Chapter 3

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Relevant Mathematical Topics

“Young man, in mathematics, you don’t understand things. You just get used to them.”

John von Neumann

3.0 Preliminaries

I chose the quote above because I disagree so strongly with it. And I suspect, so do many of you reading this book. If concepts are presented in a clear, sincerely pedagogic way, they can be understood by reasonably capable students. And in fact, they must be understood, if one wishes to make any kind of mark in any field, like physics, whose foundation is mathematical.

Our goal in this chapter, and more so, in this entire book, is to understand. As I’ve said before, I will do my best to help you do so, and if possible, you will help me and others by suggesting ways to make it all even clearer and easier for others, who come later. (See Sect. 1.5, pg. 5.)

3.0.1 Background

Certain topics in mathematics play key roles in the path integral (PI) approach to QFT. However, these topics can also be employed in the canonical quantization approach, though they are not typically used there. This can lead, and has led, to student confusion about these topics as somehow being an inextricable part of the path integral method, and that method alone. They are not.

In this chapter, we will introduce these concepts in the context of canonical quantization, primarily for QED. In Part 2 of this book, we will use them in developing QED via path integrals. Later on, we will see them again, as they are used with the path integral development of the theories of electroweak and strong interactions.

3.0.2 Chapter Overview

In this chapter, we will introduce and develop the mathematics of

- Green functions,
- Grassmann variables, and
- the generating functional.

You have probably already learned about Green functions in prior courses, but the meaning of the term in QFT is somewhat different from that of the usual mathematics course. We will explore those differences, and in so doing, refine our understanding of these widely used functions. With the QFT form of the Green function in hand, we can deduce the transition amplitude for a given interaction. That is where its value lies.

Grassmann variables are a particular type of variable that are used as part of the process for finding relevant Green functions. In that case, they are actually Grassmann fields, and are treated as classical, rather than quantum fields. Despite being classical in nature, and not operators, they have an unusual property such fields typically do not have. They anti-commute. If you are suspecting that these fields may be related to fermions, you would be correct. More on this later.

Grassmann fields are used with something called the generating functional to determine a particular Green function. The generating functional generates the Green function by taking derivatives of it with respect to fields, some of which are Grassmann fields. That Green function is then used to find the transition amplitude.
We show all this graphically in Wholeness Chart 3-1 below.

**Wholeness Chart 3-1. Finding Transition Amplitude from the Generating Functional**

1. find generating functional (via canonical or PI approach)
2. take derivatives with respect to fields (including Grassmann fields)
3. Green function
4. transition amplitude

So, our ultimate goal in this chapter will be to find the generating functional (via canonical quantization) for any particular interaction. From it, we can obtain the transition amplitude.

In Part 2 of this book we will find the generating functional via the path integral approach. From there, we will show this yields the same Green function and thus, the same transition amplitude.

There is a wholeness chart summary of the material in this chapter beginning herein on page 81 I recommend following that, step-by-step, as you proceed through the chapter.

### 3.1 Green Functions

#### 3.1.1 Usual Green Function Definition/Derivation from Mathematics

The usual Green function technique of most mathematics texts entails solving an equation of form (3-1), where \( u(x) \) is the solution, \( L \) is an operator, and \( f(x) \) is a known function.

\[
Lu(x) = f(x)
\]  
(3-1)

The Green function for this equation is \( G(x,s) \), which solves

\[
LG(x,s) = \delta(x-s).
\]  
(3-2)

The solution \( u(x) \) to (3-1) is then (see proof below)

\[
u(x) = \int G(x,s)f(s)ds.
\]  
(3-3)

**Proof**

Putting (3-3) into (3-1), then using (3-2), we have

\[
Lu(x) = L\int G(x,s)f(s)ds = \int LG(x,s)f(s)ds = \int \delta(x-s)f(s)ds = f(x).
\]  
(3-4)

End of proof

#### 3.1.2 Usual Math Definition of Green Function for QFT Field Equations

The Klein-Gordon (Scalar) Equation Green Function

\[
\frac{\partial_c \phi^a + m^2}{\partial_c \phi^a + m^2} = 0
\]  
(3-5)

The Green function for (3-5) is found from (3-6) (where a minus sign on RHS doesn’t change the essence of the issue and is conventional in QFT, we use \( y \) in place of the \( s \) of (3-2), and \( x \) and \( y \) here represent 4D spacetime positions).

\[
\left(\frac{\partial_c \phi^a + m^2}{\partial_c \phi^a + m^2} \right)G(x,y) = -\delta(x-y)
\]  
(3-6)

To solve (3-6), convert it to momentum space via a Fourier transform\(^1\) over \( x \).

\[
\left(\frac{-k^2 + m^2}{\hat{G}(k,y)} = -e^{iky}.
\]  
(3-7)

\(^1\) Our convention is the one typically used in QFT. \( \hat{f}(k) = \int d^4x f(x)e^{ikx} \) where \( kx = k^\mu x_\mu \) and \( f(x) = (2\pi)^{-4}\int d^4k \hat{f}(k)e^{-ikx}. \) Note also that from Fourier transform tables (or taking the time to prove it yourself) \( \hat{\partial_\mu f(x)} = -ik_\mu \hat{f}(k). \)
The solution to (3-7) is

$$G(k,y) = \frac{e^{iky}}{k^2 - m^2}.$$  \hspace{1cm} (3-8)

Converting (3-8) back to position space via a reverse Fourier transform, we get

$$G(x-y) = \frac{1}{(2\pi)^4} \int d^4k \frac{e^{-i(kx-y)}}{k^2 - m^2 \pm i\epsilon} = \Delta_F(x-y),$$  \hspace{1cm} (3-9)

which, when multiplied by $i$, is just the scalar Feynman propagator. (Note that we add the infinitesimal quantity $i\epsilon$ to the denominator of the integrand in (3-9) for reasons delineated in Vol. 1, pg. 76. Essentially, the integrand would otherwise blow up at $|k| = m$. After doing the work one needs to do with (3-9) [effectively integrating it], the $\epsilon$ can be taken to zero [and we actually get a suitable result].)

**The Spin 1 Green Function**

In analogy to the Klein-Gordon (scalar) case, instead of (3-5), we have the spin 1 field equation, known as the Proca equation, which for $m = 0$ becomes the Maxwell equation for photons.

$$\left(\partial_\alpha \partial^\alpha + m^2\right)A^\mu = 0.$$  \hspace{1cm} (3-10)

Do **Problem 1** to show the Proca equation Green function is (3-11).

$$G^{\mu\nu}(x-y) = \frac{g^{\mu\nu}}{(2\pi)^4} \int d^4k \frac{e^{-ik(x-y)}}{k^2 - m^2 \pm i\epsilon} = \frac{1}{4\pi^2} \left[ \frac{1}{p^2 - m^2 + i\epsilon} \right] d^4p = D_F^{\mu\nu}(x-y),$$  \hspace{1cm} (3-11)

In the massless case, (3-11) [times $i$] reduces to the familiar photon Feynman propagator.

**The Spin $\frac{1}{2}$ Green Function**

Similarly, the Green function for the Dirac equation,

$$\left( i\gamma^\alpha \partial_\alpha - m \right) \psi = 0,$$  \hspace{1cm} (3-12)

is found, after much manipulation of gamma matrices, [and after multiplying by $i$] to be what we have known as the spinor propagator,

$$G_S(x-y) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} \frac{e^{-ip(x-y)}(p^2 + m)}{p^2 - m^2 + i\epsilon} d^4p = S_F(x-y).$$  \hspace{1cm} (3-13)

We will not take the time to digress and deduce (3-13), as we have other fish to fry. Hopefully, you can just accept it without proof, given the parallels with the spin 0 and spin 1 cases.

**General Conclusion for Green Functions of Usual Mathematics and QFT Field Equations**

The Green function (defined as negative of the usual math definition and multiplied by $i$) for each particle type field equation equals the Feynman propagator. So, we can see why propagators are sometimes called Green functions.

**3.1.3 Simplified Overview of Green Function Methodology of QFT**

**A Somewhat Different Approach to Green Functions**

When students first see the more extended use of Green functions in QFT, which goes well beyond merely finding propagators, there is typically confusion, as the methodology therein seems quite different from the usual math Green functions, as described in the foregoing sections. So, we will consider two different approaches to Green functions, the usual math one (see above) and, what we will term the “QFT Green function methodology” (see below). After introducing the second of these, we will compare the two and see how, in some circumstances, they give the same result, i.e., the same Green functions. (The QFT Green function methodology results are more extensive, including not only the usual math Green function results, but more.)
Brief Overview of QFT Green Function Methodology: In Words

The QFT Green function methodology approach provides a final result, the Green function for a particular interaction, that can be readily transformed into the Feynman amplitude for that interaction. The transformation involves straightforward substitution, as explained in words in the next paragraph and in pictures in the section after this one.

As it turns out, the QFT methodology yields a Green function which is comprised of propagators (for example, \( S_F(p) \)), and only propagators. That is, there are no external line relations (for example, \( u_1(p) \)), unlike Feynman amplitudes. One obtains the appropriate Feynman amplitude simply by substituting an external line expression for each incoming/outgoing particle (which is a propagator in a Green function) in the particular interaction of interest (for example, substitute \( u_1(p) \) for \( S_F(p) \)).

There are other subtleties, such as getting the signs right on the 4-momenta, but that is the technique in its essence. Thereby, one turns a Green function into a Feynman amplitude.

Brief Overview of QFT Methodology Green Functions: In Pictures

The Green function is a mathematical entity, just as the transition amplitude is a mathematical entity. And just as transition amplitudes can be represented pictorially by Feynman diagrams, so can Green functions be represented by diagrams. Fig. 3-1 displays the relationship between a typical diagram associated with a Green function and the corresponding Feynman diagram.

