Green Functions, the Generating Functional and Propagators in the Canonical Quantization Approach

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The following is not a complete document in that it does not cover some material and does not provide every step for the material it does cover. It is not a "stand alone" presentation of Green functions and generating functionals. Rather, it is intended only as an aid for those using other books, particularly Mandl and Shaw, for certain material I did not cover in *Student Friendly Quantum Field Theory* (*SFQFT*). If I write a second book on QFT, this will all be expanded and improved pedagogically for inclusion therein.

There is a wholeness chart summary of the material beginning herein on page 2 with the title "Summary of Green Function Methodology of QFT", the link for which may be found near the bottom of the *SFQFT* home page. See "Canonical Approach" column in that chart. The path integral approach column is covered in that chart, but is not covered in this document.

Please note that I have yet to get review/feedback on this document from others, so it probably has some errors in it. Hopefully, they are only typos.

Usual Green Function Definition/Derivation from Mathematics

We want to solve an equation where $u(x)$ is the solution, L is an operator, and $f(x)$ is a known function of form

$$Lu(x) = f(x). \tag{1}$$

The Green function for this equation is $G(x,s)$, where ($L = L^*$ for L Hermitian)

$$LG(x,s) = \delta(x-s). \tag{2}$$

The solution $u(x)$ is

$$u(x) = \int G(x,s) f(s)\, ds. \tag{3}$$

Proof: Putting (3) into (1), then using (2), we have

$$Lu(x) = L\int G(x,s) f(s)\, ds = \int LG(x,s) f(s)\, ds = \int \delta(x-s) f(s)\, ds = f(x) \tag{4}$$

Usual Math Definition of Green Function for QFT Field Equations

The Klein-Gordon (Scalar) Equation Green Function

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi = 0 \tag{5}$$

The Green function for (5) is found from (where a minus sign on RHS doesn't change essence of the issue and is conventional in QFT and we use y in place of the s of (2))

$$\left(\partial_\mu \partial^\mu + m^2\right) G(x,y) = -\delta(x-y). \tag{6}$$

Solution to (6): Convert (6) to momentum space via a Fourier transform[1] over x.

$$\left(-k^2 + m^2\right)\hat{G}(k,y) = -e^{iky}. \tag{7}$$

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[1] Our convention is the one typically used in QFT. $\hat{f}(k) = \int d^4x\, f(x) e^{ikx}$ where $kx = k^\mu x_\mu$, and $f(x) = \frac{1}{(2\pi)^4}\int d^4k\, f(k) e^{-ikx}$.

Note also that from Fourier transform tables (or taking the time to prove it yourself) $\widehat{\partial_\mu f(x)} = -ik_\mu \hat{f}(k)$

The solution to (7) is

$$\hat{G}(k,y)=\frac{e^{iky}}{k^2-m^2}. \tag{8}$$

Converting (8) back to position space via a reverse Fourier transform, we get

$$G(x-y)=\frac{1}{(2\pi)^4}\int d^4k\frac{e^{-ik(x-y)}}{k^2-m^2\pm i\varepsilon}=\Delta_F(x-y), \tag{9}$$

which is just the scalar propagator. (Note that we add the infinitesimal quantity $i\varepsilon$ to the denominator of the integrand in (9) for reasons delineated in Klauber, Chap. 3, pg. 76. Essentially, the integrand would otherwise blow up at $|k| = m$. After doing the work one needs to do with (9) (effectively integrating it), the ε can be taken to zero (and we actually get a suitable result).

<u>The Maxwell Eq (Photon) Green Function</u>

In analogy to the Klein-Gordon (scalar) case, instead of (5), we have (where m =0 for photons)

$$\left(\partial_\mu\partial^\mu+m^2\right)A^\mu=0. \tag{10}$$

In similar fashion to (7) through (9), we get

$$G^{\mu\nu}(x-y)=\frac{g^{\mu\nu}}{(2\pi)^4}\int d^4k\frac{e^{-ik(x-y)}}{k^2-m^2\pm i\varepsilon}=D_F^{\mu\nu}(x-y). \tag{11}$$

<u>General Conclusion for Green Functions of Usual Mathematics and QFT Field Equations</u>

The Green function (defined as negative of usual math definition) for each particle type field equation equals the Feynman propagator.

For scalars, this is (times a factor of i)

$$iG(x-y)=i\Delta_F(x-y)=\langle 0|T\left\{\phi(x)\phi^\dagger(y)\right\}|0\rangle\ . \tag{12}$$

For photons,

$$iG^{\mu\nu}(x-y)=iD_F^{\mu\nu}(x-y)=\langle 0|T\left\{A^\mu(x)A^\nu(y)\right\}|0\rangle\ . \tag{13}$$

Similar for fermions.

So we can see why propagators are sometimes called Green functions.

Simplified Overview of Green Function Methodology of QFT

<u>A Somewhat Different Approach to Green Functions</u>

When students first see the use of Green functions in QFT, there is typically confusion, as the methodology therein seems quite different from the usual math Green functions, as described in the foregoing. So, we will consider two different approaches to Green functions, the usual math one and, what we will term the "QFT Green function methodology". After introducing the second of these, we will compare the two and see how, in some circumstances, they give the same result, i.e., the same Green functions. (The QFT Green function methodology results are more extensive, including not only the usual math Green function results, but more.)

<u>Brief Overview of QFT Methodology Green Functions: In Words</u>

The QFT methodology Green function approach, found in detail in texts such as Mandl and Shaw, provides a final result, the Green function for a particular interaction, that can be readily transformed into the Feynman amplitude for that interaction. The transformation involves straightforward substitution, as explained in words in the next paragraph and in pictures in the section after this one.

As it turns out, the QFT methodology Green function yields a final result that has only propagators (for example, $S_F\,(p)$) and no external line relations (for example, $u_1(\mathbf{p})$), unlike Feynman amplitudes. One obtains the appropriate Feynman amplitude simply by substituting an external line expression for each incoming/outgoing particle (which is a propagator in a Green function) in the particular interaction of interest (for example, substitute $u_1(\mathbf{p})$ for $S_F\,(p)$.

There are other subtleties, such as getting the signs right on the 4-momenta, but that is the technique in its essence. Thereby, one turns a Green function into a Feynman amplitude.

Brief Overview of QFT Methodology Green Functions: In Pictures

The Green function is a mathematical entity, just as the transition amplitude is a mathematical entity. And just as the transition amplitude can be represented pictorially by Feynman diagrams, so can a Green function be represented by diagrams. Fig. 1 displays the relationship between a typical Green function diagram and a Feynman diagram.

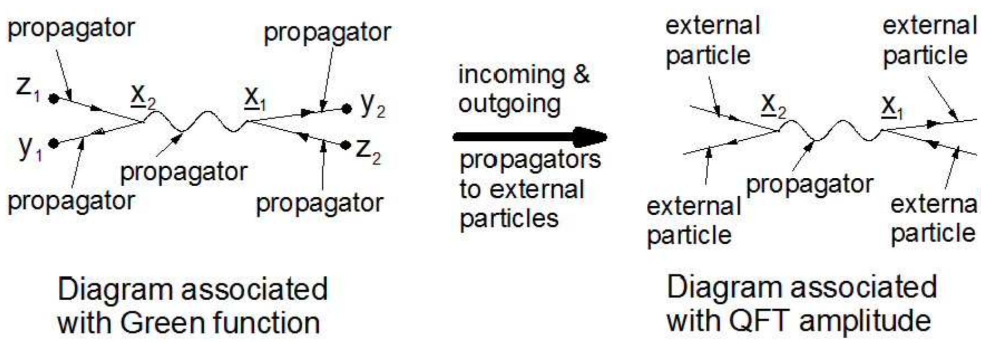

Figure 1. Changing a Green Function Diagram to a QFT Feynman Diagram for Bhabha Scattering

Note from the LHS of Fig. 1 that, as mentioned in the previous section, the Green function, and pictorially, its associated diagrams, are made up wholly of propagators. In the LHS, the incoming electron and positron of Bhabha scattering (only lowest order and only one of two ways it can occur are shown) are represented by propagators. So are the outgoing electron and positron.

