}{2!}I-\frac{b_{Im}^2}{2!}I+ia_{Im}X_3+ib_{Re}X_2+ib_{Im}X_1\ldots \end{aligned} \quad . \tag{2-69}$$

By expanding the top row LHS of (2-69) around $a_{Im} = b_{Re} = b_{Im} = 0$, we could see whether or not it matches the expansion of M in (2-69), 2nd row. We will not go through all that tedium, but draw instead on our knowledge of the other multiple parameter case $SO(3)$ where we found the equal sign with the question mark in (2-69) is actually a not-equal sign. If we wished, however, we could, with a copious amount of labor, find a matrix function to exponentiate that would give us M. That is, similar to (2-49),

$$M\left(a_{Im},b_{Re},b_{Im}\right)=e^{i\beta_i X_i}\neq e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)} \qquad \beta_i=\beta_i\left(a_{Im},b_{Re},b_{Im}\right). \tag{2-70}$$

To help in what comes next, do **Problem 17.**

Infinitesimal Case

However, for small values of a_{Im}, b_{Re}, and b_{Im} (<< 1), as can be found by doing Problem 17

$$e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)}\approx I+ia_{Im}X_3+ib_{Re}X_2+ib_{Im}X_1\approx M\left(a_{Im},b_{Re},b_{Im}\right) \qquad a_{Im},b_{Re},b_{Im}\ll 1 . \tag{2-71}$$

Exponentiation in infinitesimal case is simpler

As with prior cases, one could generate the global (finite) $SU(2)$ group by step-wise integration over infinitesimal a_{Im}, b_{Re}, and b_{Im}.

2.5.5 Another Representation of SU(2)

(2-72) below is a different representation of $SU(2)$ with three different parameters.

Do **Problem 18** to prove it.

$$M\left(\phi_1,\phi_2,\phi_3\right)=\begin{bmatrix} \cos\phi_1 e^{i\phi_2} & \sin\phi_1 e^{i\phi_3} \\ -\sin\phi_1 e^{-i\phi_3} & \cos\phi_1 e^{-i\phi_2}\end{bmatrix} \qquad a=\cos\phi_1 e^{i\phi_2} \qquad b=-\sin\phi_1 e^{-i\phi_3} \tag{2-72}$$

Another representation of SU(2)

Note that (2-6) is a special case of (2-72) where $\phi_2 = \phi_3 = 0$ (and here, $\phi_1 = -\phi$).

Finding the generators for (2-72) in the same way as we did for (2-56)

$$\begin{aligned}
X_1 &= -i\left.\frac{\partial M}{\partial\phi_1}\right|_{\phi_i=0} = -i\begin{bmatrix} -\sin\phi_1 e^{i\phi_2} & \cos\phi_1 e^{i\phi_3} \\ -\cos\phi_1 e^{-i\phi_3} & -\sin\phi_1 e^{-i\phi_2}\end{bmatrix}_{\phi_i=0} = -i\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}=\begin{bmatrix} 0 & -i \\ i & 0\end{bmatrix} \\
X_2 &= -i\left.\frac{\partial M}{\partial\phi_2}\right|_{\phi_i=0} = -i\begin{bmatrix} i\cos\phi_1 e^{i\phi_2} & 0 \\ 0 & -i\cos\phi_1 e^{-i\phi_2}\end{bmatrix}_{\phi_i=0} = -i\begin{bmatrix} i & 0 \\ 0 & -i\end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & -1\end{bmatrix} \\
X_3' &= -i\left.\frac{\partial M}{\partial\phi_3}\right|_{\phi_i=0} = -i\begin{bmatrix} 0 & i\sin\phi_1 e^{i\phi_3} \\ i\sin\phi_1 e^{-i\phi_3} & 0\end{bmatrix}_{\phi_i=0} = \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix} \qquad \text{Not relevant} \\
X_3 &= -i\left.\frac{\partial^2 M}{\partial\phi_1\partial\phi_3}\right|_{\phi_i=0} = -i\begin{bmatrix} 0 & i\cos\phi_1 e^{i\phi_3} \\ i\cos\phi_1 e^{-i\phi_3} & 0\end{bmatrix}_{\phi_i=0} = \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix} \qquad \text{3rd generator}
\end{aligned} \tag{2-73}$$

The generators for this rep are the same as for the other rep

So, we have the same generators as we did for the prior representation of $SU(2)$, (2-63), with the same structure constants in the commutation relations as in (2-64). We won't go into detail with this representation to show how it comprises a Lie algebra, how one can generate the original Lie group from it, etc., as it will not play a big role in what we do in this book. But we can infer one significant conclusion from this.

In general, different reps of same Lie group have same structure (same Lie algebra commutation rels)

In general, any representation for a given Lie group will have the same Lie algebra generators, i.e., the same structure (same structure constants in the generator commutation relations). So, for example, if we see two different 2X2 complex matrices, and we find the generators for each, if they are the same, then the two matrices are simply different representations of the same group.

This rep good for spin; prior for QFT

(2-72) has value in analyzing spin. (2-56) has value in QFT. Different reps for different steps.

Digression for Brief Look at Spin and $SU(2)$

However, we will now digress briefly to show how (2-72) can be used for spin analysis. Recall the wave function in NRQM had a two-column vector representing spin.

Brief look at how this rep of SU(2) can handle spin

$$|\psi\rangle_{\substack{spin\\up}} = Ae^{-ikx}\begin{bmatrix}1\\0\end{bmatrix} \text{ spin in } +z \text{ direction} \qquad |\psi\rangle_{\substack{spin\\down}} = Ae^{-ikx}\begin{bmatrix}0\\1\end{bmatrix} \text{ spin in } -z \text{ direction} \tag{2-74}$$

Consider the case where we rotate the spin down particle to a spin up orientation (or conversely, rotating our observing coordinate system in the opposite direction). In physical space we have rotated the z axis 180° and could use the $SO(3)$ rotation group (2-41) to rotate the 3D (pseudo) vector for spin angular momentum through 180°. However, for the manner in which we represent spin in (2-74), that would not work, as spin there is represented by a two component vector, not a three component one. But, consider the $SU(2)$ representation (2-72) where, in this case, $\phi_2 = \phi_3 = 0$, and the ϕ_1 is a rotation about the x axis (which effectively rotates the z axis through the angle ϕ_1). We actually need to take $\phi_1 = \phi/2$, where ϕ is the actual physical angle of rotation, in order to make it work, as we are about to see.

Then note what (2-72) does to the spin down wave function on the RHS of (2-74).

3D rotation in SU(2) via $\phi_1 = \phi_{3D}/2$.

$$M|\psi\rangle_{\substack{spin\\down}} = \begin{bmatrix}\cos\phi_1 & \sin\phi_1\\ -\sin\phi_1 & \cos\phi_1\end{bmatrix}Ae^{-ikx}\begin{bmatrix}0\\1\end{bmatrix} = Ae^{-ikx}\underbrace{\begin{bmatrix}\cos\frac{\phi}{2} & \sin\frac{\phi}{2}\\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2}\end{bmatrix}}_{\text{for } \phi=180^\circ}\begin{bmatrix}0\\1\end{bmatrix}$$
$$= Ae^{-ikx}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = Ae^{-ikx}\begin{bmatrix}1\\0\end{bmatrix} = |\psi\rangle_{\substack{spin\\up}}\,. \tag{2-75}$$

So, we see that the spinor (two component column vector) lives in a 2D vector space, on which the elements of the $SU(2)$ group operate. And rotations in 3D space can be characterized, in a 2D complex space, by the $SU(2)$ group. As noted earlier this has wide ranging application in analyzing spin, but we will leave further treatment of this topic to the other sources cited previously.