![Diagram associated with Green function](image1)

![Diagram associated with QFT amplitude](image2)

**Figure 3-1. Changing a Green Function Feynman Diagram to External Particles Feynman Diagram**

(for one type of Bhabha Scattering)

Note from the LHS of Fig. 3-1 that, as mentioned in the previous section, the Green function, and pictorially, its associated Feynman diagrams, are made up wholly of propagators. In the LHS, the incoming electron and positron of Bhabha scattering (only lowest order and only one of two ways it can occur are shown) are represented by propagators. The outgoing electron and positron on the RHS of Fig. 3-1 are represented in the LHS by propagators, as well. Small dots on the end of the lines in diagrams indicate the lines represent propagators, rather than external particles.

In Green function lingo, the propagators that become external particles are called legs. A Green function with \( n \) legs is called an \( n \)-point Green function. For example, the Green function corresponding to the diagram on the LHS of Fig. 3-1 is a four-point Green function.

To convert the diagram (and mathematically, the Green function) to an external particle diagram (and mathematically, to a transition amplitude), we change each leg to an external particle. Note carefully, though, that by convention all 4-momenta in a Green function are defined as pointed inward in the Green function diagram. To convert an incoming Green function propagator to an outgoing Feynman amplitude particle, we need to switch the sign on the 4-momentum.

Generally, we can conclude that both the Green function and the transition amplitude carry the same information. Either can be readily obtained from the other.

Fig. 3-2 shows us what, given the foregoing paragraphs, should not be a big surprise. That is, a Green function is comprised of many (an infinite number including higher order) terms (in pictorial form, diagrams), just as we once learned a transition amplitude is composed of many (an infinite number) of higher order terms (in pictorial form, diagrams).

All of the Feynman-like diagrams of Fig. 3-2 can typically be combined into one diagram, using a shaded circle to represent all possible internal propagator configurations of a connected diagram for the same given legs (external particles in an external particle transition amplitude). Three of these are shown in Fig. 3-3, where the topmost example represents Fig. 3-2. The second example in Fig. 3-3 is
just a dressed photon propagator, i.e., the bare propagator together with all its self-energy modifications. The third diagram is the dressed vertex.

![Diagrams for one Green function](image)

**Figure 3-2. Schematic of the Green Function Corresponding to Bhabha Scattering**

= Bhabha scattering of Fig. 3-2

= + + ...

= + + ...

**Figure 3-3. Symbolism for Combining Sub-diagrams into One Diagram**

The shaded circle symbols are also used in external particle Feynman diagrams (with external particles rather than propagator legs) to represent the sum of all connected sub-diagrams with the same external particles.

**So, What Good Is It?**

One may ask what value the Green function methodology has, as we already know how to construct amplitudes for any given interaction (at least for QED, at this point). In certain contexts, it has several key advantages.

Green functions may be less directly relatable to observable quantities than transition amplitudes, but they are easier to calculate, and the latter are readily obtained from them. Additionally, as we will see, transition amplitudes for several different interactions can be derived from a single Green function, allowing us to more readily discern relationships between these interactions.

And as we will also eventually see, QCD shares certain similarities with QED, but the more complicated form of $SU(3)$ compared to $U(1)$ makes strong force field theory far more challenging. It turns out that development of QCT using Green functions (in tandem with the PI approach) is considerably easier than it otherwise would be without them.

Additionally, advanced topics in QFT that require going beyond low-order perturbation solution methods (such as found in QCD) are, almost without exception, formulated in terms of Green functions.

Beyond that, they teach us, once again, that the physical, phenomenal world can be represented in different, yet consonant, mathematical ways.
3.1.4 The Math Behind the Green Function Methodology of QFT

Mathematical Form of the Green Function

In this sub-section, we first simply present, in (3-14), the defining expression for Green functions in the QFT methodology, written \( G^{\mu\nu\ldots} \). We then explore that relation and present examples, with different legs (incoming/outgoing particles in the associated Feynman diagrams), in order to show how Green functions as discussed in prior sections arise from (3-14).

\[
G^{\mu\nu\ldots}(x_1,x_2,...,y_1,...,z_1,...) = \frac{\langle 0 \left| T \left[ S A^\mu(x_1) A^\nu(x_2) \ldots \psi(y_2) \ldots \overline{\psi}(z_3) \ldots \right] \right| 0 \rangle}{\langle 0 \left| S \right| 0 \rangle}
\]  
(3-14)

\( S \) is the familiar \( S \) operator (not the action, which often uses the same symbol) from the canonical quantization approach to QFT; \( A^\mu, \psi, \) and \( \overline{\psi} \) are the usual interaction picture QED quantum (operator) fields from that same approach; and \( T \) indicates time ordering. Kets and bras are interaction picture states. (3-14) is the position space form of the Green function, which has a concomitant momentum-space form obtained via the usual Fourier transform methodology.

Take care to note that there is one field in (3-14) for each incoming/outgoing particle (each leg) in an interaction, and nothing (not counting \( S \)) for other particles (internal lines in a Feynman diagram).

For now, we are not going to worry about the denominator in (3-14), but focus on the numerator. Shortly, we will show how the denominator arises mathematically and ends up being little more than a phase factor. The critical part of the Green function is the numerator.

The \( S \) operator (see Vol. 1, pg. 213c with infinite integration range) is (where we underline dummy integration variables to distinguish them from the \( x_1, x_2, \ldots \) in the Green function (3-14))

\[
S = Te^{-i\int \mathcal{L}_I(x)d^4x} = Te^{i\int \mathcal{L}_I(x)d^4x} = \sum_{n=0}^{\infty} i^n \frac{1}{n!} \int T \left\{ \mathcal{L}_I(x_1) \ldots \mathcal{L}_I(x_n) \right\} d^4x_1 \ldots d^4x_n
\]  
(3-15)

For QED, this becomes

\[
S = I + ie \int \left( \overline{\psi} \gamma^\alpha A_\alpha \psi \right) d^4x_1 - \frac{e^2}{2} \int T \left\{ \left( \overline{\psi} \gamma^\alpha A_\alpha \psi \right) \left( \overline{\psi} \gamma^\beta A_\beta \psi \right) \right\} d^4x_1 d^4x_2 + \ldots
\]  
(3-16)

We can re-express (3-14) by inserting (3-16) into it, to get

\[
G^{\mu\nu\ldots}(x_1,x_2,...,y_1,...,z_1,...) = \frac{\langle 0 \left| T \left[ A^\mu(x_1) A^\nu(x_2) \ldots \psi(y_2) \ldots \overline{\psi}(z_3) \ldots \right] \right| 0 \rangle}{\langle 0 \left| S \right| 0 \rangle}
\]  
(3-17)

How Only Propagators Remain in Any Green Function

Recall from Wick’s theorem (3-18) (Vol. 1, pg.204) how we can convert the time ordering of (3-17) into normal ordering, and along the way we get terms with propagators in them. (A,B,C etc. are fields such as \( A^\mu, \psi, \) and \( \overline{\psi}. \)
Using (3-18) in (3-17) we will end up with, thanks to normal ordering, a lot of terms having destruction operators on the RHS. Any of those operating on the vacuum ket \( |0\rangle \) in the numerator of (3-17) will result in zero. We will also get terms with all creation operators and each of those will produce a ket state that does not match (i.e. is orthogonal to) the vacuum state bra.

So, the only terms surviving are those with only propagators (i.e., contractions) in them and no operators, since propagators are only numbers and not operators. These terms each have a number sandwiched between a vacuum bra and a vacuum ket, so we can take the number outside the bracket, and the bracket \( \langle 0|0 \rangle = 1 \).

Conclusion: We can forget about all terms arising in (3-17) except for those having only propagators and no operators in them. They are the only terms surviving in (3-17).

Thus, we see how the claim made earlier that Green functions comprise only propagators is true for the relation defined by (3-14). It remains to justify that these equal our familiar transition amplitudes when we substitute external particle relations for the incoming/outgoing propagators (i.e., the legs).

The Denominator of the Green Function

The Green function in position space (3-14) has a bra and a ket representing the same initial vacuum state, i.e., at \( t = -\infty \). This can be re-written more explicitly with notation specifying that the vacuum under consideration is the initial vacuum as

\[
G_{\mu \cdots \nu \cdots} (x_{1}, \ldots, x_{n}, \ldots, z_{1}, \ldots, z_{m}, \ldots) = \langle 0, t = -\infty | T \{ S \xi_{\mu} (\cdots) \phi (x_{1}) \cdots \phi (x_{n}) \cdots \overline{\phi} (z_{1}) \cdots \overline{\phi} (z_{m}) \cdots | 0, t = -\infty \rangle
\]

Examining the denominator of (3-19), we note that the operator \( S \), as typically defined (see Vol. 1 Wholeness Chart 8-4, pg. 248, last six rows), takes a given state (ket) at \( t = -\infty \) to its final state (ket) at \( t = +\infty \). Any terms in \( S \) that create particles states different from the vacuum would end up dropping out of the denominator, since the resulting ket would be orthogonal to the bra vacuum state. Similarly, any terms in \( S \) that destroy the vacuum (leaving zero) would drop out. The total \( S \) operator itself can never destroy the vacuum (and leave zero in the denominator), as one can see from its form in the above referenced wholeness chart in Vol. 1. \( S \) comprises \( e \) raised to a complex power, and when expanded, has the identity \( (S^{(0)} = I) \) as its lowest order term (which cannot annihilate the vacuum).

It is reasonable to presume that we cannot measure any difference between the vacuum at \( t = -\infty \) and the vacuum at \( t = +\infty \). Hence, the two can only differ by a phase factor, which is not detectable physically. (Phase factors disappear in probability and expectation value calculations). Thus,

\[
ket \langle 0, t = +\infty | = e^{i\phi} \langle 0, t = -\infty | \quad \text{bra} \quad \langle 0, t = +\infty | = e^{-i\phi} \langle 0, t = -\infty | \quad (3-20)
\]

and the denominator of (3-19) becomes

\[
\langle 0, t = -\infty | S | 0, t = -\infty \rangle = \langle 0, t = -\infty | 0, t = +\infty \rangle = \langle 0, t = -\infty | e^{i\phi} | 0, t = -\infty \rangle = e^{i\phi} \quad (3-21)
\]

As we claimed earlier, the denominator is, at most, a phase factor, and phase factors are irrelevant when we calculate probability. The phase \( \phi \) could, in fact, be zero, in which case the denominator (3-21) would be one, but we work here with the most general case.