To convert the diagram (and mathematically, the Green function) to a Feynman diagram (and mathematically, the transition amplitude), we merely substitute an external particle for each incoming/outgoing propagator. Generally, we can conclude that both the Green function and the transition amplitude carry the same information. Either can be readily obtained from the other.

Fig. 2 shows us what, given the foregoing paragraphs, should not be a big surprise. That is, a Green function is comprised of many (an infinite number) of higher order terms (diagrams in pictorial form), just as we once learned a transition amplitude is composed of many (an infinite number) of higher order terms (Feynman diagrams in pictorial form).

One may ask what value this has, as we already know how to construct amplitudes for any given interaction. One needs to know a lot more theory to answer this question. Full development of Green functions and their relationship to something called the generating functional can help significantly in advanced QFT. Additionally, it teaches us that the physical world can be represented in different, yet consonant, mathematical ways.

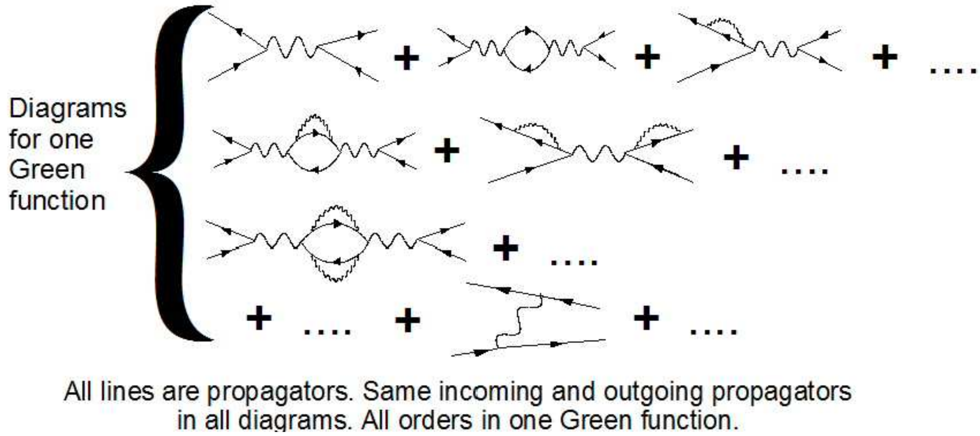

Figure 2. The Green Function Corresponding to Bhabha Scattering Includes All Orders for the Same Incoming and Outgoing Propagators

The Math Behind the Green Function Methodology of QFT

Mathematical Form of the Green Function

In this sub-section, we first simply present, in (14), the defining expression for Green functions in the QFT methodology, written $G^{\mu\nu..}(x_1, x_2, ,..y_1,..z_1)$. We then explore that relation and do a couple of examples, with different incoming/outgoing particles (called ”legs”), in order to show how the Green function as discussed in prior sections arises from (14).

$$\text{Green function } G^{\mu\nu\cdots}\left(x_1,x_2,\ldots y_1,\ldots z_1,\ldots\right)=\frac{\left\langle 0\left|T\left\{SA^{\mu}\left(x_1\right)A^{\nu}\left(x_2\right)\ldots\psi\left(y_2\right)\ldots.\bar{\psi}\left(z_3\right)\ldots\right\}\right|0\right\rangle}{\left\langle 0\left|S\right|0\right\rangle}\quad M\ \&\ S\left(12.8\right)pg\,245 \qquad (14)$$

Where S is the familiar (at least by now it should be) S operator from the canonical quantization approach to QFT; A^{μ}, ψ, and $\bar{\psi}$ are the usual QED quantum fields from that same theory; and T indicates time ordering. The notation *M & S* refers to Mandl and Shaw, 2nd ed. (2010, Wiley).

For now, we are not going to worry about the denominator in (14), but focus on the numerator. In Appendix A we provide hopefully helpful, intuitive insight into how the denominator arises mathematically and ends up being little more than a phase factor. The primary action is in the numerator. Don’t look at Appendix A until after you gain some familiarity with what is going on in the numerator.

We can re-express (14) by inserting into it the full expression (15) for the S operator (where we underline dummy integration variables to distinguish them from the x_1, x_2, … in the Green function (14), and the last line is specifically for QED),

$$\begin{aligned}S&=Te^{i\int\mathcal{L}_I(x)d^4x}=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int T\left\{\mathcal{L}_I\left(\underline{x}_1\right)..\mathcal{L}_I\left(\underline{x}_n\right)\right\}d^4\underline{x}_1..d^4\underline{x}_n\\&=I+i\int_{-\infty}^{\infty}\mathcal{L}_I\left(\underline{x}_1\right)d^4\underline{x}_1-\tfrac{1}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\left\{\mathcal{L}_I\left(\underline{x}_1\right)\mathcal{L}_I\left(\underline{x}_2\right)\right\}d^4\underline{x}_1d^4\underline{x}_2+\ldots\ldots \qquad (15)\\&=I+ie\int_{-\infty}^{\infty}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}d^4\underline{x}_1-\frac{e^2}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\left\{\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_2}\right\}d^4\underline{x}_1d^4\underline{x}_2+\ldots\ldots\end{aligned}$$

This gives us

$$G^{\mu\nu....}\left(x_1,x_2,...y_1,..\ z_1,...\right)=\frac{\left\langle 0\middle|T\left\{A^{\mu}\left(x_1\right)A^{\nu}\left(x_2\right)...\psi\left(y_2\right)....\bar{\psi}\left(z_3\right)...\right\}\middle|0\right\rangle}{\left\langle 0\middle|S\middle|0\right\rangle}+$$

$$\frac{ie\left\langle 0\middle|T\left\{\int_{-\infty}^{\infty}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}d^4\underline{x}_1 A^{\mu}\left(x_1\right)A^{\nu}\left(x_2\right)...\psi\left(y_2\right)....\bar{\psi}\left(z_3\right)...\right\}\middle|0\right\rangle}{\left\langle 0\middle|S\middle|0\right\rangle}+ \tag{16}$$

$$\frac{-\frac{e^2}{2!}\left\langle 0\middle|T\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_2}\right\}d^4\underline{x}_1 d^4\underline{x}_2 A^{\mu}\left(x_1\right)A^{\nu}\left(x_2\right)...\psi\left(y_2\right)....\bar{\psi}\left(z_3\right)...\right\}\middle|0\right\rangle}{\left\langle 0\middle|S\middle|0\right\rangle}$$

$$+......$$

<u>How Only Propagators Remain in Any Green Function</u>

Recall from Wick's theorem how we can convert the time ordering of (16) into normal ordering, and along the way we get terms with propagators in them. See Klauber, *Student Friendly Quantum Field Theory*, Chap. 7, pg. 204 for <u>Wick's theorem</u>, i.e., (where *A*,*B*,*C* etc. are fields such as A^{μ}, ψ, and $\bar{\psi}$)

$$\begin{aligned}
T\left\{(AB...)_{\underline{x}_1}.......(AB...)_{\underline{x}_n}\right\} &= N\left\{(AB...)_{\underline{x}_1}.......(AB...)_{\underline{x}_n}\right\} \\
&+N\left\{\underbrace{(AB..)_{\underline{x}_1}(AB..)_{\underline{x}_2}}\ ...\right\}+N\left\{\underbrace{(AB..)_{\underline{x}_1}(AB..)_{\underline{x}_2}}..\right\}+...+N\left\{\ ...\underbrace{(A...Z)_{\underline{x}_{n-1}}(A...Z)_{\underline{x}_n}}\right\} \\
&+N\left\{\underbrace{(ABC...)_{\underline{x}_1}(ABC...)_{\underline{x}_2}}\ ...\right\}+N\left\{\underbrace{(ABC...)_{\underline{x}_1}(ABC...)_{\underline{x}_2}}...\right\}+....
\end{aligned} \tag{17}$$

+ (all normal ordered terms with three non-equal times contractions)

+ etc.

