Generators of Translation, Rotation, and Boost

See Zwiebach, Sects. 11.5 and 11.6, pgs 226-233 and Klauber, Vol. 2, Wholeness Charts 2-2, pgs. 20-21; 6-3, pg. 172 Robert D. Klauber Jan 26, 2023

	Translation		**Lorentz Transformations (Rotations and Boosts)**	
	Particle – 1 Spatial D	**Field – 4*D***	**Particle – 4*D***	**Field – 4*D***
Dependent variable	$x=x(t)$ $p=p(t)$	$\phi^\mu=\phi^\mu(t,\mathbf{x})$ $\pi^\nu=\pi^\nu(t,\mathbf{y})$	$x^\mu=x^\mu(\tau)$ $p^\nu=p^\nu(\tau)$	$\phi^\mu=\phi^\mu(t,\mathbf{x})$ $\pi^\nu=\pi^\nu(t,\mathbf{x})$
Classical		$\delta=\delta(\mathbf{x}-\mathbf{y})$ below to save space		
Poisson brackets	$\{u,v\}=\left\{\frac{\partial u}{\partial x}\frac{\partial v}{\partial p}-\frac{\partial u}{\partial p}\frac{\partial v}{\partial x}\right\}$	$\{u,v\}=\left\{\frac{\partial u}{\partial\phi^r}\frac{\partial v}{\partial\pi_r}-\frac{\partial u}{\partial\pi_r}\frac{\partial v}{\partial\phi^r}\right\}\delta$	As 2 columns to left	As 2 columns to left
Special case	$\{x,p\}=\left\{\frac{\partial x}{\partial x}\frac{\partial p}{\partial p}-\frac{\partial x}{\partial p}\frac{\partial p}{\partial x}\right\}=1$	$\{\phi^s,\pi_t\}=\left\{\frac{\partial\phi^s}{\partial\phi^r}\frac{\partial\pi_t}{\partial\pi_r}-\frac{\partial\phi^s}{\partial\pi_r}\frac{\partial\pi_t}{\partial\phi^r}\right\}\delta$ $=\delta^s_r\delta^r_t\delta(\mathbf{x}-\mathbf{y})=\delta^s_t\delta(\mathbf{x}-\mathbf{y})$	$M^{\mu\nu}=x^\mu p^\nu-x^\nu p^\mu$ Lorentz charges, Zwiebach (11.76), x^μ and p^ν satisfy {*x*,*p*} relations	$\mathcal{M}^{\mu\nu}=\phi^\mu\pi^\nu-\phi^\nu\pi^\mu$ $M^{\mu\nu}=\int\mathcal{M}^{\mu\nu}d\mathbf{x}$ ϕ^μ and π^ν satisfy $\{\phi^\mu,\pi_\nu\}$ relations
Transformation	$x'=x+\varepsilon$ $\delta x=\varepsilon$	$\phi'^\mu=\phi^\mu+\varepsilon^\mu$ $\delta\phi^\mu=\varepsilon^\mu$	$x'^\mu=\Lambda^\mu_\nu x^\nu$ small $\to\delta x^\mu=\varepsilon^{\mu\nu}x_\nu$	$\phi'^\mu(x'^\alpha)=\Lambda^\mu_\nu\phi^\nu(\Lambda^\alpha_\beta x^\beta)\to\delta\phi^\mu=\varepsilon^{\mu\nu}$
Via Poisson brackets	$\delta x=\{x,\varepsilon p\}=\varepsilon\{x,p\}=\varepsilon$	$\delta\phi^\mu=\int\{\phi^\mu,\varepsilon^\nu\pi_\nu\}d\mathbf{y}=\varepsilon^\nu\delta^\mu_\nu=\varepsilon^\mu$	$\delta x^\mu=\left\{x^\mu,-\frac{1}{2}\varepsilon_{\alpha\beta}M^{\alpha\beta}\right\}=\varepsilon^{\mu\nu}x_\nu$	$\delta\phi^\mu=\int\left\{\phi^\mu,-\frac{1}{2}\varepsilon_{\alpha\beta}M^{\alpha\beta}\right\}d\mathbf{y}=\varepsilon^{\mu\nu}\phi_\nu$
Transform operator	p (via Poisson bracket)	π_ν (via Poisson bracket)	$-\frac{1}{2}M^{\alpha\beta}$ (via Poisson bracket)	$-\frac{1}{2}M^{\alpha\beta}$ (via Poisson bracket)
Quantum				
Commutators	$[x,p]=i$ $(\hbar=1)$	$[\phi^\mu,\pi_\nu]=i\delta^\mu_\nu\delta\to[\phi^\mu,\pi^\nu]=ig^{\mu\nu}\delta$	As 2 columns to left	As 2 columns to left
Transformation	$x'=x+\varepsilon$ $\delta x=\varepsilon$	As at left for x^μ indep variable	$x'^\mu=\Lambda^\mu_\nu x^\nu$ small $\to\delta x^\mu=\varepsilon^{\mu\nu}x_\nu$	As at left for x^μ indep variable
Via commutators	$\delta x=[x,-i\varepsilon p]=-i\varepsilon[x,p]=\varepsilon$	N/A: x^μ indep, not dep variable	$\delta x^\mu=\left[x^\mu,-\frac{1}{2}\varepsilon_{\alpha\beta}M^{\alpha\beta}\right]=\varepsilon^{\mu\nu}x_\nu$ Zwiebach (11.79)	N/A: x^μ indep, not dep variable
Transform operator	$p=-i\frac{d}{dx}$ (via commutator)	N/A for x^μ	$-\frac{1}{2}M^{\alpha\beta}$ (via commutator)	N/A for x^μ
ϕ transformation	$\phi'(t,x)=\phi(t,x+\varepsilon)$ $\phi=Ae^{-i(Et-px)}$ $\to\phi'=Ae^{-i(Et-p(x+\varepsilon))}$	$\phi'^\mu=\phi^\mu+\varepsilon^\mu=\phi^\mu+\delta\phi^\mu$ $\delta\phi^\mu=\int[\phi^\mu,-i\varepsilon^\nu\pi_\nu]d\mathbf{y}=\delta^\mu_\nu\varepsilon^\nu=\varepsilon^\mu$ Mirrors δx for particle	Scalar ϕ unchanged under Lorentz transf. Ditto for $Et-px$	$\phi'^\mu=\phi^\mu+\delta\phi^\mu$ $\delta\phi^\mu=\varepsilon^{\mu\nu}\phi_\nu$ $\delta\phi^\mu=\int\left[\phi^\mu,-\frac{1}{2}\varepsilon_{\alpha\beta}M^{\alpha\beta}\right]d\mathbf{y}=\varepsilon^{\mu\nu}\phi_\nu$
ϕ transform operator	$T_\varepsilon=e^{i\varepsilon p}$	$-i\pi_\nu$ (via field commutator)	Identity operator	$-\frac{1}{2}M^{\alpha\beta}$ (via field commutator)
	$\phi'(t,x)=T_\varepsilon\phi(t,x)$ $=T_\varepsilon Ae^{-i(Et-px)}=e^{i\varepsilon p}Ae^{-i(Et-px)}$ $=Ae^{-i(Et-p(x+\varepsilon))}=\phi(t,x+\varepsilon)$	N/A	$\phi'(x')=\phi(x)$	N/A