End of digression

Raising operation: column vector component up one level. Lowering operation: down one

We do note that the matrix operation of (2-75) is sometimes referred to as a raising operation as it raises the lower component into the upper component slot. Conversely, when an operation transfers an upper component to a lower component slot, it is called a lowering operation. We will run into these concepts again in QFT.

2.5.6 Shortcut Way to Generate the First Representation

1st SU(2) rep is easy to generate from the generators

Note that because of its particular form, our first representation (2-56) of the $SU(2)$ can be found rather easily from the Lie algebra simply by adding the generators and the identity matrix multiplied by their associated parameters. That is,

$$M = \begin{bmatrix}a_{Re}+ia_{Im} & b_{Re}+ib_{Im}\\ -b_{Re}+ib_{Im} & a_{Re}-ia_{Im}\end{bmatrix} = a_{Re}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + ia_{Im}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + ib_{Re}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + ib_{Im}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$
$$= a_{Re}I + ia_{Im}X_3 + ib_{Re}X_2 + ib_{Im}X_1 = \sqrt{1-a_{Im}^2-b_{Re}^2-b_{Im}^2}\,I + ia_{Im}X_3 + ib_{Re}X_2 + ib_{Im}X_1 \tag{2-76}$$
$$= I + ia_{Im}X_3 + ib_{Re}X_2 + ib_{Im}X_1 - \frac{a_{Im}^2}{2!}I - \frac{b_{Re}^2}{2!}I - \frac{b_{Im}^2}{2!}I + \frac{a_{Im}b_{Re}}{2!}[0] + \dots,$$

where the last line, in which we expand the dependent variable a_{Re} in terms of the independent variables, is simply our original expansion (2-61), which in terms of the generators is (2-65).

So, in this particular representation, going back and forth between the Lie group and the Lie algebra (plus the identity matrix) is relatively easy.

However, it is not so easy and simple with the second representation (2-72). In the expansion of $M(\theta_i)$ (which we didn't do), one gets terms in $\theta_i X_i$ in the infinite summation, but the original matrix had functions of $\sin\theta_1$, $\cos\theta_1$, $e^{\pm i\theta_2}$, $e^{\pm i\theta_3}$ multiplied by one another. That gets complicated in a hurry.

2nd SU(2) rep is hard to generate from the generators

As noted, in NRQM, we deal with 2nd representation (2-72). In QFT, we deal with the 1st. So, in this sense, QFT is easier. (But, probably only in that sense.)

2.6 The *SU*(3) Lie Group

The *SU*(3) Lie group is also an important group in QFT, as it is intimately involved in theoretical descriptions of the strong force. We will examine the group as represented by matrices, which are 3X3, have complex components, and may operate on three component vectors in a vector space. As you may be surmising, in strong interaction theory, the three components of the vectors will represent three quark eigenstates, each with a different color charge (eigenvalue). More on that later in the book.

The SU(3) Lie group relevant to the strong interaction

2.6.1 The Matrix Form of the *SU*(3) Lie Group

A general three-dimensional complex matrix N has form

$$N = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{23} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} \qquad n_{ij} \text{ complex, in general} \tag{2-77}$$

Since the group we will examine is special unitary, it must satisfy

$$N^\dagger N = I \qquad Det\, N = 1\ . \tag{2-78}$$

Using (2-78) with (2-77) would lead us, in similar fashion to what we did in *SU*(2), to one dependent n_{ij}, and eight independent n_{ij}, but it would require a lot of complicated, extremely tedious algebra. So, we will only outline the steps involved and take solace in the fact that the final result works in *SU*(3) applications, and so is undoubtedly correct. I do not know of anywhere in the literature where this algebraic exercise is actually carried out. I have never worked through it myself, and am content understanding that if I did, all relations like (2-78) for the matrix representation N we will work with would hold. I hope you, the reader, can be content with this, as well, but if any reader ever does work it out, please send me a copy (via the email address at the feedback link on the book website [URL opposite pg. 1]).

For the rest of this section, it may help to follow along with the parallels from *SU*(2) in Section 2.5.

The matrix

$$N = \frac{1}{2}\begin{bmatrix} 2\alpha_0 + \alpha_3 + \dfrac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & 2\alpha_0 - \alpha_3 + \dfrac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & 2\alpha_0 - \dfrac{2\alpha_8}{\sqrt{3}} \end{bmatrix} \tag{2-79}$$

can satisfy (2-78), where all α values are real, provided α_0 is the correct function of the other α_i to satisfy the determinant relationship constraint of (2-78).

$$\alpha_0 = \alpha_0(\alpha_i) \qquad i = 1,\dots 8 \qquad \text{such that } \ Det\, N = 1\ . \tag{2-80}$$

Further, α_0 must be such that for all $\alpha_i = 0$, N must $= I$. So, in that case, from (2-79), $\alpha_0(0) = 1$. This is parallel to what we found in *SU*(2).

To gain a little insight into all the algebra behind this, note that N is hermitian, so $N^{\dagger}=N$, and if, as claimed, it is also unitary (so $N^{\dagger}=N^{-1}$) then we must have

$$N^{-1}N = N^{\dagger}N = NN = I \quad \text{with} \quad Det\,N = 1\ , \tag{2-81}$$

and from a well-known law of multiplication of determinants

$$NN = I \quad \rightarrow \ \left(Det\,N\right)\left(Det\,N\right) = Det\,I = 1\,. \tag{2-82}$$

This may give us some confidence that if $Det\,N = 1$ (which the functional form of $\alpha_0(\alpha_i)$ guarantees), then N is indeed unitary.

If we define matrices λ_i as

$$\begin{gathered}\lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_2 = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \lambda_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \\ \lambda_5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix} \quad \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix} \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix},\end{gathered} \tag{2-83}$$

then

$$N = \alpha_0 I + \tfrac{1}{2}\left(\alpha_1\lambda_1 + \alpha_2\lambda_2 + \alpha_3\lambda_3 + \alpha_4\lambda_4 + \alpha_5\lambda_5 + \alpha_6\lambda_6 + \alpha_7\lambda_7 + \alpha_8\lambda_8\right) = \alpha_0 I + \alpha_i \frac{\lambda_i}{2}, \tag{2-84}$$

which parallels the first part of the second row of (2-76).

Do **Problem 19** to show that $\lambda_i/2$ are the generators of N.

We can, of course, use other bases for our Lie algebra (which is a vector space plus a second binary operation, so has elements that are bases for the vector space) than (2-83) (divided by two by convention), such as any eight independent linear combinations of the $\lambda_i/2$. However, the λ_i as defined here make things easiest in the long run and are the widely accepted convention.

Note many authors (though not all) prefer to use $\lambda_i/2$, rather than λ_i, as the generators (Lie algebra basis), and as that is the most widely used convention, we will employ it here. It can help therefore if we define a new symbol for our generators,

$$\hat{F}_i = \tfrac{1}{2}\lambda_i\,. \tag{2-85}$$

Do **Problem 20** to help in what comes next.

It turns out, if one cranks all the algebra using (2-83) and (2-85), that the following commutation relations exist between the Lie algebra generators (basis vector matrices)

$$\left[\hat{F}_i, \hat{F}_j\right] = i f_{ijk}\hat{F}_k\ , \tag{2-86}$$

where repeated indices, as usual, indicate summation. The f_{ijk} are the structure constants for $SU(3)$, but are not as simple as the structure constants for $SU(2)$, which took on the values ±1 of the Levi-Civita symbol ε_{ijk}.