Green Functions in Momentum Space

Just as we can express our transition amplitudes in position space [see, for example, Vol. 1, pg. 221, (8-26)] or in momentum space [Vol. 1, pg. 223, (8-34)], so we can express a Green function in
3.1.5 Interaction Examples of Green Functions

**Bhabha Scattering**

In Bhabha scattering we have an incoming electron, an incoming positron, an outgoing electron, and an outgoing positron. So, in (3-14), we take our legs to be the four fields corresponding to those particles. See Fig. 3-1. Note that we have no photon legs for this case, so we have no $A^\mu(x_i)$ factors and no superscripts on $G$. Our four-point Green function is

$$G(y_1, y_2, z_1, z_2) = \frac{\langle 0 | T \left[ \psi(y_1) \psi(y_2) \overline{\psi}(z_1) \overline{\psi}(z_2) \right] | 0 \rangle}{\langle 0 | S | 0 \rangle}$$

For Bhabha scattering. \hspace{1cm} (3-23)

Note, we want to create the electron virtual particle at $z_1$, so we use the quantum field $\overline{\psi}(z_1)$ for that. To create a positron at $y_1$, we use $\psi(y_1)$; to destroy an electron at $y_2$, $\psi(y_2)$; and to destroy a positron at $z_2$, $\overline{\psi}(z_2)$.

As shown earlier, the denominator in (3-23) only represents a phase factor, and in finding probabilities, we work with the square of the absolute value of transition amplitudes, where phase factors drop out. So, we will concentrate on the numerator of (3-23).

Using (3-16) in (3-23), we have

$$\langle 0 | T \left[ \psi(y_1) \psi(y_2) \overline{\psi}(z_1) \overline{\psi}(z_2) \right] | 0 \rangle \hspace{1cm} (3-24)$$

$$= \langle 0 | T \left[ \overline{\psi}(y_1) \psi(y_2) \overline{\psi}(z_1) \overline{\psi}(z_2) + i e A_\mu \psi(y_1) \overline{\psi}(y_2) \overline{\psi}(z_1) \overline{\psi}(z_2) \right] | 0 \rangle \hspace{1cm} (3-25)$$

From Wick’s theorem (3-18) and our knowledge that, due to normal ordering, only terms with propagators and no operators survive, we get (3-24) equal to

$$\langle 0 | T \left[ \psi(y_1) \psi(y_2) \overline{\psi}(z_1) \overline{\psi}(z_2) + \psi(y_1) \psi(y_2) \overline{\psi}(z_1) \overline{\psi}(z_2) \right] d^4 x_1 d^4 x_2 d^4 x_3 d^4 x_4 \hspace{1cm} (3-25)$$
Note that the second term in the penultimate row of (3-24) dropped out in going to (3-25) because there is no way we can make a photon propagator out of a single $A_\mu$ field. Note also that all quantities inside the bra and ket are numbers, not operators, so we can move those numbers outside the bracket with $(0\langle 0 \rangle = 1$, and just deal with the numbers inside the large parentheses of (3-25).

The first term in (3-25) is simply two unconnected propagators that don’t interact. That term represents a positron at $y_1$ traveling to $z_2$ and an electron at $z_1$ traveling to $y_2$. For the $S$ operator approach, such a term corresponded to no interaction between the electron and positron, which is one of the ways the particles could behave. The incoming electron and positron remain unchanged and are in the same state outgoing as they were incoming. Such a happening has a probability, just as the interaction of the two has a probability$^2$.

Here that term does not represent external particles, however, but propagators. But recall that we obtain the Feynman amplitudes by substituting external particle relations for the legs in a Green function. That is what we would do here, so we would end up with the behavior described in the foregoing paragraph, with the Green function diagrams for this term of Fig. 3-4. These diagrams are not connected, and we will discuss the ramifications of that shortly.

\[ \begin{array}{c}
\text{Figure 3-4. Disconnected Green Function Diagrams of Top Row in (3-25)}
\end{array} \]

All of the reasoning applied to the first term in (3-25) also applies to the second term.

Looking now at the third term in (3-25), we see it represents the LHS of Fig. 3-1. That is, the contractions of (3-25), are the virtual particles (propagators) of the LHS of Fig. 3-1. The factor in front of that second term in (3-25) is the same factor we have when we evaluate Feynman amplitudes. We can surmise that that term, when we replace the incoming and outgoing propagators with external particle relations (i.e., take the appropriate $S_F$ to the appropriate $u_\gamma(p_1), v_\beta(p_2)$, $\bar{u}_\gamma(p'_1)$, or $\bar{v}_\beta(p'_2)$), becomes an external particle Feynman amplitude.

Thus, that term can be written

\[
G_{3rd\, term}(y_1, y_2, z_1, z_2) = -\frac{e^2}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} iS_F(z_1 - y_2)\gamma_\alpha iS_F(z_2 - x_1) iD^\beta_F iS_F(x_2 - y_1)\gamma_\beta iS_F(z_1 - x_2) d^4\xi_1 d^4\xi_2
\]

Inserting (3-26) into (3-22), we would find the momentum space equivalent of this term in the Green function to be XXX THIS NEEDS TO BE DONE OUT HERE EXPLICITLY WHEN I HAVE TIME. PROBABLY SOME ERRORS IN DETAILS HERE, QUITE POSSIBLY IN SIGNS. SORRY. JUST PRESSED FOR TIME RIGHT NOW AND TRYING TO GET THIS POSTED BEFORE I LEAVE FOR A MONTH. RDK XXX

\[
G_{3rd\, term}(p_1, p_2, p'_1, p'_2) = -\frac{e^2}{2} iS_F(p'_1)\gamma_\alpha iS_F(p'_2) iD^\beta_S iS_F(p_2)\gamma_\beta iS_F(p_1)
\]

(3-27)

If we make the changes of (3-28) below to (3-27), we get (3-29).

\[
\begin{align*}
S_F(z_1 - x_2) \rightarrow & \quad S_F(p_1) \rightarrow u_\gamma(p_1) \quad S_F(z_2 - y_1) \rightarrow S_F(p_2) \rightarrow \bar{v}_\beta(p_2) \\
S_F(x_1 - y_2) \rightarrow & \quad S_F(p'_{G1}) \rightarrow \bar{u}_\gamma(p'_1) \quad S_F(z_2 - x_1) \rightarrow S_F(p'_2) \rightarrow v_\alpha(-p'_{G2}) = v_\alpha(p'_2) \\
p'_1, p'_2 = & \quad \text{Green function 4-momenta,} \\
p'_1 = & \quad \text{outgoing Feynman amplitude particle 4-momenta} \\
p'_1 = & \quad -p'_{G1}, p'_2 = -p'_{G2}
\end{align*}
\]

(3-28)
\[
G_{3rd\ term}(p_1, p_2; p_1', p_2') \rightarrow \frac{e^2}{2} \bar{u}_\eta(p_1')\gamma^\mu v_{\gamma_2}(p_2')iD_{\mu\nu}(k = p_1 + p_2)\bar{v}_{\gamma_2}(p_2)\gamma^\nu u_\eta(p_1)
\]

\[
S_B^{(2)}_{\text{transition amplitude}} = \sqrt{\frac{m}{VE_{p_1}}} \sqrt{\frac{m}{VE_{p_2}}} \int \frac{m}{VE_{p_1}} \int \frac{m}{VE_{p_2}} (2\pi)^4 \delta^{(4)} (p_1 + p_2 - (p_1' + p_2')) \mathcal{M}_{B_1}^{(2)}
\]

(3-29)

Feynman amplitude \( \mathcal{M}_{B_1}^{(2)} \)

Compare this to Vol. 1, (8-34), pg. 223 and we see the momentum space form of this term in the Green function corresponds to the first way Bhabha scattering can occur.

Note some subtleties in the outgoing particle momenta of (3-29). By definition, our Green function propagator legs in (3-27) have 4-momentum directed inward, i.e., from \( y_2 \) to \( x_1 \) and from \( z_2 \) to \( x_1 \). But the Feynman amplitude has them in the opposite direction, so we need a sign reversal to account for that.

Also, be careful with the numeric subscripts in position space vs momentum space. In position space numerical subscripts \( i = 1, 2, \ldots \) in \( x_i, y_i, z_i \) refer to particular field legs. In momentum space the numerical subscripts \( 1, 2, \ldots \) refer to particular external particle momenta. These are, in general, not related.

The fourth term in (3-25), via the logic of Vol. 1, pg. 258, Box 9-1, turns out to be equivalent to the third term. In brief, the coordinates in the integrations are dummy variables that can be interchanged, and along with the anti-commutation relations for fermions, the two terms turn out to be the same thing. Adding them together gives us a single term like the third one, but without the factor of 2 in the denominator.

So, from our experience (Vol. 1, Chap. 8) in converting transition amplitudes from expressions in position space to expressions in momentum space, we could go through the very lengthy procedure of carrying out the integrations in (3-25) to get its momentum space equivalent. See, for example, Vol. 1, Sect 8.4.2, pgs. 220 to 225. Or we could simply extrapolate our knowledge of Feynman rules (Vol. 1, Box 8-3, pg. 236) to express the last two terms shown explicitly in (3-25) [which are equal, as noted above] in momentum space.

Be aware that (3-25) also contains higher order terms (from other terms in \( S \)) which would be associated with the other diagrams of Fig. 3-2. It also would include the other way Bhabha scattering can occur, as in the last diagram of Fig. 3-2 plus non-displayed higher order diagrams for that other way.