Using (17) in (16) we will end up with, thanks to normal ordering, a lot of terms having destruction operators on the RHS. Any of those operating on the vacuum ket |0⟩ in the numerator of (16) will result in zero. So, the only terms surviving are those with only propagators (i.e., contractions) in them and no operators, since propagators are only numbers and not operators. These terms each have a number sandwiched between a vacuum bra and a vacuum ket, so we can take the number outside the bracket, and the bracket ⟨0|0⟩ = 1.

<u>Conclusion:</u> We can forget about all terms arising in (16) except for those having only propagators and no operators in them. They are the only terms surviving in (16).

Thus we see how the claim made earlier that Green functions comprise only operators is true for the relation defined by (14). It remains to justify that these equal our familiar transition amplitudes when we substitute external particle relations for the incoming/outgoing propagators (i.e., the legs).

<u>An Example: Bhabha Scattering</u>

In Bhabha scattering we have an incoming electron, an incoming positron, an outgoing electron, and an outgoing positron. So, in (14), we take our legs to be the four fields corresponding to those particles. See Fig. 1. Note that we have no photon legs for this case, so we have no $A^{\mu}(x_i)$ factors and no superscripts on *G*.

$$G\left(y_1,y_2,z_1,z_2\right)=\frac{\left\langle 0\middle|T\left\{S\psi\left(y_1\right)\psi\left(y_2\right)\bar{\psi}\left(z_1\right)\bar{\psi}\left(z_2\right)\right\}\middle|0\right\rangle}{\left\langle 0\middle|S\middle|0\right\rangle}\quad\text{For Bhabha scattering}. \tag{18}$$

Note we want to create the electron virtual particle at z_1, so we use the quantum field $\bar{\psi}(z_1)$ for that. To create a positron at y_1, we use $\psi(y_1)$; to destroy an electron at y_2, $\psi(y_2)$; and to destroy a positron at z_2, $\bar{\psi}(z_2)$.

From Appendix A we know that the denominator in (18) represents a phase factor, so we will concentrate on the numerator. (In finding probabilities we work with the square of the absolute value of transition amplitudes and phases of the amplitudes drop out.)

Using the last row of (15) in (18), we have

$$\text{numerator of } G\left(y_1,y_2,z_1,z_2\right)=$$

$$\langle 0|T\left\{\begin{pmatrix} I+ie\int_{-\infty}^{\infty}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}d^4\underline{x}_1 \\ -\frac{e^2}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\left\{\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_2}\right\}d^4\underline{x}_1d^4\underline{x}_2+.... \end{pmatrix}\psi\left(y_1\right)\psi\left(y_2\right)\bar{\psi}\left(z_1\right)\bar{\psi}\left(z_2\right)\right\}|0\rangle \qquad (19)$$

$$=\langle 0|T\left\{\begin{matrix} \psi\left(y_1\right)\psi\left(y_2\right)\bar{\psi}\left(z_1\right)\bar{\psi}\left(z_2\right)+ie\int_{-\infty}^{\infty}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}d^4\underline{x}_1\psi\left(y_1\right)\psi\left(y_2\right)\bar{\psi}\left(z_1\right)\bar{\psi}\left(z_2\right) \\ -\frac{e^2}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}T\left\{\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_2}\right\}d^4\underline{x}_1d^4\underline{x}_2\,\psi\left(y_1\right)\psi\left(y_2\right)\bar{\psi}\left(z_1\right)\bar{\psi}\left(z_2\right)+..... \end{matrix}\right\}|0\rangle .$$

From Wick's theorem (17) and our knowledge that, due to normal ordering, only terms with propagators and no operators survive, we get (19) as (where I apologize for the crude renditions of contractions [propagators] due to the limitations of my software)

$$\langle 0|\left\{\begin{matrix} \psi\left(y_1\right)\psi\left(y_2\right)\bar{\psi}\left(z_1\right)\bar{\psi}\left(z_2\right)+ \\ -\frac{e^2}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_1}\left(\bar{\psi}\gamma^{\mu}A_{\mu}\psi\right)_{\underline{x}_2}\psi\left(y_1\right)\psi\left(y_2\right)\bar{\psi}\left(z_1\right)\bar{\psi}\left(z_2\right)\;d^4\underline{x}_1d^4\underline{x}_2 \\ +\ \dots \end{matrix}\right\}|0\rangle . \qquad (20)$$

Note that the second term in the last row of (19) dropped out because there is no way we can make a photon propagator out of a single A^{μ} field. Note also that all quantities inside the bra and ket are numbers, not operators, so we can move those numbers outside the bracket with $\langle 0|0\rangle = 1$, and just deal with the numbers inside the large parentheses of (20).

The first term in (20) is simply two unconnected propagators that don't interact. That term represents a positron at y_1 traveling to z_2 and an electron at z_1 traveling to y_2. For the *S* operator approach, such a term corresponded to no interaction between the electron and positron, which is one of the ways the particles could behave. The incoming electron and positron remain unchanged and are in the same state outgoing as they were incoming. Such a happening has a probability, just as the interaction of the two has a probability[2].

Here that term does not represent external particles, however, but propagators. But recall that we obtain the Feynman amplitudes by substituting external particle relations for the legs in a Green function. That is what we would do here, so we would end up with the behavior described in the foregoing paragraph.

Looking now at the second term in (20), we see it represents the LHS of Fig. 1. That is, the contractions of (20), are the virtual particles (propagators) of the LHS of Fig. 1. The factor in front of that second term in (20) is the same factor we have when we evaluate Feynman amplitudes. We can surmise that that term, when we

[2] However, when we integrate over all space and time in the standard approach to QFT Feynman amplitude calculations, the particles then have to interact, as they must eventually over infinite time. Hence this term doesn't come into play in that approach. See Klauber, Chap 17, which explains the rational for integration over all time and space.

replace the incoming and outgoing propagators with external particle relations, i.e., take the appropriate S_F to the appropriate $u_r(\mathbf{p}_1)$, $v_r(\mathbf{p}_2)$, $\bar{u}_r(\mathbf{p})$, or $\bar{v}_r(\mathbf{p})$.

As examples, looking at Fig. 1 and the contractions/propagators of (20), we would make the following conversions to get the Feynman amplitude.

$$\begin{array}{ll} S_F(z_1 - \underline{x}_2) \xrightarrow[\text{space}]{\text{momentum}} S_F(p_1) \to u_{r_1}(\mathbf{p}_1) & S_F(y_1 - \underline{x}_2) \xrightarrow[\text{space}]{\text{momentum}} S_F(p_2) \to \bar{v}_{r_2}(\mathbf{p}_2) \\ S_F(\underline{x}_1 - y_2) \xrightarrow[\text{space}]{\text{momentum}} S_F(p_1') \to \bar{u}_{r_1'}(\mathbf{p}_1') & S_F(\underline{x}_1 - z_2) \xrightarrow[\text{space}]{\text{momentum}} S_F(p_2') \to v_{r_2'}(\mathbf{p}_2') \end{array} \tag{21}$$

Be aware that (20) also contains higher order terms which would be associated with the other diagrams of Fig. 2. It also would include the other way Bhabha scattering can occur, as in the last diagram of Fig. 2 plus non-displayed higher order diagrams for that other way.