The f_{ijk} do turn out to be totally anti-symmetric, like the ε_{ijk}.

$$f_{ijk} = -f_{jik} = f_{jki} = -f_{kji} = f_{kij} = -f_{ikj}\ , \tag{2-87}$$

and they take on the specific values (some of which you can check via your solution to Problem 20)

Values of *SU*(3) Structure Constants

ijk	123	147	156	246	257	345	367	458	678	Others
fijk	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0

As with $SU(2)$, we will not go through all the steps to show the $\hat{F}_i$ with the α_i comprise a Lie algebra, as they parallel what we did for $SO(2)$ and $SO(3)$. One can see that the $\hat{F}_i$ matrices and the scalars α_i form a vector space, and that with commutation as the second binary operation defined via (2-86), there is closure.

TO BE CONTINUED IN THE TEXT

2.7 Problems

1. Give your own examples of a group, field, vector space, and algebra.
2. Is a field a commutative group plus a second binary operator?
   Is a vector space a commutative group plus a scalar operation with a scalar field?
   Is an algebra a vector space plus a second binary operation?
   Does a field plus a scalar operation with a scalar field comprise an algebra? Is the algebra unital?
3. Show that 3D spatial vectors form a vector space. Then show that QM states do as well. What do we call the space of QM states?
4. With a simple sign change to (2-6), find an orthogonal group $O(2)$ that is not special orthogonal, i.e., not $SO(2)$. Show graphically what the sign change means for the operation on a vector. Does this graph help in understanding why special orthogonal transformations are more likely to represent the kinds of phenomena we see in nature? Explain your answer. For the operation $\mathbf{Av} = \mathbf{v}'$, from matrix theory, we know $|\mathbf{v}'| = |\text{Det } A\,||\mathbf{v}|$. Does this latter relation make sense for your graphical depiction? Explain.
5. Write down a 2D matrix that cannot be expressed using (2-6) and a 3D matrix than cannot be expressed using (2-12) and (2-13).
6. Show that U of (2-16) forms a group under matrix multiplication.
7. Show that for U of (2-16), $U^\dagger U = I$ and Det U =1. Then examine the group $-U$ and show that its determinant is not 1 (but – 1.)
8. Show that any unitary operation U (not just (2-16)) operating on a vector (which could be a quantum mechanical state) leaves the magnitude of the vector unchanged.
9. Is $e^{i\theta}$ a way of describing via a unitary group $U(\theta)$ acting on a complex number the same thing as the $SO(2)$ group (2-6) acting on a 2D real vector? Explain your answer mathematically. (Hint: Express the components of a 2D real vector as the real and imaginary parts of complex scalar. Then, compare the effect of $e^{i\theta}$ on that complex scalar to the effect of (2-6) on the 2D real vector.) Note $U(\theta)$ and $SO(2)$ are *different groups*. We do *not* say that $U(\theta)$ here is a representation of $SO(2)$. The two different kinds of groups can describe the same physical world phenomenon, but they are not different representations (of a particular group), as the term "representation" is employed in group theory.
10. Show (2-28) in terms of matrices.

11. Show that $X=\begin{bmatrix}0 & i\\ -i & 0\end{bmatrix}$ along with a scalar field multiplier θ comprise a Lie algebra. Use Wholeness Chart 2-1 as an aid. Note that every θX is in the set of elements comprising the algebra, and the operations are matrix addition and matrix commutation. This is considered a trivial Lie algebra. Why do you think it is?

12. Show there is no identity element for the 2nd operation (2-46) in the $SO(3)$ Lie algebra. (Hint: The identity element has to work for every element in the set, so you only have to show there is no identity for a single element of your choice.)

13. Why did we take a matrix commutation relation as our second binary operation for the algebra for our $SO(3)$ Lie group, rather than the simpler alternative of matrix multiplication? (Hint examine closure.)

14. Obtain the θ_i' values for (2-51) up to third order.

15. Show that M of (2-56) obeys the group closure property under the group operation of matrix multiplication.

16. Show that $X_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, X_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$, and $X_3=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ generate the three parameter $SU(2)$ group. (Hint: Use (2-61), along with (2-56), (2-59), and the derivative of (2-59) to get M, and prove that all elements shown in the text in that expansion can be obtained with the generators and the identity matrix. Then, presume that all other elements not shown can be deduced in a similar way, with similar results.) Then sum up the second order terms in the expansion to see if it gives you, to second order, the group matrix (2-56).

17. For $SU(2)$ with $a_{Im}, b_{Re}, b_{Im} << 1$, show $e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)}\approx I+ib_{Im}X_1+ib_{Re}X_2+ia_{Im}X_3\approx M$

18. Show that $M\left(\theta_1,\theta_2,\theta_3\right)=\begin{bmatrix}\cos\theta_1 e^{i\theta_2} & \sin\theta_1 e^{i\theta_3}\\ -\sin\theta_1 e^{-i\theta_3} & \cos\theta_1 e^{-i\theta_2}\end{bmatrix}$ is an $SU(2)$ Lie group.

19. Show that $\lambda_\imath/2$ (in (2-83) are the generators of the matrix N of (2-79). (Hint: Expand N in terms of α_i. Comparing with (2-61) to (2-63) and (2-76) may help.)

20. Show that $[\lambda_1/2, \lambda_2/2] = i\lambda_3/2$ and that $[\lambda_6/2, \lambda_7/2] = i\frac{\sqrt{3}}{2}\lambda_8 - i\frac{1}{2}\lambda_3$. Then express your results in terms of $\hat{F}_i=\frac{1}{2}\lambda_i$.

Chapter 2 Problem Solutions

Problem 1. Give your own examples of a group, field, vector space, and algebra.

Note: You will probably think of some examples different from those below. There are many correct answers to this problem. Only some, of many, possibilities are shown.

Ans. (first part, group).

Rotations in 3D, 4D, or any dimensional Euclidean space. Complex numbers under addition.

Ans. (second part, field).

Imaginary numbers under addition and multiplication. Rational real numbers under addition and multiplication. (This is a subfield of the field of all real numbers under addition and multiplication.)

Ans. (third part, vector space).

4D vectors in special relativity. Fock space in QFT.

Ans. (fourth part, algebra).

4D vectors under vector addition, vector cross product (defined via 4D Levi-Civita symbol), and scalar multiplication.

Problem 2. Is a field a commutative group plus a second binary operator? **Ans.** Yes.

Is a vector space a commutative group plus a scalar operation with a scalar field? **Ans.** Yes.

Is an algebra a vector space plus a second binary operation? **Ans.** Yes.

Does a field plus a scalar operation with a scalar field comprise an algebra? **Ans.** Yes.

Is the algebra unital? **Ans.** Yes.

Problem 3. Show that 3D spatial vectors form a vector space. Then show that QM states do as well. What do we call the space of QM states?

Ans. (first part).

3D spatial vectors have vector addition (binary operation), $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where $\mathbf{c}$ is a vector in 3D space (closure). They may be scalar multiplied, $3\mathbf{a} = \mathbf{d}$, where again $\mathbf{d}$ is a vector in 3D (closure).

Both operations are associative, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$ and $2{\bullet}(3{\bullet}4) = (2{\bullet}3){\bullet}4$.

The identity for vectors is zero. $\mathbf{a} + 0 = \mathbf{a}$. For scalars, it is unity, $2{\bullet}1 = 1{\bullet}2 = 2$.

The inverse for any vector $\mathbf{a}$ is $-\mathbf{a}$, $\mathbf{a} + (-\mathbf{a}) = 0$; for any scalar x, $1/x$, $x{\bullet}(1/x) = 1$.

The binary operation is commutative, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$; as is scalar multiplication $xy = yx$.

Thus, 3D spatial vectors form a vector space. Note they also have an inner product, which can be feature of a vector space, but is not necessary.

Ans. (second part). We represent states in QM with the ket notation,

States have state/vector addition (binary operation), $|\mathbf{a}\rangle + |\mathbf{b}\rangle = |\mathbf{c}\rangle$, where $|\mathbf{c}\rangle$ is a state (closure). (We assume suitable normalization.) They may be scalar multiplied, $x|\mathbf{a}\rangle = |\mathbf{d}\rangle$, where again $|\mathbf{d}\rangle$ is a state (closure).