Beyond those terms, however, (3-25) contains yet more terms, one of which we show in (3-30).

\[
\langle 0 \mid \left\{ -\frac{e^2}{2i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\bar{\psi} \gamma^\alpha A_{\alpha \beta} \psi)(x_1) (\bar{\psi} \gamma^\beta A_{\beta \gamma} \psi)(x_2) \psi(y_1) \psi(y_2) \bar{\psi}(z_1) \bar{\psi}(z_2) \right\} d^4x_1 d^4x_2 \langle 0 \rangle \quad (3-30)
\]

The corresponding Green function diagram for (3-30) is

\[\text{Fig. 3-5. Disconnected Green Function Diagrams of (3-30), Part of (3-25)}\]

And, of course, there are more disconnected diagrams in (3-25).

Show some more such diagrams by doing Problem 2.
Express (3-30) in momentum space by doing Problem XX
Møller Scattering

Note that the same Green function (3-25) could also represent Møller (electron-electron) scattering, as the four legs in (3-25) could create electron propagators beginning at \( z_1 \) and \( z_2 \), then end two electron propagators at \( y_1 \) and \( y_2 \). This parallels what we learned about transition amplitudes for these two processes in QED. (See Vol. 1, Wholeness Chart 8-4, pg. 250 for \( S_\beta(2) \) and pg. 225.)

That Green function could, of course, also represent positron-positron scattering.

The Photon Propagator

Consider the photon propagator, with Feynman diagrams shown in Fig. 3-6 to the first two orders.

![Figure 3-6. The Photon Propagator to First Two Orders](image)

Now consider the Green function (3-17) with only two photon legs (a two-point Green function), where again we focus on the numerator.

\[
G^{\mu\nu}(x_1, x_2) = \frac{\langle 0 | T \left\{ A^\mu(x_1) A^\nu(x_2) \right\} | 0 \rangle}{\langle 0 | S | 0 \rangle} +
\]

\[
ie \langle 0 | T \left\{ \int_{-\infty}^{\infty} \left( \bar{\psi} \gamma^\alpha A_\alpha \psi \right) d^4 x_1 A^\mu(x_1) A^\nu(x_2) \right\} | 0 \rangle
\]

\[
\left. - \frac{e^2}{2} \langle 0 | T \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \bar{\psi} \gamma^\alpha A_\alpha \psi \right)_{x_1} \left( \bar{\psi} \gamma^\beta A_\beta \psi \right)_{x_2} d^4 x_1 d^4 x_2 A^\mu(x_1) A^\nu(x_2) \right\} | 0 \rangle \right. 
\]

\[
\frac{\langle 0 | S | 0 \rangle}{\langle 0 | S | 0 \rangle}
\]

(3-31)

In converting the time ordering to normal ordering via Wick’s theorem, the second term will drop out, as it has an odd number of photon fields, and only with an even number can a non-zero term result. Thus, we end up with

\[
\text{numerator of } G^{\mu\nu}(x_1, x_2) =
\]

\[
\left\{ A^\mu(x_1) A^\nu(x_2) - \frac{e^2}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \bar{\psi} \gamma^\alpha A_\alpha \psi \right)_{x_1} \left( \bar{\psi} \gamma^\beta A_\beta \psi \right)_{x_2} A^\mu(x_1) A^\nu(x_2) d^4 x_1 d^4 x_2 \right\}
\]

\[
\langle 0 | 0 \rangle
\]

(3-32)

Again, this equals simply the quantities inside the large parentheses, since they are numbers and \( \langle 0 | 0 \rangle = 1 \).

Note we are not converting the leg propagators to external particles, since we are dealing here simply with a propagator (photon propagator) and not an external particle interaction.

We should be able to see that the terms shown in (3-32) are simply the lowest order propagator (first term) and the second order correction (second term), shown diagrammatically in Fig. 3-6. These are just the relation derived in the standard QFT canonical approach for the photon propagator to second order.

Thus, we see that the Green function for two photon legs is simply the photon propagator. (We showed this to second order and can surmise that if we included yet higher order terms in (3-31), we would get the higher order terms for the photon propagator as found in standard QFT without Green functions calculations.)
Green Functions That Vanish

Note that unless the Green function (3-14) has the same number of $\psi$ legs as $\bar{\psi}$ legs, it will equal zero, because otherwise the ket would either be destroyed or not match the bra. This is true since $S$ always has the same number of each of those types of fermion field operator. This means Green functions corresponding to the two LHS graphs of Fig. 3-7 equal zero. And it means electron number must be conserved, which we know it is.

Green functions with no fermion legs and an odd number of photon legs, such as those represented by the two RHS graphs of Fig. 3-7, will also vanish. This is a result of Furry’s theorem. See Vol. 1, pg. 341. For each such configuration, there will be another configuration with opposite sign on the transition amplitude and thus, the Green function, so the two configurations will cancel one another.

![Figure 3-7. Examples of Vanishing Green Functions](image)

Bottom line: The Green function vanishes when
1) the number of $\psi$ legs $\neq$ number of $\bar{\psi}$ legs, or
2) there are no fermion legs and an odd number of photon legs

3.1.6 Connected Green Functions

For any QED case with more than three legs (such as (3-25) with four), we will have connected and disconnected graphs. We can represent this diagrammatically for (3-25) as shown in Fig. 3-8.

![Figure 3-8. Decomposition of the Green Function (3-25) into Connected and Disconnected Parts](image)

Disconnected diagrams represent two or more independent processes which do not contain any additional information beyond that which is already contained in their individual component connected processes. As a result, focus is usually directed to the connected Green function $G_c$.

$$G_c^{\mu \nu \ldots}(x_1, \ldots) = G^{\mu \nu \ldots}(x_1, \ldots) - \sum \text{disconnected processes}$$

Disconnected Vacuum Processes in the Green Function Denominator and Numerator

We will not expound on it in depth here, as it is not critical and might take us a bit afield, but it can be shown$^3$ that the denominator of the Green function (3-19) represents vacuum bubble processes. (See Vol. 1, pgs. 234-235.) These contributions lead to a certain quantity in the denominator. The disconnected vacuum bubble processes in the numerator lead to a multiplicative factor there. It turns out that the quantity in the denominator and the multiplicative factor in the numerator are equal, which may feel right intuitively, since they are both vacuum bubble contributions. The bottom line: they cancel each other out, effectively meaning we can ignore vacuum bubble processes in the Green function. This provides justification for keeping the denominator in the definition (3-14).

3.1.7 Another Example

Consider a different (four-point) Green function, where we continue our convention in this chapter of using spacetime argument symbolism of $x_i$ for $A^\mu$, $y_i$ for $\psi$, and $z_i$ for $\bar{\psi}$.

---

In (3-34), we have purposely placed $S$ at the end of the string of operators to make a point. In moving $S$ to the left side of the relation, we would exchange the operators in $S$ sequentially with the operator legs. Every boson-boson exchange, and every boson-fermion exchange, would leave the relation unchanged because those fields commute. Each exchange for fermions with fermions would produce a minus sign factor, because fermion field anti-commute. But there are two fermion legs, so such exchanges, in total, would produce no change in the relation. (See Vol. 1, pgs. 72 and 118.) We note this because you will often see a Green function in the literature presented in a form like (3-34), with $S$ on the right.

That said, let us turn attention to (3-34).

Do Problem 3 to show (3-35).

The lowest order contribution to (3-34) is

$$G^{(0)\mu\nu}(x_1,x_2,y_1,z_1) = A^\mu(x_1) A^\nu(x_2) \psi(y_1) \overline{\psi}(z_1) = iD_F^{\mu\nu}(x_1-x_2)iS_F(y_1-z_1), \quad (3-35)$$

which, if you made the sketch asked for in Problem 3, can be seen graphically to correspond to two disconnected propagators, one fermionic and one photonic.

In general, like all Green functions, (3-34) can be decomposed into connected and disconnected parts, as shown graphically in Fig. 3-9.

[Figure 3-9. Decomposition of Green Function (3-34) Into Connected and Disconnected Parts]

Do Problem 4 to show (3-36).

XXX IN THE FUTURE, DEVELOPMENT OF THE FOLLOWING WILL BE EXPANDED XXX

The second order part of the connected Green function of (3-34) in momentum space is found by calculating the connected Green function in position space, $G^{\mu\nu}(x_1,...,x_n)$, then Fourier transforming that to momentum space. The result is

$$G^{(2)\mu\nu}_c(k_1,k_2,p_1,p_2) = iD_F^{\gamma\beta}(k_2)iS_F(-p_2)(ie\gamma_\beta)iS_F(p_1+k_1)(ie\gamma_\alpha)iS_F(p_1)iD_F^{\alpha\mu}(k_1) + iD_F^{\mu\alpha}(k_1)iS_F(-p_2)(ie\gamma_\alpha)iS_F(p_1+k_1)(ie\gamma_\beta)iS_F(p_1)iD_F^{\beta\nu}(k_2), \quad (3-36)$$

which is represented in Feynman diagrams in Fig. 3-10.

[Figure 3-10. Feynman Diagrams for the Connected 2nd Order Green Function of (3-36)]

The minus signs on $p_2$ in (3-36) come in because the argument of the fermion propagator is always in the sense of the fermion line arrows in Feynman diagrams, and here the inward momentum for the Green function is in the opposite direction of the line arrows in the diagram.
(3-36) could instead be obtained directly from the Feynman diagrams using Feynman rules.

