Gravity and E/M Units in Higher Dimensions

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Gravity

	D = 4				D ≥ 4		
	$d = \mathrm{D} - 1 = 3$				$d = \mathrm{D} - 1 \geq 3$ $\quad c_d = \dfrac{\Gamma(d/2)}{2\pi^{d/2}}$		
	Quantity	**Units**	**Point Source**		**Quantity**	**Units**	**Point Source**
	$\mathbf{F} = m\mathbf{g}$	F (dynes)	$F = -\dfrac{GmM}{r^2}$		$\mathbf{F}^{(D)} = m\mathbf{g}^{(D)}$	Same as 4D	$F^{(D)} = -c_d \dfrac{G^{(D)}mM}{r^{d-1}}$
	$G = \dfrac{Fr^2}{mM}$	$\dfrac{F \cdot l^2}{m^2} \left(\dfrac{\text{dynes} \cdot l^2}{\text{g}^2} = \dfrac{\text{ergs} \cdot \text{cm}}{\text{g}^2} \right)$	N/A		$G^{(D)} = \dfrac{Fr^{d-1}}{mM} = Gr^{d-3}$	$\dfrac{F \cdot l^{d-1}}{m^2} \left(\dfrac{\text{dynes} \cdot l^{d-1}}{\text{g}^2} = \dfrac{\text{ergs} \cdot \text{cm}^{d-2}}{\text{g}^2} \right)$	N/A
	M	m (g)	“		M	Same as 4D	“
	test mass m	m (g)	“		test mass m	Same as 4D	“
	$\mathbf{g} = \dfrac{\mathbf{F}}{m} = -\nabla V_g$	$\dfrac{F}{m} = \dfrac{Energ}{m \cdot l} \left(\dfrac{\text{dynes}}{\text{g}} = \dfrac{\text{ergs}}{\text{g} \cdot \text{cm}} \right)$	$\mathbf{g} = -\dfrac{GM}{r^2} \dfrac{\mathbf{r}}{\lvert\mathbf{r}\rvert}$		$\mathbf{g}^{(D)} = \dfrac{\mathbf{F}}{m} = -\nabla V_g^{(D)}$	Same as 4D	$\mathbf{g}^{(D)} = -c_d \dfrac{G^{(D)}M}{r^{d-1}} \dfrac{\mathbf{r}}{\lvert\mathbf{r}\rvert}$
	V_g	$\dfrac{E}{m} \left(\dfrac{\text{ergs}}{\text{g}} \right)$	$V_g = -\dfrac{GM}{r}$		$V_g^{(D)}$	Same as 4D	$V_g^{(D)} = -\dfrac{c_d}{d-2} \dfrac{G^{(D)}M}{r^{d-2}}$
Newton grav law	$\nabla \bullet \mathbf{g} = -4\pi G \rho_m$	$\dfrac{F}{m \cdot l} \left(\dfrac{\text{dynes}}{\text{g} \cdot l} = \dfrac{\text{ergs}}{\text{g} \cdot \text{cm}^2} \right)$	$\nabla \bullet \mathbf{g} = -4\pi GM \delta^{(3)}(\mathbf{r})$		$\nabla \bullet \mathbf{g}^{(D)} = -4\pi G^{(D)} \rho_m^{(D)}$	Same as 4D	$\nabla \bullet \mathbf{g}^{(D)} = -4\pi G^{(D)} M \delta^{(d)}(\mathbf{r})$
	$\nabla^2 V_g = 4\pi G \rho_m$	“	$\nabla^2 V_g = 4\pi GM \delta^{(3)}(\mathbf{r})$		$\nabla^2 V_g^{(D)} = 4\pi G^{(D)} \rho_m^{(D)}$	Same as 4D	$\nabla^2 \mathbf{g}^{(D)} = 4\pi G^{(D)} M \delta^{(d)}(\mathbf{r})$
	ρ_m	$\dfrac{m}{l^3} \left(\dfrac{\text{g}}{\text{cm}^3} \right)$	$M\delta(r)$		$\rho_m^{(D)}$	$\dfrac{m}{l^d} \left(\dfrac{\text{g}}{\text{cm}^d} \right)$	$M\delta^{(3)}(\mathbf{r})$

For each spatial dimension greater than 3 (D greater than 4), the units of $G^{(D)}$ increase by a factor of length (cm) and the units of mass density are reduced by a factor of length (cm) in the denominator. All other quantities keep the same units as in $3d$ (= $4D$).