Both operations are associative, $|\mathbf{a}\rangle + (|\mathbf{b}\rangle + |\mathbf{c}\rangle) = (|\mathbf{a}\rangle + |\mathbf{b}\rangle) + |\mathbf{c}\rangle$ and $2{\bullet}(3{\bullet}4) = (2{\bullet}3){\bullet}4$.

The identity for vectors is zero. $|\mathbf{a}\rangle + 0 = |\mathbf{a}\rangle$. For scalars, it is unity, $2{\bullet}1 = 1{\bullet}2 = 2$.

The inverse for any vector $|\mathbf{a}\rangle$ is $-|\mathbf{a}\rangle$, $|\mathbf{a}\rangle + (-|\mathbf{a}\rangle) = 0$; for any scalar x, $1/x$, $x{\bullet}(1/x) = 1$.

The binary operation is commutative, $|\mathbf{a}\rangle + |\mathbf{b}\rangle = |\mathbf{b}\rangle + |\mathbf{a}\rangle$; as is scalar multiplication $xy = yx$.

Thus, QM states form a vector space. Note they also have an inner product, which can be a feature of a vector space, but is not necessary. In QM, we typically normalize the state vectors via this inner product such that the integral $\langle\mathbf{a}|\mathbf{a}\rangle$ over all space is one.

Ans. (third part). Hilbert space.

Problem 4. With a simple sign change to (2-6) XXX be sure of eq num on final print XXX, find an orthogonal group $O(2)$ that is not special orthogonal, i.e., not $SO(2)$. Show graphically what the sign change means for the operation on a vector. Does this graph help in understanding why special orthogonal transformations are more likely to represent the kinds of phenomena we see in nature? Explain your answer. For the operation $\mathbf{Av} = \mathbf{v}'$, from matrix theory, we know $|\mathbf{v}'| = |DetA||\mathbf{v}|$. Does this latter relation make sense for your graphical depiction. Explain.

Ans (first part). Consider the negative of (2-6), i.e.

$$N(\theta) = -M(\theta) = -\begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}$$

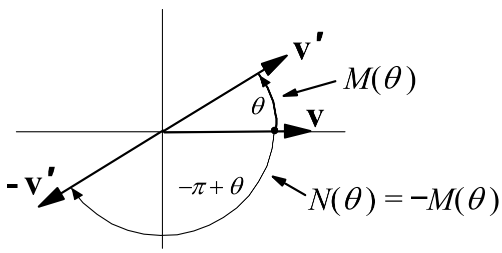
Det N is obviously – 1, since Det M = 1, so the group is not special.

$$N^T N = (-1)\begin{bmatrix} cos\,\theta & sin\,\theta \\ -sin\,\theta & cos\,\theta \end{bmatrix}(-1)\begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}$$

$$= \begin{bmatrix} cos^2\,\theta + sin^2\,\theta & -cos\,\theta\, sin\,\theta + sin\,\theta\, cos\,\theta \\ -sin\,\theta\, cos\,\theta + cos\,\theta\, sin\,\theta & sin^2\,\theta + cos^2\,\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

So, $N^T = N^{-1}$ and the group N is orthogonal (but not special).

Ans (second part).



Ans (third part). Yes. We most commonly deal with rotations through an angle θ, and almost never with rotation through some specific other function of θ, like $-\pi + \theta$. Pinning down the orthogonal transformation to a special orthogonal (in matrix representation, with determinant = +1) constrains the rotation transformation to a simple rotation through θ.

Ans (fourth part). Yes. We know the determinants of both M and N have magnitude 1. And we know the magnitude of $\mathbf{v}$ remains unchanged under the action of either M or N. So, $|\mathbf{v}'| = |\text{Det } A\,||\mathbf{v}|$ here means $|\mathbf{v}'| = |\mathbf{v}|$.

Problem 5. Write down a 2D matrix that cannot be expressed using (2-6) XXX be sure of eq num on final print XXX and a 3D matrix than cannot be expressed using (2-12) and (2-13) XXX be sure of eq num on final print XXX.

There are, of course, many answers to this question. We give two below.

Ans. (1st part)

The 2D matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ cannot be expressed, for any } \theta \text{, via } M(\theta) = \begin{bmatrix} cos\,\theta & -sin\,\theta \\ sin\,\theta & cos\,\theta \end{bmatrix}.$$

Ans. (2nd part)

The 3D matrix

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ cannot be expressed, for any } \theta_1,\ \theta_2, \text{ or } \theta_3 \text{ via } A(\theta_1,\theta_2,\theta_3) = A_1(\theta_1)A_2(\theta_2)A_3(\theta_3), \text{ where}$$

$$A_1(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\,\theta_1 & -sin\,\theta_1 \\ 0 & sin\,\theta_1 & cos\,\theta_1 \end{bmatrix} \quad A_2(\theta_2) = \begin{bmatrix} cos\,\theta_2 & 0 & -sin\,\theta_2 \\ 0 & 1 & 0 \\ sin\,\theta_2 & 0 & cos\,\theta_2 \end{bmatrix} \quad A_3(\theta_3) = \begin{bmatrix} cos\,\theta_3 & -sin\,\theta_3 & 0 \\ sin\,\theta_3 & cos\,\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Problem 6. Show that U of (2-16) XXX be sure of eq num on final print XXX forms a group under matrix multiplication. U is

$$U(\theta)=\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$

Ans. Any $U(\theta_1)U(\theta_2) = U(\theta_1 + \theta_2)$ is contained in the set, so there is closure. Matrix multiplication is always associative. The identity element is $U(0)$. The inverse for any $U(\theta)$ is in the set and is equal to $U(-\theta) = U^{\dagger}(\theta)$.

$$U(\theta)U^{-1}(\theta)=U(\theta)U(-\theta)=U(\theta)U^{\dagger}(\theta)=\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 7. Show that for U of (2-16) XXX be sure of eq numXXX, $U^{\dagger}U = I$ and Det U =1. Then examine the group $-U$ and show that its determinant is not 1 (but – 1.)

Ans. (first part). $U^{-1}(\theta) = U(-\theta) = U^{\dagger}(\theta)$. So, from Problem 6, $U^{\dagger}(\theta)\, U(\theta) = 1$.

Ans. (second part).

$$Det\left(-U(\theta)\right)=-e^{i\theta}e^{-i\theta}=-1.$$

Problem 8. Show that any unitary operation U (not just (2-16) XXX be sure of eq numXXX) operating on a vector (which could be a quantum mechanical state) leaves the magnitude of the vector unchanged.

Ans. Label the original vector v (of any dimension) and the transformed vector (under the unitary operation U) v'. So, $v' = Uv$. Then the magnitude of v' is the positive square root of

$$|v'|^2 = v'^{\dagger}v' = \left(v^{\dagger}U^{\dagger}\right)(Uv) = v^{\dagger}U^{\dagger}Uv = v^{\dagger}U^{-1}Uv = v^{\dagger}v = |v|^2.$$

The property of unitary transformations, $U^{\dagger}U = I$, means the vector magnitude is invariant under U. If the vector happens to be a normalized QM state, this means the total probability remains unity under the action of the transformation U.

Problem 9. Is $e^{i\theta}$ a way of describing via a unitary group $U(\theta)$ acting on a complex number the same thing as the $SO(2)$ group XXX (2-6) XXX acting on a 2D real vector? Explain your answer mathematically. (Hint: Express the components of a 2D real vector as the real and imaginary parts of complex scalar. Then, compare the effect of $e^{i\theta}$ on that complex scalar to the effect of XXX (2-6) XXX on the 2D real vector.)