Note that (3-36) can be expressed in short hand notation, given that the order of photon propagators is unimportant and \( D_F^{\alpha \mu} (k) = D_F^{\alpha \mu} (k) \).

\[
G_c^{(2)\mu \nu} (k_1, k_2, p_1, p_2) = i D_F^{\mu \alpha} (k_1) i S_F (-p_2) (i e \gamma_\beta) i S_F (p_1 + k_1) (i e \gamma_\alpha) i S_F (p_1) i D_F^{\beta \nu} (k_2) \\
+ i D_F^{\mu \alpha} (k_1) i S_F (-p_2) (i e \gamma_\alpha) i S_F (p_1 + k_2) (i e \gamma_\beta) i S_F (p_1) i D_F^{\beta \nu} (k_2) \\
= i D_F^{\mu \alpha} (k_1) i S_F (-p_2) \Gamma_{\alpha \beta} (k_1, k_2, p_1, p_2) i S_F (p_1) i D_F^{\beta \nu} (k_2).
\]

where

\[
\Gamma_{\alpha \beta} (k_1, k_2, p_1, p_2) = (i e \gamma_\alpha) i S_F (p_1 + k_1) (i e \gamma_\alpha) + (i e \gamma_\alpha) i S_F (p_1 + k_2) (i e \gamma_\beta) \\
= -e^2 (\gamma_\beta i S_F (p_1 + k_1) \gamma_\alpha + \gamma_\alpha i S_F (p_1 + k_2) \gamma_\beta).
\]

(3-38) is called a vertex function, which readers of Vol. 1 have seen before on pgs. 459-460 therein. These are helpful in streamlining notation when a given interaction can be represented by more than one Feynman diagram (where, of course, the incoming and outgoing particles/propagators are the same in each diagram.)

**The Feynman Amplitude for Compton Scattering**

Momentum-space Green functions follow Feynman rules just as Feynman amplitudes do. Consider the case of Compton scattering,

\[
\gamma (k, r) + e^- (p, s) \rightarrow \gamma (k', r') + e^- (p', s'),
\]

where we obtain the Feynman amplitude from the Green function (3-36) with the following changes.

\[
k_1 = k \quad p_1 = p \quad k_2 = -k' \quad p_2 = -p',
\]

and for second order

\[
i D_F^{\mu \alpha} (k_1) \rightarrow e_\nu^\alpha (k) \quad i S_F (p_1) \rightarrow u_s (p) \quad i D_F^{\beta \nu} (k_2) \rightarrow e_\mu^\beta (k') \quad i S_F (-p_2) \rightarrow \bar{u}_{s'} (p').
\]

Thus, (3-37) becomes

\[
\mathcal{M} (k, k', p, p') = e_\nu^\alpha (k) \bar{u}_{s'} (p') \Gamma_{\alpha \beta} (k, -k', p, -p') u_s (p) e_\mu^\beta (k'),
\]

the Feynman amplitude for Compton scattering. (Compare with Vol. 1, pg. 228.)

**Higher Order Relations**

For higher orders in perturbation theory, relations (3-41) have to be modified using the relations in Vol. 1, pg. 370, Wholeness Chart 14-4, Column IX, as follows.

\[
i D_F^{\mu \alpha} (k_1) \rightarrow \left( Z_f^{n \text{th}} \right)^{1/2} e_\nu^\alpha (k) \\
i S_F (p_1) \rightarrow \left( Z_f^{n \text{th}} \right)^{1/2} u_s (p)
\]

(3-43)

For second order, (3-43) reduces to (3-41).

**3.1.8 Crossing**

Note that the single Green function (3-34) [equivalently for second order, (3-36)] can represent several different interactions, such as
\[ \gamma + e^- \rightarrow \gamma + e^- \] (Compton scattering)
\[ \gamma + e^+ \rightarrow \gamma + e^+ \] (Compton positron scattering)
\[ \gamma + \gamma \rightarrow e^+ + e^- \] (fermion pair production)
\[ e^+ + e^- \rightarrow \gamma + \gamma \] (fermion pair annihilation). \hspace{1cm} (3-44)

We simply need to take different initial and final particles when we transition from the Green function to the Feynman amplitude.

Interactions linked in this way are called crossed reactions; and the relations between them, crossing relations. Crossing is typically emphasized in the Green function methodology, but it is actually inherent in the S operator methodology of Vol. 1, as well. See Wholeness Chart 8-4, therein, pg. 250, where (3-45) is one term in the S operator.

\[ S_B^{(2)} = -e^2 \int d^4x_1 d^4x_2 N \left\{ (\bar{\psi} \mathcal{A} \psi)_{x_1} (\bar{\psi} \mathcal{A} \psi)_{x_2} \right\} \] \hspace{1cm} (3-45)

(3-45) can give rise to a transition amplitude, i.e., an S matrix element, (and concomitantly, a Feynman amplitude) for any one of (3-44).

3.1.9 Comparing the Standard Math Green Function with QFT Green Function Methodologies

Green Function Free Photon Propagator

From the result of (3-32), we concluded that
\[ G^\mu\nu(x_1, x_2) = D_D^{\mu\nu}(x_1 - x_2) + \frac{D_D^{\mu\nu}(x_1 - x_2)}{k^2 + i\epsilon}, \] \hspace{1cm} (3-46)

where
\[ D_D^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\epsilon}. \] \hspace{1cm} (3-47)

Green Function by Two Different Approaches = Propagator

So now we see the connection between the usual math Green function approach and the QFT Green function methodology. Using the former with the QFT field equations, we get the no loops term in (3-46), i.e., (3-11) or (3-11) containing the familiar photon Feynman propagator, and for that reason alone, we can see why it is common to call the propagator a “Green function”. It is the usual math Green function for Maxwell’s equation.

But there is another reason. The QFT Green function methodology also gives rise to the propagator as we have known it (apart from a factor of i), \( iD_D^{\mu\nu} \), in (3-46). The full propagator (including higher order terms) of (3-46) is a Green function, as found via the QFT Green function methodology.

Greater Extent of QFT Green Function Methodology

As shown above, we can use the QFT Green function methodology to obtain the math form of the propagators. However, the QFT Green function methodology has a wider range of application. At its core is (3-14), and that relation can be used to find a Green function for any interaction, in addition to just those of the free propagators (as we did for Bhabha scattering in the first example above.)

3.1.10 Visualizing Why the QFT Green Function Works

To understand the QFT Green function a little more intuitively, consider the bra relationship of (3-20) re-written as
\[ \langle 0, t = -\infty | = e^{i\theta} \langle 0, t = +\infty | \] \hspace{1cm} (3-48)

and substitute that into the numerator of (3-19) along with (3-21) into the denominator. We get
\[ G^{\mu\nu}(x_1, y_1, z_1, \ldots) = \frac{e^{i\theta} \langle 0, t = +\infty | T \left\{ S A^\mu(x_1) \ldots \psi(y_1) \ldots \bar{\psi}(z_1) \ldots \right\} | 0, t = -\infty \rangle}{e^{i\theta}}, \] \hspace{1cm} (3-49)

where the phase factors cancel.
(3-49) acts more like the amplitudes we are familiar with. That is, it takes an initial state at an earlier time \((t = -\infty, \text{here})\) to a later time \((t = +\infty, \text{here})\). In prior work, to find an amplitude, we generally started with a state of certain particles, which was acted on by the operators in \(S\) to produce a final state of particles. The time ordered \(S\) operator worked fine for that.

Here in (3-49), on the other hand, we start with a vacuum state at \(t = -\infty\) and end with a vacuum state at \(t = +\infty\). So, to get anything meaningful, we have to create some particles. The fields \(A^\mu (x_1) \ldots \psi (y_1) \ldots \overline{\psi} (z_1) \ldots\) can be used for that. Which ones we use depends on the problem at hand. If we want to examine a problem with an initial electron and positron, we would use \(\psi (y_1)\) to create an electron at \(y_1\) and \(\overline{\psi} (z_1)\) to create a positron at \(z_1\).

Then, the \(S\) operator would take those fields forward in time and cause them to interact (e.g., to annihilate creating a virtual photon which then transforms into an outgoing electron and positron, as in Bhabha scattering). But then we have to get back to the vacuum again (at \(t = +\infty\)). So, we use additional fields from \(A^\mu (x_1) \ldots \psi (y_1) \ldots \overline{\psi} (z_1) \ldots\) to destroy the resulting electron and positron and leave the vacuum. In our example, we would have a field therein of form \(\overline{\psi} (z_1')\) to destroy the positron at \(z_1'\) and \(\psi (y_1')\) to destroy the electron at \(y_1'\).

But note, the \(S\) operator is a unitary operator that can be expressed as a sequence of unitary operations, i.e., using \(U\) to represent these operations (see Vol. 1, Chap. 18, where that symbol is also employed)

\[
S = U(+\infty, -\infty) = U(+\infty, y_1') U(y_1', z_1') U(z_1', y_1) U(y_1, z_1) U(z_1, -\infty) .
\] (3-50)

Since \(S\) and the other operators in (3-49) are time ordered, each of the \(U\) operators in \(S\) of (3-50) operates in the appropriate time sequence. In our prior example, \(U(z_1, -\infty)\) takes the initial vacuum to the time at \(z_1\), where \(\overline{\psi} (z_1)\) creates an electron. \(U(y_1, z_1)\) then takes that state to \(y_1\), where \(\psi (y_1)\) creates a positron (\(y_1\) and \(z_1\) could be the same 4D point), etc. until we arrive at the final vacuum state.

(3-14) [the same as (3-19)] is just a different mathematical way to express the amplitude associated with this unfolding of events. But in it, the final bra is different in that it represents the identical vacuum state of the ket, both being at \(t = -\infty\). As noted earlier, the denominator of (3-14) [the same as (3-19)] effectively cancels out the difference in the final kets of the two methodologies.

### 3.2 Grassmann Fields

TO BE DONE.

Bottom line: Grassmann fields, though just numeric (classical, not quantum operator) fields, anti-commute. This gives them different properties for many operations, such as derivatives using the chain rule. They are closely associated with fermions and are used as a calculational device for obtaining Green functions from the generating functional. Grassmann fields are needed to obtain Green functions that have fermionic fields in them.

The presentation below of the generating functional initially focuses on bosonic fields and ignores fermionic fields. In time, it will be expanded to include Grassmann fields and Green functions with fermion fields.

Mandl & Shaw develop Grassmann fields theory on pgs. 259-263.

### 3.3 The Generating Functional

Another, useful way of determining Green functions employs use of something called the generating functional defined below in (3-51). We will show how this is done after explaining what we mean by the symbols in (3-51).