<u>Another example: The photon propagator</u>

Consider the photon propagator, with Feynman diagrams shown in Fig. 3 to the first two orders

Figure 3. The Photon Propagator to First Two Orders

Now consider the Green function (16) with only two photon legs, where again we focus on the numerator.

$$\begin{aligned} G^{\mu\nu}(x_1,x_2) = {} & \frac{\langle 0|T\left\{A^\mu(x_1)A^\nu(x_2)\right\}|0\rangle}{\langle 0|S|0\rangle} + \\ & \frac{ie\langle 0|T\left\{\int_{-\infty}^{\infty}\left(\bar{\psi}\gamma^\mu A_\mu\psi\right)_{\underline{x}_1} d^4\underline{x}_1 A^\mu(x_1)A^\nu(x_2)\right\}|0\rangle}{\langle 0|S|0\rangle} + \\ & \frac{-\frac{e^2}{2!}\langle 0|T\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left(\bar{\psi}\gamma^\mu A_\mu\psi\right)_{\underline{x}_1}\left(\bar{\psi}\gamma^\mu A_\mu\psi\right)_{\underline{x}_2}\right\} d^4\underline{x}_1 d^4\underline{x}_2 A^\mu(x_1)A^\nu(x_2)\right\}|0\rangle}{\langle 0|S|0\rangle} \\ & + \ldots\ldots \end{aligned} \tag{22}$$

In converting the time ordering to normal ordering via Wick's theorem, the second term will drop out, as it has an odd number of photon fields, and only with an even number can a non-zero term result. Thus, we end up with

$$\text{numerator of } G^{\mu\nu}(x_1,x_2) =$$

$$\langle 0|\left\{ \begin{array}{l} \underbrace{A^\mu(x_1)A^\nu(x_2)} - \frac{e^2}{2!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\bar{\psi}\gamma^\alpha A_\alpha\psi)_{\underline{x}_1}(\bar{\psi}\gamma^\beta A_\beta\psi)_{\underline{x}_2} A^\mu(x_1)A^\nu(x_2)\, d^4\underline{x}_1 d^4\underline{x}_2 \\ \\ + \ldots \end{array} \right\}|0\rangle \tag{23}$$

Again this equals simply the quantities inside the large parentheses, since they are numbers and $\langle 0|0\rangle = 1$.

Note we are not converting the leg propagators to external particles, since we are dealing here simply with a propagator (photon propagator) and not an external particle interaction.

We should be able to see that the terms shown in (23) are simply the lowest order propagator (first term) and the second order correction (second term), shown diagrammatically in Fig. 3. These are just the relation derived in the standard QFT canonical approach for the photon propagator to second order.

Thus, we see that the Green function for two photon legs is simply the photon propagator. (We showed this to second order and can surmise that if we included yet higher order terms in (22), we would get the higher order terms for the photon propagator as found in standard QFT without Green functions calculations.

Comparing the Standard Math Green Function with QFT Green Function Methodologies

Green Function Free Photon Propagator

From the result of (23), we concluded that

$$G^{\mu\nu}(x,y) = \underbrace{D_F^{\mu\nu}(x-y)}_{\text{no loops}} + \underbrace{(\text{higher order})}_{\substack{\text{fermion-antifermion}\\ \text{loops}}} \quad , \tag{24}$$

where

$$D_F^{\mu\nu}(k) = \frac{-g^{\mu\nu}}{k^2 + i\varepsilon} . \tag{25}$$

Green Function by Two Different Approaches = Propagator

So now we see the connection between the usual math Green function approach and the QFT Green function methodology. Using the former with the QFT field equations, we get the no loops term in (24), i.e., (11) or (13), containing the familiar photon Feynman propagator, and for that reason alone, we can see why it is common to call the propagator a “Green function”. It is the usual math Green function for Maxwell’s equation.

But there is another reason. The QFT Green function methodology also gives rise to the propagator as we have known it (apart from a factor of *i*), $iD_F^{\mu\nu}$, in (24). The full propagator (including higher order terms) of (24) is a Green function, as found via the QFT Green function methodology.

Greater Extent of QFT Green Function Methodology

As shown above, we can use the QFT Green function methodology to obtain the math form of the propagators. However, the QFT Green function methodology has a wider range of application. At its core is (14), and that relation can be used to find a Green function for any interaction, in addition to just those of the free propagators (as we did for Bhabha scattering in the first example above.)

Summary of Generating Functional

Another, useful way of determining Green functions employs use of something called the generating functional defined below in (26). We will show how this is done after explaining what we mean by the symbols in (26).

Definition of generating functional

$$Z[J_k, \sigma, \bar{\sigma}] = \frac{\langle 0|S'|0\rangle}{\langle 0|S|0\rangle} \qquad M\,\&\,S(12.83) \quad pg\,265\,, \tag{26}$$

where

$$S' = T\left\{e^{i\int \mathcal{L}_I'(\underline{x})d^4\underline{x}}\right\}$$

$$= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty}T\left\{\mathcal{L}_I'(\underline{x}_1)\mathcal{L}_I'(\underline{x}_2)...\mathcal{L}_I'(\underline{x}_n)\right\}d^4\underline{x}_1 d^4\underline{x}_2...d^4\underline{x}_n \quad M\ \&\ S(12.82)\ pg265 \qquad (27)$$

$$\mathcal{L}_I' = \mathcal{L}_I + \mathcal{L}_S = \mathcal{L}_I + \underbrace{J_\kappa(\underline{x})A^\kappa(\underline{x}) + \bar{\sigma}_\alpha(\underline{x})\psi_\alpha(\underline{x}) + \bar{\psi}_\alpha(\underline{x})\sigma_\alpha(\underline{x})}_{\mathcal{L}_S} \quad M\ \&\ S(12.77\ \&12.78a)\ pg264$$

Note the newly introduced field sources $J_\kappa(\underline{x})....\bar{\sigma}_\alpha(\underline{x})....\sigma_\alpha(\underline{x})$ in $\mathcal{L}_S$ above are classical fields (just functions, not operators), whereas the fields $A^\mu(\underline{x}).... \psi_\alpha(\underline{x}).... \bar{\psi}_\alpha(\underline{x})$ are quantum (operator) fields. Note further that $J_\kappa(x)$ is NOT the fermion 4-current of QFT, i.e., it is not $\bar{\psi}\gamma_\kappa\psi$. It is a new (fictitious) entity that will be used as a helpful tool. In what follows, we don't solve for, or use, algebraic forms of $J_\kappa(\underline{x})....\bar{\sigma}_\alpha(\underline{x})....\sigma_\alpha(\underline{x})$. We just employ those field sources as symbols that aid us in developing useful relations.

Definition and Example of Functional Differentiation of a Function

We first note the following.

$$\frac{\delta J_\sigma(x'')}{\delta J_\mu(x)} = \delta^\mu_\sigma\delta^{(4)}(x''-x) \qquad M\ \&\ S(12.55)\ \ pg\,259 \qquad (28)$$

(28) may look strange in the sense that it means the (functional) variation of a function with respect to itself (i.e., when $\sigma = \mu$ and $x = x''$) is infinite, when we might expect it to be one. However, it works because all functional operations are involved with integration. Using (28), as we will see in at least one example, works. It leads to a consistent theory.