Note $U(\theta)$ and $SO(2)$ are *different groups*. We do *not* say that $U(\theta)$ here is a representation of $SO(2)$. The two different kinds of groups can describe the same physical world phenomenon, but they are not different representations (of a particular group), as the term "representation" is employed in group theory)

Ans. Consider

$$v_i=\begin{bmatrix} a \\ b \end{bmatrix} \quad \text{and} \quad v = a+ib\text{, with}$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} a' \\ b' \end{bmatrix}=\begin{bmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{bmatrix}$$

where the vector has been rotated through θ in the 2D plane, and

$$e^{i\theta}(a+ib)=a'+ib'=(\cos\theta+i\sin\theta)(a+ib)=(a\cos\theta-b\sin\theta)+i(a\sin\theta+b\cos\theta)$$

where the complex number has been rotated through θ in the complex plane. a' and b' are the same in both.

The act of rotation in a plane is described mathematically by two different types of groups, one is an $SO(2)$ group; the other, a unitary group $U(1)$. The $U(1)$ formulation is *not* a *different representation* of the $SO(2)$ formulation, but a completely different type of group (which happens here to characterize the same thing in the physical world.)

Problem 10. Show XXX check eq num (2-27) XXX in terms of matrices.

Ans.

$$\xrightarrow[\text{form}]{\text{abstract}} \quad \mathbf{Cv} = (\mathbf{A}\otimes\mathbf{B})\mathbf{v} = (\mathbf{A}\otimes\mathbf{B})(\mathbf{w}\otimes\mathbf{y}) = \mathbf{Aw}\otimes\mathbf{By} = (\mathbf{w}''\otimes\mathbf{y}''') = \mathbf{v}'$$

$$\xrightarrow[\text{notation}]{\text{in matrix}} \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\right)\left(\begin{bmatrix} w_1 \\ w_2 \\ w_3\end{bmatrix}\otimes\begin{bmatrix} y_1 \\ y_2\end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3\end{bmatrix} \otimes \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix} y_1 \\ y_2\end{bmatrix}$$

$$= \begin{bmatrix} w_1'' \\ w_2'' \\ w_3''\end{bmatrix} \otimes \begin{bmatrix} y_1''' \\ y_2'''\end{bmatrix} = \begin{bmatrix} w_1''y_1''' & w_1''y_2''' \\ w_2''y_1''' & w_2''y_2''' \\ w_3''y_1''' & w_3''y_2'''\end{bmatrix} = \begin{bmatrix} v_{11}' & v_{12}' \\ v_{21}' & v_{22}' \\ v_{31}' & v_{32}'\end{bmatrix} \xrightarrow[\text{notation}]{\text{in abstract}} = \mathbf{v}' \; .$$

Note that $\mathbf{w}\otimes\mathbf{y}$ in a matrix representation is often written as a column vector times a row vector, as we did in XXX (2-26). But that would have been confusing above, since **A** operates on **y** in a matrix representation with **y** as a column vector. The best way to keep things straight is to use index notation, as in in XXX (2-27).

Problem 11. Show that $X = \begin{bmatrix} 0 & i \\ -i & 0\end{bmatrix}$ along with a scalar field multiplier θ comprise a Lie algebra. Use Wholeness Chart 2-1 as an aid. Note that every θX is in the set of elements comprising the algebra, and the operations are matrix addition and matrix commutation. This is considered a trivial Lie algebra. Why do you think it is?

Ans. (first part)

An algebra has two binary operations (matrix addition and commutation of two matrices here) and one scalar multiplication operation. We need to check that these operations satisfy the requirements for an algebra of Wholeness Chart 2-1.

First, look at the first binary operation of matrix addition with scalar field multiplication.

Closure: Every possible matrix addition between elements in the set results in an element (matrix) in the same set.

For different θ_i, $\theta_1 X + \theta_2 X = \theta_3 X$ is a member of the set.

Associative: $\theta_1 X + (\theta_2 X + \theta_3 X) = (\theta_1 X + \theta_2 X) + \theta_3 X$. Matrix addition is associative.

Identity: For $\theta' = 0$, $\theta X + \theta' X = \theta X$, so $\theta' X = [0]_{2X2}$ is the identity element for matrix addition.

Inverse: Each element θX of the set has an inverse $-\theta X$, since $\theta X - \theta X = \theta' X = [0]_{2X2}$ (the identity element).

Commutation: $\theta_1 X + \theta_2 X = \theta_2 X + \theta_1 X$. Matrix addition is commutative.

<u>Conclusion:</u> Under addition and scalar multiplication by θ, the set X comprises a vector space, and satisfies the requirements for one of the operations of an algebra.

Second, look at the second binary operation of matrix commutation.

Closure: $[\theta_1 X, \theta_2 X] = \theta_1\theta_2\,[X, X] = \theta_1\theta_2\,[0]_{2X2} = \theta' X =$ element in the set (where $\theta' = 0$).

Third, look at both binary operations together.

Distributive: From the general relation $\mathbf{A}\circ(\mathbf{B} \;✢\; \mathbf{C}) = \mathbf{A}\circ\mathbf{B} \;✢\; \mathbf{A}\circ\mathbf{C} \;\rightarrow\; [A,B+C] = [A,B] + [A,C]$

$$[A,B+C] = [\theta_1 X, \theta_2 X + \theta_3 X] = \theta_1 X(\theta_2 X + \theta_3 X) - (\theta_2 X + \theta_3 X)\,\theta_1 X$$

$$= \theta_1 X\,\theta_2 X + \theta_1 X\,\theta_3 X - \theta_2 X\theta_1 X - \theta_3 X\theta_1 X = [\theta_1 X, \theta_2 X] + [\theta_1 X, \theta_3 X] = [A,B] + [A,C]$$

<u>Conclusion:</u> The set θX under matrix addition, matrix commutation, and scalar field multiplication is an algebra. It is a Lie algebra because every element in the set is a smooth, continuous function of the smooth, continuous θ.

Further, regarding the 2nd binary operation, one sees from the analysis below that this particular algebra is associative, non-unital, and commutative.

Associative: $[\theta_1 X,[\theta_2 X,\theta_3 X]] = \theta_1\theta_2\theta_3\,[X,[X,X]] = 0 = [[\theta_1 X,\theta_2 X],\theta_3 X]$. 2nd operation is associative in this case.

Identity: An element A would be the identity element under commutation, if and only if, $[A, X] = X$. But since every $A = \theta X$, we must have $[A, X] = \theta[X, X] = 0$. So, no identity element exists, and this algebra is non-unital.

Inverse: If there is no identity element, there is no meaning for an inverse. (The algebra is non-unital.)

Commutation (Is the binary operation of matrix commutation itself commutative?): $[\theta_1 X, \theta_2 X] - [\theta_2 X, \theta_1 X] = 0$?

$[\theta_1 X, \theta_2 X] - [\theta_2 X, \theta_1 X] = \theta_1\theta_2 XX - \theta_2\theta_1 XX - \theta_2\theta_1 XX + \theta_1\theta_2 XX = 0$. The 2nd operation is commutative.

Ans. (second part)

It is trivial because there is only a single matrix X from which all elements of the Lie algebra are formed.

Problem 12. Show there is no identity element for the 2nd operation XXX check eq number (2-45) in the $SO(3)$ Lie algebra. (Hint: The identity element has to work for every element in the set, so you only have to show there is no identity for a single element of your choice.)

Ans. Let's choose X_3. Under the 2nd operation, we need to show there is no element I, such that $\mathbf{I}\circ\mathbf{X}_3 = \mathbf{X}_3$, or more specifically for our 2nd operation,

$$-i\left[I, X_3\right] = X_3 . \tag{A}$$

There are at least two ways to prove this. The quickest is to realize I must be some linear combination of X_1, X_2, and X_3. For X_3, the LHS of equation (A) equals zero, which does not equal the RHS value of X_3. For X_1, the LHS via XXX check eq num (2-44) is proportional to X_2, and thus is not X_3. For X_2, the LHS is proportional to X_1, and thus is not X_3. So, there is no I that satisfies (A).