#### 3.3.1 Definition of Generating Functional

\[
Z[J_k, \sigma, \overline{\sigma}] = \frac{\langle 0 | S' | 0 \rangle}{\langle 0 | S' | 0 \rangle} \quad M & S (12.83) \ pg \ 265 ,
\] (3-51)

where
Section 3.3. The Generating Functional

\[ S' = \left\{ e^{i S'} \right\} \]

\[ = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left( \prod_{i=1}^{n} \mathcal{L}_i \right) d^4 x_1 \cdots d^4 x_n \quad \text{M \& S (12.82) pg 265} \]

\[ \mathcal{L}_i' = \mathcal{L}_i + \mathcal{L}_S \]

\[ = \mathcal{L}_i + J_k(x) A^\mu(x) + \sigma_\alpha(x) \psi_\alpha(x) + \overline{\psi}_\alpha(x) \sigma_\alpha(x) \quad \text{M \& S (12.77 \& 12.78a) pg 264} \]

Note the newly introduced field sources \( J_k(x) \) in \( \mathcal{L}_S \) above are classical fields (just functions, not operators), whereas the fields \( A^\mu(x) \) are quantum (operator) fields. Note further that \( J_k(x) \) is NOT the fermion 4-current of QFT, i.e., it is not \( \overline{\psi} \gamma_\mu \psi \). It is a new (fictitious) entity that will be used as a helpful tool. In what follows, we don't solve for, or use, algebraic forms of \( J_k(x) \) in \( \mathcal{L}_S \). We just employ those field sources as symbols that aid us in developing useful relations.

3.3.2 Definition and Example of Functional Differentiation of a Function

We first note the following.

\[ \frac{\delta J_k(x')}{\delta J_\mu(x)} = \delta_\mu^\mu \delta^{(4)}(x' - x) \quad \text{M \& S (12.55) pg 259} \quad (3-53) \]

(3-53) may look strange in the sense that it means the (functional) variation of a function with respect to itself (i.e., when \( \sigma = \mu \) and \( x = x' \)) is infinite, when we might expect it to be one. However, it works because all functional operations are involved with integration. Using (3-53), as we will see in at least one example, works. It leads to a consistent theory.

As such an example where we use (3-53) with a functional, consider

\[ \frac{\delta \mathcal{S}^{\mu}}{\delta J_\mu(y)} \left( \int f(y') J_\sigma(y') dy' \right) = \left( \int f(y') \frac{\delta J_\sigma(y')}{\delta J_\mu(y)} dy' \right) = \left( \int f(y') \frac{\partial}{\partial g(y')} g(y') dy' \right) = \delta_\sigma^\sigma f(y) \quad (3-54) \]

Note this parallels more elementary calculus, i.e.,

\[ \frac{\partial}{\partial g(y)} f(y') g(y') = f(y') \frac{\partial g(y')}{\partial g(y)} = f(y') / \partial g(y') \frac{\partial g(y')}{\partial y} = f(y') \delta_\sigma^\sigma = f(y) \quad (3-55) \]

Key Functional Derivative

From (3-51) with (3-52) and (3-53), (sorry for missing steps)

\[ \frac{1}{i} \frac{\delta \mathcal{S}^{\mu}}{\delta J_\mu(x)} = \left\langle 0 \left| \frac{1}{i} \frac{\delta \mathcal{S}^{\mu}}{\delta J_\mu(x)} \left| 0 \right\rangle \right\rangle = \left\langle 0 \left| T \{ S' A^\mu(x) \} \right| 0 \right\rangle \quad \text{M \& S (12.90) pg 266} \quad (3-56) \]

with similar relations for \( \overline{\sigma}(x) \) and \( \sigma(x) \).

3.3.3 Relation Between Green Function and Generating Functional

The relationship (3-57) below is proven by substituting (3-51) into (3-57), using (3-56) (including the relations for \( \overline{\sigma} \) and \( \sigma \)) and comparing with (3-14). (again, sorry for missing steps)

\[ G^{\mu \nu \cdots}(x_1 \cdots y_1 \cdots z_1 \cdots) = (-1)^n \left( \frac{1}{i} \right)^n \frac{\delta^n \mathcal{S}^{\mu}}{\delta J_\mu(x_1) \cdots \delta \overline{\sigma}(y_1) \cdots \delta \sigma(z_1)} \quad \text{M \& S (12.91) pg 266} \quad (3-57) \]

\( n = \) total derivatives \( \bar{n} = \) derivatives with respect to \( \sigma \) fields.

3.3.4 Generating Functional Special Case: Free Field Photon Propagator

(See Mandl & Shaw, Sect 12.5.1 (pg 267-269))
The free photon propagator case consists of a (virtual) single photon propagating, but not interacting with anything. We define the symbol $Z_0$ as the free field generating functional where the subscript “0” signifies free field, no interactions.

The Mandl and Shaw treatment of this seems to me like it would be hard to follow. The treatment below is more straightforward and probably easier to understand, at least conceptually.

**Free photon propagator $Z_0$ from general $Z$ definition**

For the free field case, $S \rightarrow S_0 = 1$, the general definition for generating functional (3-51) becomes

$$Z_0[J, \sigma, \bar{\sigma}] = \frac{\langle 0 | S_0 | 0 \rangle}{\langle 0 | S_0 | 0 \rangle} = \frac{\langle 0 | S_0 | 0 \rangle}{\langle 0 | 1 \rangle} = \langle 0 | S_0 | 0 \rangle \quad M & S (12.94) \quad pg 267$$

(3-58)

In (3-52), the interaction Lagrangian $L_I = 0$, so $L' = L_S$. Thus,

$$S_0' = T \left\{ e^{i \int L'_S(x) dx} \right\} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T \left\{ L'_S(x_1) L'_S(x_2) \cdots L'_S(x_n) \right\} d^4 x_1 d^4 x_2 \cdots d^4 x_n \quad M & S (12.93) \quad pg 267.$$

(3-59)

Substituting the value for $L_S$ in (3-52) into (3-59), we have

$$S_0' = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} T \left\{ \left( J_\mu(x_1) A^\mu(x_1) + \bar{\sigma}_a(x_1) \psi_a(x_1) + \bar{\psi}_a(x_1) \sigma_a(x_1) \right) \cdots \right\} d^4 x_1 d^4 x_2 \cdots d^4 x_n$$

(3-60)

Because we seek a relationship for the free photon field, we now ignore the fermion fields, i.e., take $\sigma = \bar{\sigma} = 0$. Later, we will generalize our result to include both fermions and photons. We also simplify by only looking at the first few terms in (3-60). Doing that, we get

$$S_0' = 1 + i \int T \left\{ J_\mu(x_1) A^\mu(x_1) \right\} dx_1 - \frac{1}{2} \int T \left\{ J_\mu(x_1) A^\mu(x_1) J_\nu(x_2) A^\nu(x_2) \right\} d^4 x_1 d^4 x_2 + \cdots$$

(3-61)

When we use Wick’s theorem to convert the time ordering to normal ordering, the terms with just propagators arising from the quantum fields in (3-61) (the photon fields) will be just numbers. With (3-61) in (3-58), these will survive (not be equal to zero). Any terms with operators (the quantum photon fields) left in them, upon normal ordering will have destruction operators on the RHS of (3-61). In (3-58), these will act on the vacuum ket and result in zero. Hence we can ignore all terms in (3-61) that have operators left after using Wick’s theorem. That is, we only have to be concerned with the terms arising that consist solely of one or more contractions (propagators).

Restricting our effective $S_0'$ to this for use in (3-58), we have (note that we need an even number of photon fields in the second line of (3-61) to give rise to a contraction)

$$\text{effective terms in } S_0' = 1 - \frac{1}{2} \int T \left\{ J_\mu(x_1) A^\mu(x_1) A^\nu(x_2) J_\nu(x_2) \right\} d^4 x_1 d^4 x_2 + \cdots$$

(3-62)

Thus, from (3-58) (where $Z_0$ is now only a function of $J$ without fermion sources),

$$Z_0[J, \sigma, \bar{\sigma}] = Z_0[J_k] = \langle 0 | S_0' | 0 \rangle = S_0' \langle 0 | 0 \rangle = S_0'$$

(3-63)

Compare the RHS of this to our result (3-73) from the appendix,
\[
e^{-\frac{i}{2} \int [J_\mu(x) D^{\mu\nu}_F(x-y) J_\nu(y)] d^4 x d^4 y} = 1 - \frac{i}{2} \int \overline{J_\mu(x)} D^{\mu\nu}_F (x-y) J_\nu(y) d^4 x d^4 y + \ldots \quad (3-73)
\]

with the definition (3-72), also from the appendix,
\[
\int [J_\sigma(x^*) D^{\rho\sigma}_F(x^*-x')] J_\rho(x') d^4 x^* d^4 x' = \text{symbol} \left[ J_\mu D^{\mu\rho}_F J_\rho \right]. \quad (3-72)
\]

and we see that, at least to lowest order, for only photons and no fermions, (3-63) is
\[
Z_0 [J_{k, \sigma, \bar{\sigma}}] \rightarrow Z_0 \left[ J_k \right] = e^{-\frac{i}{2} \int [J_\sigma(x^*) D^{\rho\sigma}_F(x^*-x')] J_\rho(x') d^4 x^* d^4 x'} = e^{-\frac{i}{2} \int [J_\sigma D^{\rho\sigma}_F J_\rho].} \quad (3-64)
\]

Hopefully, we can accept that if we carried out the above steps and those of Appendix B for the higher order terms, the relation (3-64) still holds.