As such an example where we use (28) with a functional, consider

$$\frac{\delta}{\delta J_\nu(y)}\overbrace{\left(\int f(y')J_\sigma(y')dy'\right)}^{\text{functional}} = \left(\int f(y')\frac{\delta J_\sigma(y')}{\delta J_\nu(y)}dy'\right) = \left(\int f(y')\delta^\nu_\sigma\delta(y-y')dy'\right) = \delta^\nu_\sigma f(y)\,, \quad (29)$$

Note this parallels more elementary calculus, i.e.,

$$\frac{\partial}{\partial g(y)}f(y')g(y') = f(y')\frac{\partial g(y')}{\partial g(y)} = f(y')\frac{\partial g(y')}{\partial g(y')}\frac{\partial y'}{\partial y} = f(y')\delta^{y'}_y = f(y) \qquad (30)$$

Key Functional Derivative

From (26) with (27) and (28), (sorry for missing steps)

$$\frac{1}{i}\frac{\delta Z[J_k,\sigma,\bar{\sigma}]}{\delta J_\mu(x)} = \frac{\langle 0|\frac{1}{i}\dfrac{\delta S'}{\delta J_\mu(x)}|0\rangle}{\langle 0|S|0\rangle} = \frac{\langle 0|T\left\{S'A^\mu(x)\right\}|0\rangle}{\langle 0|S|0\rangle} \qquad M\ \&\ S(12.90)\ \ pg\,266 \qquad (31)$$

with similar relations for $\bar{\sigma}(x)$ and $\sigma(x)$

Relation Between Green Function and Generating Functional

The relationship (32) below is proven by substituting (26) into (32), using (31) (including the relations for $\bar{\sigma}$ and σ), and comparing with (14). (again, sorry for missing steps)

$$G^{\mu....}(x_1,...y_1,..\ z_1,...) = (-1)^{\bar{n}}\left(\frac{1}{i}\right)^n\left.\frac{\delta^n Z[J_k,\sigma,\bar{\sigma}]}{\delta J_\mu(x_1)...\ \delta\bar{\sigma}(y_1)...\ \delta\sigma(z_1)}\right|_0 \qquad M\ \&\ S(12.91)\ \ pg\,266 \qquad (32)$$

n = total derivatives $\qquad \bar{n}$ = derivatives with respect to σ fields.

Generating Functional Special Case: Free Field Photon Propagator (See Mandl & Shaw, Sect 12.5.1 (pg 267-269))

The free photon propagator case consists of a (virtual) single photon propagating, but not interacting with anything. We define the symbol Z_0 as the free field generating functional where the subscript "0" signifies free field, no interactions.

The Mandl and Shaw treatment of this seems to me like it would be hard to follow. The treatment below is more straightforward and probably easier to understand, at least conceptually.

<u>Free photon propagator Z_0 from general Z definition</u>

For the free field case, $S \rightarrow S_0 = 1$, the general definition for generating functional (26) becomes

$$Z_0\left[J_k,\sigma,\bar{\sigma}\right]=\frac{\langle 0|S_0'|0\rangle}{\langle 0|S_0|0\rangle}=\frac{\langle 0|S_0'|0\rangle}{\langle 0|1|0\rangle}=\langle 0|S_0'|0\rangle \qquad M\,\&\,S(12.94)\quad pg\,267 \tag{33}$$

In (27), the interaction Lagrangian $\mathcal{L}_I = 0$, so $\mathcal{L}' = \mathcal{L}_S$. Thus,

$$\begin{aligned} S_0' &= T\left\{e^{i\int \mathcal{L}_I'(x)d^4x}\right\} \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty} T\left\{\mathcal{L}_S'(x_1)\mathcal{L}_S'(x_2)...\mathcal{L}_S'(x_n)\right\}d^4x_1d^4x_2...\,d^4x_n \quad M\,\&\,S(12.93)\;pg\,267. \end{aligned} \tag{34}$$

Substituting the value for $\mathcal{L}_S$ in (27) into (34), we have

$$S_0'=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty}T\left\{\begin{array}{l}\left(J_\mu(x_1)A^\mu(x_1)+\bar{\sigma}_\alpha(x_1)\psi_\alpha(x_1)+\bar{\psi}_\alpha(x_1)\sigma_\alpha(x_1)\right)\\ \times\left(J_\mu(x_2)A^\mu(x_2)+\bar{\sigma}_\alpha(x_2)\psi_\alpha(x_2)+\bar{\psi}_\alpha(x_2)\sigma_\alpha(x_2)\right)\times......\\ \times\left(J_\mu(x_n)A^\mu(x_n)+\bar{\sigma}_\alpha(x_n)\psi_\alpha(x_n)+\bar{\psi}_\alpha(x_n)\sigma_\alpha(x_n)\right)\times......\end{array}\right\}d^4x_1d^4x_2...\,d^4x_n\,. \tag{35}$$

Because we seek a relationship for the free photon field, we now ignore the fermion fields, i.e., take $\sigma = \bar{\sigma} = 0$. Later, we will generalize our result to include both fermions and photons. We also simplify by only looking at the first few terms in (35). Doing that, we get

$$\begin{aligned} S_0' &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty}.....\int_{-\infty}^{\infty}T\left\{J_\mu(x_1)A^\mu(x_1)J_\nu(x_2)A^\nu(x_2)J_\kappa(x_3)A^\kappa(x_3)......\right\}d^4x_1d^4x_2...\,d^4x_n \\ &= 1+i\int T\left\{J_\mu(x_1)A^\mu(x_1)\right\}dx_1-\frac{1}{2}\iint T\left\{J_\mu(x_1)A^\mu(x_1)J_\nu(x_2)A^\nu(x_2)\right\}d^4x_1d^4x_2+...... \end{aligned} \tag{36}$$

When we use Wick's theorem to convert the time ordering to normal ordering, the terms with just propagators arising from the quantum fields in (36) (the photon fields) will be just numbers. With (36) in (33), these will survive (not be equal to zero). Any terms with operators (the quantum photon fields) left in them, upon normal ordering will have destruction operators on the RHS of (36). In (33), these will act on the vacuum ket and result in zero. Hence we can ignore all terms in (36) that have operators left after using Wick's theorem. That is, we only have to be concerned with the terms arising that consist solely of one or more contractions (propagators).

Restricting our effective S'_0 to this for use in (33), we have (note that we need an even number of photon fields in the second line of (36) to give rise to a contraction)

$$\text{effective terms in } S_0' = 1 - \frac{1}{2}\iint J_\mu(x_1)\underbrace{A^\mu(x_1)A^\nu}(x_2)J_\nu(x_2)d^4x_1d^4x_2 + \ ......$$
$$= 1 - \frac{1}{2}\iint J_\mu(x_1)iD_F^{\mu\nu}(x_2 - x_1)J_\nu(x_2)d^4x_1d^4x_2 + \ ...... \qquad (37)$$

Thus, from (33) (where Z_0 is now only a function of J without fermion sources,

$$Z_0[J_k,\sigma,\bar{\sigma}] = Z_0[J_k] = \langle 0|S_0'|0\rangle = S_0'\langle 0|0\rangle = S_0' = 1 - \frac{i}{2}\iint J_\mu(x_1)D_F^{\mu\nu}(x_2 - x_1)J_\nu(x_2)d^4x_1d^4x_2 + \ ......(38)$$

Compare the RHS of this to our result (52) from Appendix B,

$$e^{-\frac{i}{2}\iint J_\mu(x)D_F^{\mu\nu}(x-y)J_\nu(y)d^4xd^4y} = 1 - \frac{i}{2}\iint J_\mu(x)D_F^{\mu\nu}(x-y)J_\nu(y)d^4xd^4y \ + ... \qquad (52)$$

with the definition (51)

$$\iint J_\sigma(x'')D_F^{\sigma\rho}(x''-x')J_\rho(x')d^4x''d^4x' = \text{symbol}\left[J_\sigma D_F^{\sigma\rho}J_\rho\right], \qquad (51)$$

and we see that, at least to lowest order, for only photons and no fermions, (38) is

$$Z_0[J_k,\sigma,\bar{\sigma}] \rightarrow Z_0[J_k] = e^{-\frac{i}{2}\iint J_\sigma(x'')D_F^{\sigma\rho}(x''-x')J_\rho(x')d^4x''d^4x'} = e^{-\frac{i}{2}\left[J_\sigma D_F^{\sigma\rho}J_\rho\right]}. \qquad (39)$$

Hopefully, we can accept that if we carried out the above steps and those of Appendix B for the higher order terms, the relation (39) still holds.