The second way is to write out the matrix components of I as the unknowns I_{mn}, then carry out the matrix multiplication on the LHS of (A) with the known components of X_3 and set the result equal to X_3. If you do this, you should find two relations

$$-I_{11} + I_{22} = i \qquad \text{and} \qquad -I_{11} + I_{22} = -i \quad . \tag{B}$$

(B) has no solution, so there is no I that solves (A).

Problem 13. Why did we take a matrix commutation relation as our second binary operation for the algebra for our $SO(3)$ Lie group, rather than the simpler alternative of matrix multiplication? (Hint examine closure.)

Ans. We showed on pg. XXX check pg 18 that the second binary operation XXX (2-45) led to closure. Consider if instead of that operation, we used simple matrix multiplication of elements in the set. We only have to examine two particular elements, as we shall see below.

From XXX (2-42),

$$X_1 X_2 = i^2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is not in the set. It cannot be formed by a linear combination of X_1, X_2, and X_3.

Hence, matrix multiplication would not give us closure, a requirement for both operations in an algebra.

Problem 14. Obtain the θ_i' values for XXX (2-50) XXX up to third order.

Ans. From the first and last parts of XXX (2-50) XXX

$$\begin{aligned}
i\theta_1' X_1 &= i\theta_1 X_1 - i\tfrac{1}{12}\theta_1\theta_2^2 X_1 + \dots &\rightarrow\quad& \theta_1' \approx \theta_1 - \tfrac{1}{12}\theta_1\theta_2^2 \\
i\theta_2' X_2 &= i\theta_2 X_2 - i\tfrac{1}{12}\theta_1^2\theta_2 X_2 + \dots &\rightarrow\quad& \theta_2' \approx \theta_2 - \tfrac{1}{12}\theta_1^2\theta_2 \\
i\theta_3' X_3 &= \;\;0 \;\; - i\tfrac{1}{2}\theta_1\theta_2 X_3 + \dots &\rightarrow\quad& \theta_3' \approx -\tfrac{1}{2}\theta_1\theta_2 \;.
\end{aligned}$$

Problem 15. Show that M of XXX (2-55) obeys the group closure property under the group operation of matrix multiplication.

Ans. For two group elements

$$\underset{\sim}{M} = \begin{bmatrix} f & g \\ -g^* & f^* \end{bmatrix} \qquad \hat{M} = \begin{bmatrix} c & d \\ -d^* & c^* \end{bmatrix},$$

$$\underset{\sim}{M}\hat{M} = \begin{bmatrix} f & g \\ -g^* & f^* \end{bmatrix}\begin{bmatrix} c & d \\ -d^* & c^* \end{bmatrix} = \begin{bmatrix} fc - gd^* & fd + gc^* \\ -g^* c - f^* d^* & f^* c^* - g^* d \end{bmatrix} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ -\tilde{b}^* & \tilde{a}^* \end{bmatrix} = \tilde{M}.$$

$\tilde{M}$ is in the group defined by XXX (2-55), provided $\tilde{a}\tilde{a}^* + \tilde{b}\tilde{b}^* = 1$. From the determinant rule for matrices $\left(Det\underset{\sim}{M}\right)\left(Det\hat{M}\right) = \left(Det\tilde{M}\right)$, $(1)(1) = Det\tilde{M} = 1$, so $\tilde{a}\tilde{a}^* + \tilde{b}\tilde{b}^* = 1$, so we have closure.

Problem 16. Show that $X_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$, and $X_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ generate the three parameter $SU(2)$ group. (Hint: Use XXX (2-60) along with XXX (2-55), XXX (2-58), and the derivatives of XXX (2-58) to get M, and prove that all elements shown in the text in that expansion can be obtained with the generators and the identity matrix. Then, presume that all other elements not shown can be deduced in a similar way, with similar results.) Then sum up the second order terms in the expansion to see if it gives you, to second order, the group matrix XXX (2-55).

Ans. (first part) From

$$M\left(a_{Im}, b_{Re}, b_{Im}\right) = \underbrace{M(0,0,0)}_{I} + a_{Im} \underbrace{\left.\frac{\partial M}{\partial a_{Im}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}}_{iX_3} + b_{Re} \underbrace{\left.\frac{\partial M}{\partial b_{Re}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}}_{iX_2} + b_{Im} \underbrace{\left.\frac{\partial M}{\partial b_{Im}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}}_{iX_1} \qquad \text{XXX (2-60)}$$

$$+ \frac{\left(a_{Im}\right)^2}{2!} \left.\frac{\partial^2 M}{\partial a_{Im}^2}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} + \frac{a_{Im} b_{Re}}{2!} \left.\frac{\partial^2 M}{\partial a_{Im}\partial b_{Re}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} + \ldots,$$

we see that the first row above can be generated with the generators X_i and the identity. We will show that the two terms in the second row can be as well. Start with

$$M = \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} = \begin{bmatrix} a_{Re} + ia_{Im} & b_{Re} + ib_{Im} \\ -b_{Re} + ib_{Im} & a_{Re} - ia_{Im} \end{bmatrix}, \qquad \text{XXX (2-55)}$$

and

$$\frac{\partial a_{Re}}{\partial a_{Im}} = -\frac{a_{Im}}{\sqrt{1 - a_{Im}^2 - b_{Re}^2 - b_{Im}^2}} = -\frac{a_{Im}}{a_{Re}}, \qquad \text{XXX (2-58)}$$

to get

$$\frac{\partial^2 a_{Re}}{\partial a_{Im}^2} = \frac{\partial}{\partial a_{Im}}\left(-\frac{a_{Im}}{a_{Re}}\right) = \frac{\partial}{\partial a_{Im}} \frac{-a_{Im}}{\sqrt{1 - a_{Im}^2 - b_{Re}^2 - b_{Im}^2}}$$

$$= \frac{-1}{\sqrt{1 - a_{Im}^2 - b_{Re}^2 - b_{Im}^2}} - \frac{1}{2}\frac{\left(-a_{Im}\right)}{\left(1 - a_{Im}^2 - b_{Re}^2 - b_{Im}^2\right)^{3/2}}\left(-2a_{Im}\right) = -\frac{1}{a_{Re}} - \frac{a_{Im}^2}{a_{Re}^3}.$$

The matrix part of the first term in the second row is

$$\left.\frac{\partial^2 M}{\partial a_{Im}^2}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = \left.\begin{bmatrix} -\frac{1}{a_{Re}} - \frac{a_{Im}^2}{a_{Re}^3} & 0 \\ 0 & -\frac{1}{a_{Re}} - \frac{a_{Im}^2}{a_{Re}^3} \end{bmatrix}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

which is the identity times negative unity. To get the matrix part of the second term in the second row of XXX (2-60), we need

$$\frac{\partial^2 a_{Re}}{\partial a_{Im}\partial b_{Re}} = \frac{\partial}{\partial b_{Re}}\left(-\frac{a_{Im}}{a_{Re}}\right) = \frac{\partial}{\partial b_{Re}}\frac{-a_{Im}}{\sqrt{1-a_{Im}^2-b_{Re}^2-b_{Im}^2}} = -\frac{1}{2}\frac{\left(-a_{Im}\right)}{\left(1-a_{Im}^2-b_{Re}^2-b_{Im}^2\right)^{3/2}}\left(-2b_{Re}\right) = -\frac{a_{Im}b_{Re}}{a_{Re}^3}.$$

Then,

$$\left.\frac{\partial^2 M}{\partial a_{Im}\partial b_{Re}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = \left.\begin{bmatrix}\frac{\partial^2 a_{Re}}{\partial a_{Im}\partial b_{Re}} & 0\\ 0 & \frac{\partial^2 a_{Re}}{\partial a_{Im}\partial b_{Re}}\end{bmatrix}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = \left.\begin{bmatrix}-\frac{a_{Im}b_{Re}}{a_{Re}^3} & 0\\ 0 & -\frac{a_{Im}b_{Re}}{a_{Re}^3}\end{bmatrix}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},$$

and there is no contribution from this term. Extrapolating these results, one can surmise that no additional matrices are needed other than the three generators and the identity matrix in order to generate any element in the $SU(2)$ group.