Electromagnetism

	D = 4				**D ≥ 4**		
	d **= D – 1 = 3**				d **= D – 1 ≥ 3** $\quad c_d = \dfrac{\Gamma(d/2)}{2\pi^{d/2}}$		
	Quantity	**Units**	**Point Source**		**Quantity**	**Units**	**Point Source**
	$\mathbf{F} = q\mathbf{E}$	F (dynes)	$\mathbf{F} = \dfrac{qQ}{4\pi r^2}\dfrac{\mathbf{r}}{\lvert\mathbf{r}\rvert}$		$\mathbf{F}^{(D)} = q\mathbf{E}^{(D)}$	$\mathbf{F}^{(D)}$ same as 4D	$\mathbf{F}^{(D)} = c_d \dfrac{qQ}{r^{d-1}}\dfrac{\mathbf{r}}{\lvert\mathbf{r}\rvert}$
	No constant in e/m comparable to G in gravity				No constant in e/m like G in gravity. So, units of q (and Q) must change		
	Q	q (esu)	N/A		$Q^{(D)}$	To cancel r factors, $q^{(D)}Q^{(D)} = (\text{esu})^2 r^{d-3}$ $q^{(D)}$ units $\rightarrow q\sqrt{l^{d-3}} = \text{esu}\cdot\text{cm}^{(d-3)/2}$	N/A
	test charge q	q (esu)	"		test charge $q^{(D)}$	"	"
	$\mathbf{E} = \dfrac{\mathbf{F}}{q} = -\nabla\Phi$	$\dfrac{F}{q} = \dfrac{Energ}{q\cdot l}\left(\dfrac{\text{dynes}}{\text{esu}} = \dfrac{\text{ergs}}{(\text{esu})\cdot\text{cm}}\right)$	$\mathbf{E} = \dfrac{Q}{4\pi r^2}\dfrac{\mathbf{r}}{\lvert\mathbf{r}\rvert}$		$\mathbf{E}^{(D)} = \dfrac{\mathbf{F}}{q^{(D)}} = -\nabla\Phi^{(D)}$	$\dfrac{F}{q^{(D)}} = \dfrac{Energ}{q^{(D)}\cdot l}\left(\dfrac{\text{dynes}}{\text{esu}\cdot\text{cm}^{(d-3)/2}}\right)$	$\mathbf{E}^{(D)} = -c_d\dfrac{Q}{r^{d-1}}\dfrac{\mathbf{r}}{\lvert\mathbf{r}\rvert}$
	Φ	$\dfrac{Energ}{q}\left(\dfrac{\text{ergs}}{\text{esu}}\right)$	$\Phi = \dfrac{Q}{4\pi r}$		$\Phi^{(D)}$	$\dfrac{Energ}{q^{(D)}} = \left(\dfrac{\text{ergs}}{\text{esu}\cdot\text{cm}^{(d-3)/2}}\right)$	$\Phi^{(D)} = \dfrac{c_d}{d-2}\dfrac{Q}{r^{d-2}}$
Max eq	$\nabla\bullet\mathbf{E} = \rho$	$\dfrac{F}{q\cdot l}\left(\dfrac{\text{dynes}}{\text{esu}\cdot l} = \dfrac{\text{ergs}}{\text{esu}\cdot\text{cm}^2}\right)$	$\nabla\bullet\mathbf{E} = Q\,\delta^{(3)}(\mathbf{r})$		$\nabla\bullet\mathbf{E}^{(D)} = \rho^{(D)}$	$\dfrac{Energ}{q^{(D)}\cdot l^2} =$ $\left(\dfrac{\text{ergs}}{\text{esu}\cdot\text{cm}^{(d-3)/2}\cdot\text{cm}^2} = \dfrac{\text{ergs}}{\text{esu}\cdot\text{cm}^{(d+1)/2}}\right)$	$\nabla\bullet\mathbf{E}^{(D)} = Q\,\delta^{(d)}(\mathbf{r})$
	$\nabla^2\Phi = -\rho$	"	$\nabla^2\Phi = -Q\delta^{(3)}(\mathbf{r})$		$\nabla^2\Phi^{(D)} = \rho^{(D)}$	This is same as above and (in different, but equivalent, units) below.	$\nabla^2\Phi^{(D)} = -Q\,\delta^{(d)}(\mathbf{r})$
	ρ	$\dfrac{Q}{l^3}\left(\dfrac{\text{esu}}{\text{cm}^3}\right)$	$-Q\,\delta(r)$		$\rho^{(D)}$	$\dfrac{q^{(D)}}{l^d}\left(\text{esu}\cdot\text{cm}^{(d-3)/2}\dfrac{1}{\text{cm}^d} = \dfrac{\text{esu}}{\text{cm}^{(d+3)/2}}\right)$	$Q\,\delta^{(d)}(\mathbf{r})$

In the force expressions (see first and last rows for both type forces) in D > 4, we get extra factors of r in the denominator on the RHS. In gravity theory, we can absorb those dimensions into the definition of $G^{(D)}$. In doing this, units for all other quantities, like potential, force per unit mass, ρ_m, and M remain unchanged.

In e/m, there is no constant in the force expressions like G in gravity. If we assume the units for force do not change in higher dimensions in e/m, as it is in gravity, we have to absorb these extra units into our higher dimension definition of charge

Since there are two factors of charge in the last block of the first row in the e/m chart above, the charge units must be $(\text{esu})\cdot\sqrt{\text{cm}^{(d-3)}} = (\text{esu})\cdot\text{cm}^{(d-3)/2}$. Since Q appears in all other relations as a single factor, all of those other relations must change units from what they had in 4D. Charge density, for example, is no longer esu divided by (higher D) volume, so **E** and Φ units are weird, as well. (Mass density in gravity however scales directly with higher D volume alone.

For each extra dimension above $d = 3$, charge units increase by a factor of the square root of length, in cgs, $\text{cm}^{1/2}$.

On the other hand, if we had wanted to keep Φ (and thus **E**) as having the same units, then we need to define charge units differently than we did above, and then **F** units are weird.