Showing Z_0 Yields Green Function for Free Photon Propagator

From the definition (32), where the Green function we seek has no fermions ($\bar{n} = 0$) and two photons (n = 2), so it is

$$G_0^{\mu\nu}(x-y) = \left(\frac{1}{i}\right)^2 \left.\frac{\delta^2 Z_0[J_k]}{\delta J_\mu(x)\delta J_\nu(y)}\right|_0 \qquad M \,\&\, S\,(12.108)\ \ pg\,269 \qquad (40)$$

Inserting (39) with (62) into (40), we find, to lowest order,

$$\left(\frac{1}{i}\right)^2 \left.\frac{\delta^2 Z_0[J_k]}{\delta J_\mu(x)\delta J_\nu(y)}\right|_0 = -\left.\frac{\delta^2 e^{-\frac{i}{2}\left[J_\sigma D_F^{\sigma\rho}J_\rho\right]}}{\delta J_\mu(x)\delta J_\nu(y)}\right|_0$$
$$= \underbrace{-\left.\frac{\delta^2(1)}{\delta J_\mu(x)\delta J_\nu(y)}\right|_0}_{=0} + \underbrace{\left(\frac{\delta^2}{\delta J_\mu(x)\delta J_\nu(y)}\frac{i}{2}\overbrace{\iint J_\sigma(x')D_F^{\sigma\rho}(x'-y')J_\rho(y')d^4x'd^4y'}^{\left[J_\mu D_F^{\mu\nu}J_\nu\right]}\right)_0}_{\text{Term X}} + ... \qquad (41)$$

From the results of the first and last lines of (61) in Appendix B, we see term X in (41) is

$$\text{Term X} = \left(\frac{\delta}{\delta J_\nu(y)} \left(\frac{\delta}{\delta J_\mu(x)} \frac{i}{2} \iint J_\sigma(x') D_F^{\sigma\rho}(x'-y') J_\rho(y') d^4x' d^4y' \right)\right)_{J=0}$$

$$= \frac{i}{2}\left(\frac{\delta}{\delta J_\nu(y)} \left(\iint \frac{\delta\ J_\sigma(x')}{\delta J_\mu(x)} D_F^{\sigma\rho}(x'-y') J_\rho(y') d^4x' d^4y' + \iint J_\sigma(x') D_F^{\sigma\rho}(x'-y') \frac{\delta\ J_\rho(y')}{\delta J_\mu(x)} d^4x' d^4y' \right)\right)_{J=0}$$

$$= \frac{i}{2}\left(\frac{\delta}{\delta J_\nu(y)} \left(\iint \delta_\sigma^\mu \delta(x-x') D_F^{\sigma\rho}(x'-y') J_\rho(y') d^4x' d^4y' + \iint J_\sigma(x') D_F^{\sigma\rho}(x'-y') \delta_\rho^\mu \delta(x-y') d^4x' d^4y' \right)\right)_{J=0}$$

$$= \frac{i}{2}\left(\frac{\delta}{\delta J_\nu(y)} \left(\int\ D_F^{\mu\rho}(x-y') J_\rho(y') d^4y' + \iint J_\sigma(x') D_F^{\sigma\mu}(x-x') d^4x' \right)\right)_{J=0} \quad \begin{pmatrix} \text{via sym in } D_F^{\sigma\mu} \text{ indices \&} \\ \text{switch dummy indices} \\ \rho \to \sigma, \text{ get next row} \end{pmatrix}$$

$$= \frac{i}{2}\left(\frac{\delta}{\delta J_\nu(y)} \left(\int\ D_F^{\mu\sigma}(x-y') J_\sigma(y') d^4y' + \iint D_F^{\mu\sigma}(x-x') J_\sigma(x') d^4x' \right)\right)_{J=0} \quad \begin{pmatrix} \text{switching dummy} \\ \text{integration variable } x' \to y' \\ \text{gives next row} \end{pmatrix}$$

$$= i\left(\frac{\delta}{\delta J_\nu(y)} \left(\int\ D_F^{\mu\sigma}(x-y') J_\sigma(y') d^4y' \right)\right)_{J=0} . \qquad (42)$$

So, Term X is

$$\text{Term X} = i\left(\int\ D_F^{\mu\sigma}(x-y') \frac{\delta\ J_\sigma(y')}{\delta J_\nu(y)} d^4y' \right)_{J=0} = i\left(\int\ D_F^{\mu\sigma}(x-y') \delta_\sigma^\nu \delta(y-y') d^4y' \right)_{J=0}$$
$$= iD_F^{\mu\nu}(x-y). \qquad (43)$$

Thus, from (40) and (41),

$$G_0^{\mu\nu}(x-y) = \left(\frac{1}{i}\right)^2 \frac{\delta^2 Z_0[J_k]}{\delta J_\mu(x) \delta J_\nu(y)}\Bigg|_0 = iD_F^{\mu\nu}(x-y) + (\text{higher order terms}) \qquad (44)$$

Result (44) shows that the usual Green function of mathematics for Maxwell's equation (11) equals the lowest order Green function from the QFT Green function methodology.

Generalizing Free Field Photon Propagator to Include Fermions

The above procedure for photons can be carried out in parallel for fermions. (We need to use classical Grassman source fields σ and $\bar{\sigma}$ instead of the classical photon source field J^μ, as we did above. Grassman variables do not commute like ordinary variables [such as the more usual classical fields] do, and there is a fair amount of study involved in learning about their algebraic and calculus related behavior. We do not do this here.) Carrying out the above procedure for Grassman source fields, and incorporating the result with what we obtained in (39) for photon source fields, we find

$$Z_0[J_k] \to Z_0[J_k, \sigma, \bar{\sigma}] = Z_0[J_k] Z_0[\sigma, \bar{\sigma}] = e^{-\frac{i}{2}\left[J_\alpha D_F^{\alpha\rho} J_\rho\right]} e^{-\frac{i}{2}\left[\sigma S_F \bar{\sigma}\right]}. \qquad (45)$$

Using (45) in (32) with only Grassman field derivatives (and all $J_\kappa = 0$), would give us a Green function equivalent to the free fermion propagator.

Appendix A. Some Background for (14)

To visualize what is happening in (14) (repeated below),

$$\text{position space } \; G^{\mu....}(x,...y,..\,z,...)=\frac{\langle 0|T\left\{SA^{\mu}(x)...\psi(y)....\bar{\psi}(z)...\right\}|0\rangle}{\langle 0|S|0\rangle} \qquad M\,\&\,S\,(12.8)\;\; pg\,245\,, \qquad (14)$$

where the bras and kets are all vacuum states at $t = -\infty$, consider the following expression, which is the same as the numerator of (14) except the bra is at $t = +\infty$,

$$\langle 0,t=+\infty|T\left\{SA^{\mu}(x)...\psi(y)....\bar{\psi}(z)...\right\}|0,t=-\infty\rangle\,. \qquad (46)$$

We know, since the entire S ($= U(+\infty, -\infty)$) operator takes a given state to its final state, and the vacuum at $t = -\infty$ can only vary from the vacuum at $t = +\infty$, by a phase factor,

$$\text{kets }\; |0,t=+\infty\rangle = e^{i\phi}|0,t=-\infty\rangle \qquad \text{not relevant here, but noted } \rightarrow \langle 0,t=+\infty|S|0,t=-\infty\rangle = 1$$

$$\text{bras }\; \langle 0,t=+\infty| = e^{-i\phi}\langle 0,t=-\infty|$$

$$\underbrace{\langle 0,t=-\infty|S|0,t=-\infty\rangle}_{\text{use this}} = \langle 0,t=-\infty||0,t=+\infty\rangle = \langle 0,t=-\infty|e^{i\phi}|0,t=-\infty\rangle = e^{i\phi} = \frac{1}{e^{-i\phi}} = \underbrace{\frac{\langle 0,t=-\infty|}{\langle 0,t=+\infty|}}_{\text{with this}} \qquad (47)$$

$$\langle 0,t=+\infty| = \frac{\langle 0,t=-\infty|}{\langle 0,t=-\infty|S|0,t=-\infty\rangle}$$

Using the last line of (47) in (46) yields (14). So (46) is the same as (14), i.e.,

$$\begin{aligned} G^{\mu....}(x,...y,..\,z,...) &= \frac{\langle 0,t=-\infty|T\left\{SA^{\mu}(x)...\psi(y)....\bar{\psi}(z)...\right\}|0,t=-\infty\rangle}{\langle 0,t=+\infty|S|0,t=-\infty\rangle} \\ &= \langle 0,t=-\infty|T\left\{SA^{\mu}(x)...\psi(y)....\bar{\psi}(z)...\right\}|0,t=-\infty\rangle\,. \end{aligned} \qquad (48)$$

To visualize what (14), the first line of (48), means, it is easier to use the second line of (48),which acts more like the amplitudes we are familiar with. That is, it takes an initial state at an earlier time ($t = -\infty$, here) to a later time ($t = +\infty$, here).