Ans. (second part) Summing up the matrices in XXX (2-60), including terms in b_{Re}^2 and b_{Im}^2, which parallel the a_{Im}^2 term, we have

$$\begin{aligned}M\left(a_{Im},b_{Re},b_{Im}\right) &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + ia_{Im}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} + ib_{Re}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + ib_{Im}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\ &\quad - \frac{a_{Im}^2}{2!}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \frac{b_{Re}^2}{2!}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \frac{b_{Im}^2}{2!}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \ldots,\\ &= \begin{bmatrix}\left(1-\frac{1}{2}a_{Im}^2-\frac{1}{2}b_{Re}^2-\frac{1}{2}b_{Im}^2\right)+ia_{Im} & b_{Re}+ib_{Im}\\ -b_{Re}+ib_{Im} & \left(1-\frac{1}{2}a_{Im}^2-\frac{1}{2}b_{Re}^2-\frac{1}{2}b_{Im}^2\right)-ia_{Im}\end{bmatrix} + \ldots.\end{aligned}$$

Then, from XXX (2-56),

$$a_{Re} = \sqrt{1-a_{Im}^2-b_{Re}^2-b_{Im}^2} = 1-\frac{1}{2}a_{Im}^2-\frac{1}{2}b_{Re}^2-\frac{1}{2}b_{Im}^2+\ldots,$$

we see that the series does reproduce M of XXX (2-55), at least to second order, and we can presume to all orders.

Problem 17. For $SU(2)$ with a_{Im}, b_{Re}, $b_{Im} << 1$, show $e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)} \approx I + ib_{Im}X_1 + ib_{Re}X_2 + ia_{Im}X_3 \approx M$.

Ans. From XXX (2-64), $M\left(a_{Im},b_{Re},b_{Im}\right) = I + ia_{Im}X_3 + ib_{Re}X_2 + ib_{Im}X_1 - \frac{a_{Im}^2}{2!}I - \frac{b_{Re}^2}{2!}I - \frac{b_{Im}^2}{2!}I + \frac{a_{Im}b_{Re}}{2!}[0] + \ldots$

which for $a_{Im}, b_{Re}, b_{Im} << 1$ becomes $M\left(a_{Im},b_{Re},b_{Im}\right) \approx I + ia_{Im}X_3 + ib_{Re}X_2 + ib_{Im}X_1$, equivalent to the RHS of the problem statement. We then need to show the LHS is approximately equal, in the limit, to the same expansion.

$$\begin{aligned}e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)} &= \underbrace{\left.e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}}_{I}\\ &+ a_{Im}\left.\frac{\partial e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)}}{\partial a_{Im}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} + b_{Re}\left.\frac{\partial e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)}}{\partial b_{Re}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} + b_{Im}\left.\frac{\partial e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)}}{\partial b_{Im}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}}\\ &+ \frac{\left(a_{Im}\right)^2}{2!}\left.\frac{\partial^2 e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)}}{\partial a_{Re}^2}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} + \frac{a_{Im}b_{Re}}{2!}\left.\frac{\partial^2 e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)}}{\partial a_{Im}\partial b_{Re}}\right|_{\substack{a_{Im}=b_{Re}\\=b_{Im}=0}} + \ldots,\end{aligned}$$

All the terms after the second row go to zero in the limit of small a_{Im}, b_{Re}, b_{Im}. Evaluating the derivatives, we have

$$e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)} \approx I + ia_{Im}X_3 + ib_{Re}X_2 + ib_{Im}X_1 = I + ib_{Im}X_1 + ib_{Re}X_2 + ia_{Im}X_3\,,$$

which is the relation in the problem statement. So, for small a_{Im} ,b_{Re} ,b_{Im},

$$e^{i\left(b_{Im}X_1+b_{Re}X_2+a_{Im}X_3\right)} \approx M = \begin{bmatrix} a_{Re}+ia_{Im} & b_{Re}+ib_{Im} \\ -b_{Re}+ib_{Im} & a_{Re}-ia_{Im} \end{bmatrix} \approx \begin{bmatrix} 1+ia_{Im} & b_{Re}+ib_{Im} \\ -b_{Re}+ib_{Im} & 1-ia_{Im} \end{bmatrix}.$$

Problem 18. Show that $M\left(\phi_1,\phi_2,\phi_3\right) = \begin{bmatrix} cos\,\phi_1 e^{i\phi_2} & sin\,\phi_1 e^{i\phi_3} \\ -sin\,\phi_1 e^{-i\phi_3} & cos\,\phi_1 e^{-i\phi_2} \end{bmatrix}$ is an $SU(2)$ group.

Ans. For $SU(2)$, we need to show $M^\dagger M = I$ and $Det\ M = 1$. For a Lie algebra, we need to show that $M(0,0,0) = I$, i.e., there is no change in any vector M might operate on when the parameters are all zero.

$$M^\dagger M = \begin{bmatrix} cos\,\phi_1 e^{-i\phi_2} & -sin\,\phi_1 e^{i\phi_3} \\ sin\,\phi_1 e^{-i\phi_3} & cos\,\phi_1 e^{i\phi_2} \end{bmatrix}\begin{bmatrix} cos\,\phi_1 e^{i\phi_2} & sin\,\phi_1 e^{i\phi_3} \\ -sin\,\phi_1 e^{-i\phi_3} & cos\,\phi_1 e^{-i\phi_2} \end{bmatrix}$$

$$= \begin{bmatrix} \left(cos\,\phi_1\right)^2 + \left(sin\,\phi_1\right)^2 & cos\,\phi_1 e^{-i\theta_2}\,sin\,\phi_1 e^{i\phi_3} - sin\,\phi_1 e^{i\phi_3}\,cos\,\phi_1 e^{-i\phi_2} \\ sin\,\phi_1 e^{-i\phi_3}\,cos\,\phi_1 e^{i\phi_2} - cos\,\phi_1 e^{i\phi_2}\,sin\,\theta_1 e^{-i\phi_3} & \left(sin\,\phi_1\right)^2 + \left(cos\,\phi_1\right)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$DetM = \left(cos\,\phi_1\right)^2 + \left(sin\,\phi_1\right)^2 = 1$$

$$M\left(\phi_1 = 0, \phi_2 = 0, \phi_3 = 0\right) = \begin{bmatrix} cos(0)e^0 & sin(0)e^0 \\ -sin(0)e^0 & cos(0)e^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Problem 19. Show that $\lambda_i/2$ (in XXX (2-82) are the generators of the matrix N of XXX (2-78). (Hint: Expand N in terms of α_i. Comparing with XXX (2-60) to XXX (2-62) and XXX (2-75) may help.)