**Showing \(Z_0\) Yields Green Function for Free Photon Propagator**

From the definition (3-57), where the Green function we seek has no fermions (\(\bar{n} = 0\)) and two photons (\(n = 2\), so it is
\[
G_0^{\mu\nu}(x-y) = \left[ \frac{1}{i} \right]^2 \left| \frac{\delta^2 Z_0 [J_k]}{\delta J_\mu(x) \delta J_\nu(y)} \right|_0 \quad M & S (12.108) \quad pg 269 \quad (3-65)
\]

Inserting (3-64) with (3-83) into (3-65), we find, to lowest order,
\[
\left( \frac{1}{i} \right)^2 \left[ \frac{\delta^2 Z_0 [J_k]}{\delta J_\mu(x) \delta J_\nu(y)} \right]_0 = -\left( \frac{\delta^2}{\delta J_\mu(x) \delta J_\nu(y)} \right)_0 + \frac{\left[ J_\mu D^{\mu\nu}_F J_\nu \right]}{\delta J_\mu(x) \delta J_\nu(y)} \frac{\delta^2}{\delta J_\mu(x) \delta J_\nu(y)} \frac{i}{2} \int [J_\sigma(x^*) D^{\rho\sigma}_F (x^*-y')] J_\rho(y') d^4 x^* d^4 y' \quad (3-66) \quad \text{Term X}
\]

From the results of the first and last lines of (3-82) in Appendix B, we see term X in (3-66) is
\[
\text{Term X} = \left( \frac{\delta}{\delta J_\nu(y)} \left( \frac{\delta}{\delta J_\mu(x)} \frac{i}{2} \int [J_\sigma(x^*) D^{\rho\sigma}_F (x^*-y')] J_\rho(y') d^4 x^* d^4 y' \right) \right)_{\nu=0} = 2 \left( \frac{\delta}{\delta J_\nu(y)} \left( \frac{\delta}{\delta J_\mu(x)} \frac{i}{2} \int [J_\sigma(x^*) D^{\rho\sigma}_F (x^*-y')] J_\rho(y') d^4 x^* d^4 y' \right) \right)_{\nu=0} = \frac{i}{2} \left( \frac{\delta}{\delta J_\nu(y)} \left( \int [D^{\rho\sigma}_F (x-y')] J_\rho(y') d^4 y' + \int J_\sigma(x') D^{\rho\sigma}_F (x-y') \frac{\delta}{\delta J_\mu(x)} (x-y') d^4 x' d^4 y' \right) \right)_{\nu=0} = \frac{i}{2} \left( \frac{\delta}{\delta J_\nu(y)} \left( \int [D^{\rho\sigma}_F (x-y')] J_\rho(y') d^4 y' + \int J_\sigma(x') D^{\rho\sigma}_F (x-x') d^4 x' \right) \right)_{\nu=0} = \frac{i}{2} \left( \frac{\delta}{\delta J_\nu(y)} \left( \int [D^{\rho\sigma}_F (x-y')] J_\rho(y') d^4 y' + \int J_\sigma(x') J_\rho(y') d^4 y' \right) \right)_{\nu=0} \quad \text{via sym in } D^{\rho\mu}_F \text{ indices & switch dummy indices } \rho \rightarrow \sigma, \text{ get next row}
\]

\[
= \frac{i}{2} \left( \frac{\delta}{\delta J_\nu(y)} \left( \int [D^{\rho\sigma}_F (x-y')] J_\rho(y') d^4 y' + \int J_\sigma(x') J_\rho(y') d^4 y' \right) \right)_{\nu=0} \quad \text{switching dummy integration variable } x' \rightarrow y'
\]

\[
= \frac{i}{2} \left( \frac{\delta}{\delta J_\nu(y)} \left( \int [D^{\rho\sigma}_F (x-y')] J_\rho(y') d^4 y' \right) \right)_{\nu=0} = \frac{i}{2} \left( \frac{\delta}{\delta J_\nu(y)} \left( \int [D^{\rho\sigma}_F (x-y')] J_\rho(y') d^4 y' \right) \right)_{\nu=0} \quad (3-67)
\]

So, Term X is
Term $X = i \left[ \int D_F^{\mu \sigma} (x-y') \frac{\delta J \sigma (y)}{\delta J \nu (y)} d^4 y' \right] \overset{J=0}{=} i \left( \int D_F^{\mu \sigma} (x-y') \delta_{\mu \nu} \delta (y-y') d^4 y' \right) \overset{J=0}{=} i D_F^{\mu \nu} (x-y).$

Thus, from (3-65) and (3-66),

$$G_0^{\mu \nu} (x-y) = \left( \frac{1}{i} \right) \frac{\delta^2 Z_0 [J_k]}{\delta J \mu (x) \delta J \nu (y)} \overset{(3-65)}{=} i D_F^{\mu \nu} (x-y) \overset{(3-66)}{=} \text{higher order terms}$$

Result (3-69) shows that the usual Green function of mathematics for Maxwell’s equation (3-11) equals the lowest order Green function from the QFT Green function methodology.

### 3.3.5 Generalizing Free Field Photon Propagator to Include Fermions

The above procedure for photons can be carried out in parallel for fermions. (We need to use classical Grassman source fields $\sigma$ and $\bar{\sigma}$ instead of the classical photon source field $J^\mu$, as we did above. Grassman variables do not commute like ordinary variables [such as the more usual classical fields] do, and there is a fair amount of study involved in learning about their algebraic and calculus related behavior. We do not do this here.) Carrying out the above procedure for Grassman source fields, and incorporating the result with what we obtained in (3-64) for photon source fields, we find

$$Z_0 [J_k] \rightarrow Z_0 [J_k, \sigma, \bar{\sigma}] = Z_0 [J_k] Z_0 [\sigma, \bar{\sigma}] = e^{-i \frac{1}{2} J_a D_F^{\mu \rho} J^\rho} \overset{e^{-i \frac{1}{2} [\sigma S_F \bar{\sigma}]}}{=}. \quad (3-70)$$

Using (3-70) in (3-57) with only Grassman field derivatives (and all $J_\kappa = 0$), would give us a Green function equivalent to the free fermion propagator.

### 3.4 Chapter Summary

See Wholeness Chart 3-2.
### Wholeness Chart 3-2

**Summary of QED Green Functions and Generating Functional**

<table>
<thead>
<tr>
<th></th>
<th><strong>Canonical Approach</strong></th>
<th><strong>Path Integral Approach</strong></th>
<th><strong>Comments</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>S operator</strong></td>
<td>$S = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ T \left( \mathcal{L}_1^T (x_1) \ldots \mathcal{L}_n^T (x_n) \right) \right] d^4 x_1 \ldots d^4 x_n$</td>
<td>To be filled in later in book.</td>
<td>Fields $A^\mu, \psi, \overline{\psi}$ are operators in canonical approach; functions, not operators, in P.I.</td>
</tr>
<tr>
<td><strong>Transition amplitude</strong></td>
<td>$S_{fi} = \langle f</td>
<td>S</td>
<td>i \rangle = \delta_{fi}$</td>
</tr>
<tr>
<td><strong>Green functions</strong></td>
<td>$G^{\mu \nu}(x_1, y_1, z_1, \ldots) = \frac{\langle 0</td>
<td>T \left( \sum_{n=0}^{\infty} A^i (x) \psi (y) \overline{\psi} (z) \ldots \right)</td>
<td>0 \rangle}{\langle 0</td>
</tr>
<tr>
<td></td>
<td>$\text{Ignore denominator above for connected Feynman diagrams}$</td>
<td></td>
<td>When evaluated both approaches give same result. Proof later.</td>
</tr>
<tr>
<td><strong>Generating functional Z</strong></td>
<td>$Z[J_k, \sigma, \overline{\sigma}] = \frac{\langle 0</td>
<td>S'</td>
<td>0 \rangle}{\langle 0</td>
</tr>
<tr>
<td></td>
<td>$S' = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ T \left( \mathcal{L}_1^T (x_1) \ldots \mathcal{L}_n^T (x_n) \right) \right] d^4 x_1 \ldots d^4 x_n$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mathcal{L}<em>i' = \mathcal{L}<em>i + \mathcal{L}</em>{S}$ where $\mathcal{L}</em>{S} = J_k A^i + \overline{\sigma} \psi + \overline{\psi} \sigma$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| **Green function from generating functional** | $G^{\mu \nu} = \left(-1\right)^{\n} \left( \frac{1}{i} \right)^n \frac{\delta^n Z[J_k, \sigma, \overline{\sigma}]}{\delta J^\mu (x_1) \ldots \delta \sigma (y_1) \ldots \delta \overline{\sigma} (z_1)} | \left| 0 \right\rangle$ means $J = \overline{\sigma} = \sigma = 0$ after taking derivatives | Yet to show two approaches equiv. \n
Defs: $n = \text{tot num source fields}$ \n$\overline{n} = \text{tot num of } \sigma \text{source fields}$ \n$\overline{n} = \text{tot num deriv wrt } \sigma$ \n$\text{order} = \text{number of vertices in graph (Green funct has all orders in it due to } \mathcal{L}_i)$

**Significance**

Knowing $G^{\mu \nu}$ for a case with particular fields $A^\mu, \psi, \overline{\psi}$, one knows the amplitude for that case. \nKnowing $Z$, one knows all possible $G^{\mu \nu}$. $Z$ contains the whole theory.
3.5 Problems

1. Find the Green function for the Proca equation \( \left( \partial_\alpha \partial^\alpha + m^2 \right) A^\mu = 0 \). (Hint: Consider the case where instead of zero on the RHS, we had a function \( f^\mu(x) \), and the solution is found via \( A^\mu(x) = \int G^{\mu\nu}(x,y) f^\nu(y) \, ds \).

2. Show a set of disconnected diagrams arising from a term in the Green function for Bhabha scattering other than the ones shown explicitly in this chapter.

3. Show that the lowest order of (3-34) is (3-35), and sketch the Green function diagrams for that lowest order contribution.