In earlier work, to find an amplitude, we generally started with a state of certain particles, which was acted on by the operators in S to produce a final state of particles. The time ordered S operator worked fine for that.

Here (2^{nd} row of (48)), on the other hand, we start with a vacuum state at $t = -\infty$ and end with a vacuum state at $t = +\infty$. So to get anything meaningful, we have to create some particles. The fields $A^{\mu}(x)...\psi(y)....\bar{\psi}(z)...$ can be used for that. Which ones we use depends on the problem at hand. If we want to examine a problem with an initial electron and positron, we would use $\bar{\psi}(z)$ to create an electron at z and $\psi(y)$ to create a positron at y.

Then the S operator would take those fields forward in time and cause them to interact (e.g., to annihilate creating a virtual photon which then transforms into an outgoing electron and positron, as in Bhabha scattering). But then we have to get back to the vacuum again (at $t = +\infty$). So we use additional fields from $A^{\mu}(x)...\psi(y)....\bar{\psi}(z)...$ to destroy the resulting electron and positron and leave the vacuum. In our example, we would have a field therein of form $\bar{\psi}(z')$ to destroy the positron at z' and $\psi(y')$ to destroy the electron at y'.

But note the S operator can be expressed as

$$S = U\left(+\infty, -\infty\right) = U\left(+\infty, y'\right)U\left(y', z'\right)U\left(z', y\right)U\left(y, z\right)U\left(z, -\infty\right). \qquad (49)$$

Since S and the other operators in the 2nd row of (48) are time ordered, each of the U operators in S of (48) operates in the appropriate time sequence. In our prior example, $U(z, -\infty)$ takes the initial vacuum to the time at z, where $\bar{\psi}(z)$ creates an electron. $U(y, z)$ then takes that state to y, where $\psi(y)$ creates a positron (y and z could be the same 4D point), etc. until we arrive at the final vacuum state.

The first row of (48) is just a different mathematical way to express the amplitude associated with this unfolding of events. But in it, the final bra is different in that it represents the identical vacuum state of the ket, both being at $t = -\infty$.

Definition note: The fields in $A^{\mu}\left(x\right)...\psi\left(y\right)....\bar{\psi}\left(z\right)...$ are called "legs". For reasons that are obvious when one sees they show up as, i.e., correspond to, external particles in Green function diagrams.

Additional note: Unless the final and initial states are equal (48) will equal zero. We can only get the same final and initial vacuum states in the first row of (48) if we have the same number of fields $\bar{\psi}$ fields as ψ fields. Since in QED, $\mathcal{L}_I = e\bar{\psi}\not{A}^{\mu}\psi$, S will always contain the same number of each, so we must have the same number of each in $A^{\mu}\left(x\right)...\psi\left(y\right)....\bar{\psi}\left(z\right)...$, i.e., in the legs.

Since there is only one photon field in $\mathcal{L}_I$, one could consider that any number of photon fields could exist as legs, i.e., in $A^{\mu}\left(x\right)...\psi\left(y\right)....\bar{\psi}\left(z\right)...$. However, as Mandl and Shaw show on pg. 247 (last paragraph), for the special case with no fermion legs, an odd number of photon legs, (48) will vanish. So for the case of no fermion legs, there must be an even number of photon field legs.

Appendix B: Exponential Expansion of a Functional

Note first the streamlined notation [*AKB*]

$$[AKB] = \iint A(x)K(x,y)B(y)d^4xd^4y \qquad (50)$$

For *A*=*B*= the field source J_σ (with different dummy subscript) in the source Lagrangian $\mathcal{L}_S$ of (27), and *K* = the no-loop photon propagator $D_F^{\sigma\rho}$, (50) becomes the special case

$$\left[J_\sigma D_F^{\sigma\rho} J_\rho\right] = \iint J_\sigma(x) D_F^{\sigma\rho}(x-y) J_\rho(y) d^4xd^4y\,. \qquad (51)$$

The entire purpose of this appendix is to show

$$e^{-\frac{i}{2}\left[J_\mu D_F^{\mu\nu} J_\nu\right]} = 1 - \frac{i}{2}\left[J_\mu D_F^{\mu\nu} J_\nu\right] + \text{........}$$

or

$$e^{-\frac{i}{2}\iint J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y) d^4xd^4y} = 1 - \frac{i}{2}\iint J_\mu(x) D_F^{\mu\nu}(x-y) J_\nu(y) d^4xd^4y + \text{........} \qquad (52)$$

Note how (52) parallels the elementary calculus relation

$$e^x = 1 + x + \text{.......} \qquad (53)$$

Relation (52) is all you need to remember from this appendix. The proof follows

Proof of (52).

Recall (28),

$$\frac{\delta J_\sigma(x'')}{\delta J_\mu(x)} = \delta_\sigma^\mu \delta^{(4)}(x''-x) \qquad M \,\&\, S(12.55)\;\; pg\, 259\,, \qquad (28)$$

Also, recall that for a simple function *g*(*x*),

$$g(x) = g(0) + \left.\frac{dg(x)}{dx}\right|_{x=0} x + \frac{1}{2}\left.\frac{d^2g(x)}{dx^2}\right|_{x=0} x^2 + ... = g(0) + g'(0)x + \frac{1}{2}g''(0)x^2 + ... \qquad (54)$$

For $g(x) = e^{f(x)} = e^{f(0)} + f'(0)e^{f(0)}x + \frac{1}{2}\left(f''(0)e^{f(0)} + f'(0)f'(0)e^{f(0)}\right)x^2 + ...$

One might expect an analogous relation to hold for functionals, as follows, where the primes indicate functional derivatives with respect to *J*,

$$\text{For} \quad e^{F[J]} = e^{F[0]} + F'(0)e^{F[0]}J + \frac{1}{2}\left(F''(0)e^{F[0]} + F'(0)F'(0)e^{F[0]}\right)J^2 + ... \qquad (\text{might expect})\,. \quad (55)$$

However, since *F*[*J*] is an integral over spacetime and is a number, the LHS of (55) is a number, not a function of space. But each *J* in (55), would have a form such as $J_\mu(x)$ or $J_\nu(y)$, i.e., it would be function of spacetime position. Thus the RHS of (55), as written, would be a function of spacetime position (*x*, *y*,.. for example). But it can't be, since the LHS is not a function, but a pure number. From this, we can glean that the terms in (55) must be included inside the integrals over *x*, *y*, etc.