Ans. For

$$N = \frac{1}{2}\begin{bmatrix} 2\alpha_0 + \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & 2\alpha_0 - \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & 2\alpha_0 - \frac{2\alpha_8}{\sqrt{3}} \end{bmatrix}$$

and

$$N\left(\alpha_i\right) = \underbrace{N\left(\alpha_i = 0\right)}_{I} + \alpha_1 \left.\frac{\partial N}{\partial \alpha_1}\right|_{\alpha_i=0} + \alpha_2 \left.\frac{\partial N}{\partial \alpha_2}\right|_{\alpha_i=0} + \alpha_3 \left.\frac{\partial N}{\partial \alpha_3}\right|_{\alpha_i=0} + \alpha_4 \left.\frac{\partial N}{\partial \alpha_4}\right|_{\alpha_i=0}$$

$$+ \alpha_5 \left.\frac{\partial N}{\partial \alpha_5}\right|_{\alpha_i=0} + \alpha_6 \left.\frac{\partial N}{\partial \alpha_6}\right|_{\alpha_i=0} + \alpha_7 \left.\frac{\partial N}{\partial \alpha_7}\right|_{\alpha_i=0} + \alpha_8 \left.\frac{\partial N}{\partial \alpha_0}\right|_{\alpha_i=0}$$

$$+ \frac{\left(\alpha_1\right)^2}{2!}\left.\frac{\partial^2 N}{\partial \alpha_1^2}\right|_{\alpha_i=0} + \frac{\alpha_1\alpha_2}{2!}\left.\frac{\partial^2 N}{\partial \alpha_1 \partial \alpha_1}\right|_{\alpha_i=0} + \ldots,$$

we get

$$\alpha_1 \frac{\partial N}{\partial \alpha_1}\bigg|_{\alpha_i=0} = \alpha_1 \begin{bmatrix} \frac{\partial \alpha_0}{\partial \alpha_1} & 0 & 0 \\ 0 & \frac{\partial \alpha_0}{\partial \alpha_1} & 0 \\ 0 & 0 & \frac{\partial \alpha_0}{\partial \alpha_1} \end{bmatrix} + \alpha_1 \frac{1}{2} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\lambda_1} \qquad \alpha_2 \frac{\partial N}{\partial \alpha_2}\bigg|_{\alpha_i=0} = \alpha_2 \begin{bmatrix} \frac{\partial \alpha_0}{\partial \alpha_2} & 0 & 0 \\ 0 & \frac{\partial \alpha_0}{\partial \alpha_2} & 0 \\ 0 & 0 & \frac{\partial \alpha_0}{\partial \alpha_2} \end{bmatrix} + \alpha_2 \frac{1}{2} \underbrace{\begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\lambda_2}$$

$$\alpha_3 \frac{\partial N}{\partial \alpha_3}\bigg|_{\alpha_i=0} = \alpha_3 \begin{bmatrix} \frac{\partial \alpha_0}{\partial \alpha_3} & 0 & 0 \\ 0 & \frac{\partial \alpha_0}{\partial \alpha_3} & 0 \\ 0 & 0 & \frac{\partial \alpha_0}{\partial \alpha_3} \end{bmatrix} + \alpha_3 \frac{1}{2} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\lambda_3} \qquad \text{etc. for other } \alpha_i \frac{\partial N}{\partial \alpha_i}\bigg|_{\alpha_i=0} \text{ terms,}$$

and

$$\frac{(\alpha_1)^2}{2!} \frac{\partial^2 N}{\partial \alpha_1^2}\bigg|_{\alpha_i=0} = \frac{(\alpha_1)^2}{2!} \begin{bmatrix} \frac{\partial^2 \alpha_0}{\partial \alpha_1^2} & 0 & 0 \\ 0 & \frac{\partial^2 \alpha_0}{\partial \alpha_1^2} & 0 \\ 0 & 0 & \frac{\partial^2 \alpha_0}{\partial \alpha_1^2} \end{bmatrix} + \alpha_1 \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{(\alpha_1)^2}{2!} \frac{\partial^2 \alpha_0}{\partial \alpha_1^2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\alpha_1 \alpha_2}{2!} \frac{\partial^2 N}{\partial \alpha_1 \partial \alpha_1}\bigg|_{\alpha_i=0} = \frac{\alpha_1 \alpha_2}{2!} \frac{\partial^2 \alpha_0}{\partial \alpha_1 \partial \alpha_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad + \text{ similar for other 2nd derivative terms.}$$

Putting all of the above together, we have

$$N = \left(1 + \alpha_i \frac{\partial \alpha_0}{\partial \alpha_i} + \frac{(\alpha_i)^2}{2!} \frac{\partial^2 \alpha_0}{\partial \alpha_i^2} + \frac{\alpha_k \alpha_l}{2!} \frac{\partial^2 \alpha_0}{\partial \alpha_k \partial \alpha_l}\bigg|_{k \neq l} + \ldots \right) I + \alpha_i \frac{\lambda_i}{2} = \alpha_0 I + \alpha_i \frac{\lambda_i}{2},$$

which is XXX (2-83). The $\lambda_i/2$ are the $SU(3)$ generators for the matrix N.

Problem 20. Show that $[\lambda_1/2, \lambda_2/2] = i\lambda_3/2$ and that $[\lambda_6/2, \lambda_7/2] = i\frac{\sqrt{3}}{2}\lambda_8 - i\frac{1}{2}\lambda_3$. Then express your results in terms of $\hat{F}_i = \frac{1}{2}\lambda_i$.

Ans. (first part).

$$\left[\frac{\lambda_1}{2},\frac{\lambda_2}{2}\right]=\underbrace{\frac{1}{2}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{\lambda_1/2}\underbrace{\frac{1}{2}\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}}_{\lambda_2/2}-\underbrace{\frac{1}{2}\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}}_{\lambda_2/2}\underbrace{\frac{1}{2}\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}}_{\lambda_1/2}=\frac{1}{4}\begin{bmatrix}i&0&0\\0&-i&0\\0&0&0\end{bmatrix}-\frac{1}{4}\begin{bmatrix}-i&0&0\\0&i&0\\0&0&0\end{bmatrix}=i\underbrace{\frac{1}{2}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}}_{\lambda_3/2}$$

Ans. (second part).

$$\left[\frac{\lambda_6}{2},\frac{\lambda_7}{2}\right]=\underbrace{\frac{1}{2}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\lambda_6/2}\underbrace{\frac{1}{2}\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix}}_{\lambda_7/2}-\underbrace{\frac{1}{2}\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix}}_{\lambda_7/2}\underbrace{\frac{1}{2}\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}}_{\lambda_6/2}=\frac{1}{4}\begin{bmatrix}0&0&0\\0&i&0\\0&0&-i\end{bmatrix}-\frac{1}{4}\begin{bmatrix}0&0&0\\0&-i&0\\0&0&i\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}0&0&0\\0&i&0\\0&0&-i\end{bmatrix}=i\frac{\sqrt{3}}{2}\underbrace{\frac{1}{2}\frac{1}{\sqrt{3}}\begin{bmatrix}1&0&0\\0&1&0\\0&0&-2\end{bmatrix}}_{\lambda_8/2}-i\frac{1}{2}\underbrace{\frac{1}{2}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&0\end{bmatrix}}_{\lambda_3/2}=i\frac{\sqrt{3}}{2}\frac{\lambda_8}{2}-i\frac{1}{2}\frac{\lambda_3}{2}.$$

Ans. (third part).

$$\left[\frac{\lambda_1}{2},\frac{\lambda_2}{2}\right]=i\frac{\lambda_3}{2}\quad\rightarrow\quad\left[\hat{F}_1,\hat{F}_2\right]=i\hat{F}_3 \qquad\qquad \left[\frac{\lambda_6}{2},\frac{\lambda_7}{2}\right]=i\frac{\sqrt{3}}{2}\frac{\lambda_8}{2}-i\frac{1}{2}\frac{\lambda_3}{2}\quad\rightarrow\quad\left[\hat{F}_6,\hat{F}_7\right]=i\frac{\sqrt{3}}{2}\hat{F}_8-i\frac{1}{2}\hat{F}_3$$