4. TO COME

Appendix: Exponential Expansion of a Functional

Note first the streamlined notation \([AKB]\)

\[
[AKB] = \int A(x) K(x, y) B(y) \, d^4 x \, d^4 y \quad (3-71)
\]

For \( A=B= \) the field source \( J_\sigma \) (with different dummy subscript) in the source Lagrangian \( L_S \) of (3-52), and \( K = \) the no-loop photon propagator \( D_F^{\rho\rho} \), (3-71) becomes the special case

\[
\left[ J_\sigma D_F^{\rho\rho} J_\rho \right] = \int J_\sigma(x) D_F^{\rho\rho}(x-y) J_\rho(y) \, d^4 x \, d^4 y \quad (3-72)
\]

The entire purpose of this appendix is to show

\[
e^{-\frac{i}{2} \left[ J_\mu D_F^{\mu\nu} J_\nu \right]} = 1 - \frac{i}{2} \left[ J_\mu D_F^{\mu\nu} J_\nu \right] + \ldots \quad (3-73)
\]

or

\[
e^{-\frac{i}{2} \int J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y) \, d^4 x \, d^4 y} = 1 - \frac{i}{2} \int J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y) \, d^4 x \, d^4 y + \ldots \quad (3-73)
\]

Note how (3-73) parallels the elementary calculus relation

\[
e^x = 1 + x + \ldots \quad (3-74)
\]

Relation (3-73) is all you need to remember from this appendix. The proof follows

Proof of (3-73).

Recall (3-53),

\[
\frac{\delta J_\sigma(x^\prime)}{\delta J_\mu(x)} = \delta_\mu^\nu \delta^{(4)}(x^\prime - x) \quad M & S (12.55) \text{ pg 259}, \quad (3-53)
\]

Also, recall that for a simple function \( g(x) \),

\[
g(x) = g(0) + \frac{d g(x)}{d x} \bigg|_{x=0} x + \frac{1}{2} \frac{d^2 g(x)}{d x^2} \bigg|_{x=0} x^2 + \ldots = g(0) + g'(0) x + \frac{1}{2} g''(0) x^2 + \ldots \quad (3-75)
\]

For

\[
g(x) = e^{f(x)} = e^{f(0)} + f'(0) e^{f(0)} x + \frac{1}{2} \left( f''(0) e^{f(0)} + f'(0) f'(0) e^{f(0)} \right) x^2 + \ldots
\]

One might expect an analogous relation to hold for functionals, as follows, where the primes indicate functional derivatives with respect to \( J \),

\[
e^{f[J]} = e^{f[0]} + f'(0) e^{f[0]} J + \frac{1}{2} \left( f''(0) e^{f[0]} + f'(0) f'(0) e^{f[0]} \right) J^2 + \ldots \quad \text{(might expect)} \quad (3-76)
\]
However, since \( F[J] \) is an integral over spacetime and is a number, the LHS of (3-76) is a number, not a function of space. But each \( J \) in (3-76), would have a form such as \( J_\mu(x) \) or \( J_\nu(y) \), i.e., it would be function of spacetime position. Thus the RHS of (3-76), as written, would be a function of spacetime position \( (x, y, \ldots) \) for example. But it can’t be, since the LHS is not a function, but a pure number. From this, we can glean that the terms in (3-76) must be included inside the integrals over \( x, y, \ldots \) etc.

Thus, the correct form for the expansion of \( e^{F[J]} \) is

\[
\begin{align*}
e^{F[J]} &= e^{F[0]} + \int \left( \frac{\delta}{\delta J_\mu(x)} F(J) \right) J_\mu(x) d^4x \\
&+ \frac{1}{2} \iint \left( \frac{\delta}{\delta J_\mu(x)} \frac{\delta}{\delta J_\nu(y)} F(J) \right) J_\mu(x)J_\nu(y) d^4xd^4y \quad \text{(actual form of expansion) (3-77)} \\
&+ \frac{1}{2} \iint \left( \frac{\delta}{\delta J_\mu(x)} F(J) \right) \left( \frac{\delta}{\delta J_\nu(y)} F(J) \right) J_\mu(x)J_\nu(y) d^4x d^4x' + \ldots \\
\end{align*}
\]

As an example of the application of (3-77), we examine a form for \( F \) we will find very useful. That is the functional we looked at in (3-72) above (with different dummy integration variables that will help in the proof),

\[
F[J] = \frac{i}{2} \left[ J_\sigma D_F^{op} J_\rho \right] = \frac{i}{2} \iint J_\sigma(x^\prime) D_F^{op} (x^\prime - x) J_\rho(x) d^4x d^4x' \quad (3-78)
\]

With (3-78) in (3-77), noting that all \( F[J] \) where \( J = 0 \), have \( F[J] = 0 \), and thus

\[
e^{F[J]} = e^{F[0]} = e^0 = 1,
\]

we find

\[
e^{F[J]} = e^{F[0]} e^{J_\sigma D_F^{op} J_\rho / 2} = \frac{1}{2} \iint J_\sigma(x^\prime) D_F^{op} (x^\prime - x) J_\rho(x) d^4x d^4x'.
\]

\[
= 1 + \int \left( \frac{\delta}{\delta J_\mu(x)} \left[ \frac{1}{2} \iint d^4x d^4x' J_\sigma(x^\prime) D_F^{op} (x^\prime - x) J_\rho(x) \right] \right) J_\mu(x) d^4x \\
\]

\[
= \frac{1}{2} \iint \left( \frac{\delta^2}{\delta J_\mu(x) \delta J_\nu(y)} \left[ \frac{1}{2} \iint d^4x d^4x' J_\sigma(x^\prime) D_F^{op} (x^\prime - x) J_\rho(x) \right] \right) J_\mu(x)J_\nu(y) d^4xd^4y \\
\]

\[
+ \frac{1}{2} \iint \left( \frac{\delta}{\delta J_\mu(x)} \left[ \frac{1}{2} \iint d^4x d^4x' J_\sigma(x^\prime) D_F^{op} (x^\prime - x) J_\rho(x) \right] \right) J_\mu(x)J_\nu(y) d^4xd^4y \\
\]

\[
+ \frac{1}{2} \iint \left( \frac{\delta}{\delta J_\nu(y)} \left[ \frac{1}{2} \iint d^4x d^4x' J_\sigma(x^\prime) D_F^{op} (x^\prime - x) J_\rho(x) \right] \right) J_\mu(x)J_\nu(y) d^4xd^4y \\
\]

\[
(3-79)
\]
Carrying out the functional derivatives for each term above using (3-53), we get the following.

**Term A**

$$\text{Term A} = \int \left[ \frac{\delta}{\delta J_\mu (x)} \left( -\frac{i}{2} \right) \int J_\sigma (x^*) D^{\rho \rho}_F (x^* - x') J_\rho (x') d^4 x^* d^4 x' \right] J_\mu (x) d^4 x$$

$$= \left( -\frac{i}{2} \right) \int \left[ \int \left( \delta^{(4)} \delta^{(4)} (x - x^*) \right) D^{\rho \rho}_F (x^* - x') J_\rho (x') d^4 x^* d^4 x' \right] J_\mu (x) d^4 x$$

$$+ \left( -\frac{i}{2} \right) \int \left[ \int J_\sigma (x^*) D^{\rho \rho}_F (x^* - x') \left( \delta^{(4)} (x - x') \right) d^4 x^* d^4 x' \right] J_\mu (x) d^4 x$$

$$= -\frac{i}{2} \int \left( \int D^{\rho \rho}_F (x - x') J_\rho (x') d^4 x' + \int J_\sigma (x^*) D^{\rho \rho}_F (x^* - x') d^4 x^* \right) J_\mu (x) d^4 x$$

$$= -\frac{i}{2} \int \left( \int J_\sigma (x^*) D^{\rho \rho}_F (x^* - x') d^4 x^* \right) J_\mu (x) d^4 x$$

$$= -i \int \left( \int J_\sigma (x^*) D^{\rho \rho}_F (x^* - x') d^4 x^* \right) J_\mu (x) d^4 x$$

$$= -i \int J_\sigma (x^*) D^{\rho \rho}_F (x^* - x') d^4 x^* J_\mu (x) d^4 x'$$

$$= 0,$$ since $J_\nu (x') = 0.$

**Term C**

From (3-80) and (3-81), we should be able to see right away (I hope) that both factors in Term C of (3-79) equal zero. So the only thing we have to worry about (at least to lowest order) is Term B.

**Term B**

$$\text{Term B} = \int \left[ \frac{\delta^2}{\delta J_\mu (x) \delta J_\nu (y)} \left( -\frac{i}{2} \right) \int d^4 x^* d^4 x' J_\sigma (x^*) D^{\rho \rho}_F (x^* - x') J_\rho (x') \right] J_\mu (x) J_\nu (y) d^4 x d^4 y$$

$$= \frac{i}{2} \int \left[ \int \left( \delta^{(4)} \delta^{(4)} (x - x') \right) J_\mu (x) J_\nu (y) d^4 x d^4 y \right]$$

$$= \frac{i}{2} \int \left[ \int J_\sigma (x^*) D^{\rho \rho}_F (x^* - x') J_\rho (x') \right] J_\mu (x) J_\nu (y) d^4 x d^4 y$$

$$= -\frac{i}{2} \int \left( \int D^{\rho \rho}_F (x - x') \delta^{(4)} (x - y) d^4 x' \right) J_\mu (x) J_\nu (y) d^4 x d^4 y$$

$$= -\frac{i}{2} \int \left( D^{\rho \rho}_F (x - y) \right) J_\mu (x) J_\nu (y) d^4 x d^4 y.$$

Returning to (3-79), we see

$$e^{-\frac{i}{2} \int J_\mu (x) D^{\rho \rho}_F (x - y) J_\rho (y) d^4 x d^4 y} = 1 - \frac{i}{2} \int J_\mu (x) D^{\rho \rho}_F (x - y) J_\rho (y) d^4 x d^4 y + \ldots$$

or

$$e^{-\frac{i}{2} \left[ J_\mu D^{\rho \rho}_F J_\nu \right]} = 1 - \frac{i}{2} \left[ J_\mu D^{\rho \rho}_F J_\nu \right] + \ldots$$

as we stated in (3-73).