Thus, the correct form for the expansion of $e^{F[J]}$ is

$$e^{F[J]} = e^{F[0]} + \underbrace{\int \left(e^{F[J]} \frac{\delta}{\delta J_\mu(x)} F(J) \right)_{J=0} J_\mu(x) d^4x}_{\text{Term A}}$$

$$+ \underbrace{\tfrac{1}{2} \iint \left(e^{F[J]} \frac{\delta}{\delta J_\mu(x)} \frac{\delta}{\delta J_\nu(y)} F(J) \right)_{J=0} J_\mu(x) J_\nu(y) d^4x d^4y}_{\text{Term B}} \qquad \begin{pmatrix}\text{actual form} \\ \text{of expansion}\end{pmatrix} \quad (56)$$

$$+ \underbrace{\tfrac{1}{2} \iint \left(\left(e^{F[J]} \frac{\delta}{\delta J_\mu(x)} F(J) \right) \left(\frac{\delta}{\delta J_\nu(y)} F(J) \right) \right)_{J=0} J_\mu(x) J_\nu(y) d^4x'' d^4x'}_{\text{Term C}} + \ldots.$$

As an example of the application of (56), we examine a form for F we will find very useful. That is the functional we looked at in (51) above (with different dummy integration variables that will help in the proof),

$$F[J] = -\frac{i}{2}\left[J_\sigma D_F^{\sigma\rho} J_\rho \right] = -\frac{i}{2} \iint J_\sigma(x'') D_F^{\sigma\rho}(x''-x') J_\rho(x') d^4x'' d^4x' . \qquad (57)$$

With (57) in (56), noting that all $F[J]$ where $J = 0$, have $F[J] = 0$, and thus $e^{F[J]}\Big|_{J=0} = e^{F[0]} = e^0 = 1$, we find

$$e^{F[J]} = e^{-\frac{i}{2}\left[J_\sigma D_F^{\sigma\rho} J_\rho \right]} = e^{-\frac{i}{2}\iint J_\sigma(x'') D_F^{\sigma\rho}(x''-x') J_\rho(x') d^4x'' d^4x'}$$

$$= 1 + \underbrace{\int \left(\frac{\delta}{\delta J_\mu(x)} \overbrace{\left(-\frac{i}{2} \right) \iint d^4x'' d^4x' J_\sigma(x'') D_F^{\sigma\rho}(x''-x') J_\rho(x')}^{F[J]} \right)_{J=0} J_\mu(x) d^4x}_{\text{Term } A}$$

$$+ \underbrace{\frac{1}{2} \iint \left(\frac{\delta^2}{\delta J_\mu(x) \delta J_\nu(y)} \overbrace{\left(-\frac{i}{2} \right) \iint d^4x'' d^4x' J_\sigma(x'') D_F^{\sigma\rho}(x''-x') J_\rho(x')}^{F[J]} \right)_{J=0} J_\mu(x) J_\nu(y) d^4x d^4y}_{\text{Term } B}$$

$$+ \underbrace{\frac{1}{2} \iint \left(\frac{\delta}{\delta J_\mu(x)} \overbrace{\left(-\frac{i}{2} \right) \iint d^4x'' d^4x' J_\sigma(x'') D_F^{\sigma\rho}(x''-x') J_\rho(x')}^{F[J]} \right)_{J=0} \times \left(\frac{\delta}{\delta J_\nu(y)} \overbrace{\left(-\frac{i}{2} \right) \iint d^4x'' d^4x' J_\sigma(x'') D_F^{\sigma\rho}(x''-x') J_\rho(x')}^{F[J]} \right)_{J=0} J_\mu(x) J_\nu(y) d^4x d^4y}_{\text{Term C}} \qquad (58)$$

$$+ \ldots\ldots ,$$

Carrying out the functional derivatives for each term above using (28), we get the following.

Term A

$$\text{Term } A=\int\left(\frac{\delta}{\delta J_\mu(x)}\overbrace{\left(-\frac{i}{2}\right)\iint J_\sigma(x'')D_F^{\sigma\rho}(x''-x')J_\rho(x')d^4x''d^4x'}^{F[J]}\right)_{J=0}J_\mu(x)d^4x$$

$$=\left(-\frac{i}{2}\right)\int\left(\iint\left(\delta_\sigma^\mu\delta^{(4)}(x-x'')\right)D_F^{\sigma\rho}(x''-x')J_\rho(x')d^4x''d^4x'\right)_{J=0}J_\mu(x)d^4x$$

$$+\left(-\frac{i}{2}\right)\int\left(\iint J_\sigma(x'')D_F^{\sigma\rho}(x''-x')\left(\delta_\rho^\mu\delta^{(4)}(x-x')\right)d^4x''d^4x'\right)_{J=0}J_\mu(x)d^4x$$

$$=-\frac{i}{2}\int\left(\int D_F^{\mu\rho}(x-x')J_\rho(x')d^4x'+\int J_\sigma(x'')D_F^{\sigma\mu}(x-x'')d^4x''\right)_{J=0}J_\mu(x)d^4x\quad\left(D_F^{\mu\rho}\text{ sym in }\rho\text{ and }\mu\text{ for next line}\right)\qquad(59)$$

$$=-\frac{i}{2}\int\left(\int D_F^{\rho\mu}(x-x')J_\rho(x')d^4x'+\int J_\sigma(x'')D_F^{\sigma\mu}(x-x'')d^4x''\right)_{J=0}J_\mu(x)d^4x\quad\left(\begin{array}{l}\text{change dummy indices}\\ \rho\to\nu,\ \sigma\to\nu;\text{ dummy}\\ \text{variable }x''\to x'\text{ for next line}\end{array}\right)$$

$$=-\frac{i}{2}\int\left(\int J_\nu(x')D_F^{\nu\mu}(x-x')d^4x'+\int J_\nu(x')D_F^{\nu\mu}(x-x')d^4x'\right)_{J=0}J_\mu(x)d^4x$$

$$=-i\int\left(\int J_\nu(x')D_F^{\nu\mu}(x-x')d^4x'\right)_{J=0}J_\mu(x)d^4x.$$

$$\begin{aligned}\text{Term } A&=\left(-i\iint J_\nu(x')D_F^{\nu\mu}(x-x')d^4xd^4x'\right)_{J=0}J_\mu(x)\\&=0,\quad\text{since}\quad J_\nu(x')=0.\end{aligned}\qquad(60)$$

Term C

From (59) and (60), we should be able to see right away (I hope) that both factors in Term C of (58) equal zero. So the only thing we have to worry about (at least to lowest order) is Term B.

Term B

$$\text{Term } B=\frac{1}{2}\iint\left(\frac{\delta^2}{\delta J_\mu(x)\delta J_\nu(y)}\overbrace{\left(-\frac{i}{2}\right)\iint d^4x''d^4x'J_\sigma(x'')D_F^{\sigma\rho}(x''-x')J_\rho(x')}^{F[J]}\right)_{J=0}J_\mu(x)J_\nu(y)d^4xd^4y$$

$$(\text{from term } A)=-\frac{i}{2}\iint\left(\frac{\delta}{\delta J_\nu(y)}\int D_F^{\mu\rho}(x''-x')J_\rho(x')d^4x''d^4x'\right)_{J=0}J_\mu(x)J_\nu(y)d^4xd^4y$$

$$=-\frac{i}{2}\iint\left(\int D_F^{\mu\rho}(x-x')\delta_\rho^\nu\delta^{(4)}(y-x')d^4x'\right)_{J=0}J_\mu(x)J_\nu(y)d^4xd^4y\qquad(61)$$

integrate over x',

$$=-\frac{i}{2}\iint\left(D_F^{\mu\nu}(x-y)\right)_{J=0}J_\mu(x)J_\nu(y)d^4xd^4y.$$

Returning to (58), we see

$$\begin{aligned}e^{-\frac{i}{2}\iint J_\mu(x)D_F^{\mu\nu}(x-y)J_\nu(y)d^4xd^4y}&=1-\frac{i}{2}\iint J_\mu(x)D_F^{\mu\nu}(x-y)J_\nu(y)d^4xd^4y\ +........\\ \text{or}\qquad e^{-\frac{i}{2}\left[J_\mu D_F^{\mu\nu}J_\nu\right]}&=1-\frac{i}{2}\left[J_\mu D_F^{\mu\nu}J_\nu\right]\ +........\end{aligned}\qquad(62)$$

as we stated in (52).

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