Notes

Recall from Klauber Vol. 1, Chaps. 1 and 2, that a basic postulate for quantization (going from classical theory to quantum theory) is the taking of the classical Poisson brackets over into commutators (with an extra factor of i and $\hbar = 1$). This is what we do in this chart. It is virtually never noted in texts that the generator of translation, so often referred to quantum theory, has a direct analogue in classical theory. The difference is simply that for one we use commutators, and for the other, Poisson brackets. The parallel between the classical and quantum realms extends beyond merely translation to the general Lorentz (including rotation) transformation.

For the 1d (x and t) particle case, in quantum theory, the ϕ translation operator comprises a Lie group, with continuous parameter ε. As there is only one parameter in this case, it is a $U(1)$ group. The operator p is then a generator of the associated Lie algebra.

Note that in 3d, since $[p_i, p_j] = 0$ the Lie algebra generators for different spatial dimensions ($i = 1,2,3$) all commute. So, they don't collectively form a higher degree Lie algebra. There are simply three different, independent $U(1)$ Lie groups/algebras (for translation), each acting on its own without regard to the others. We will see this is not the case for rotation, or for Lorentz boosts. The operators there do not commute, and their non-zero commutation relations lead to higher degree Lie groups.

Similar logic applies to 4D fields in translation. Each of the four components of a field may each be translated independent of the others.

For Lorentz transformations for a particle, there are six independent $M^{\mu\nu}$, three for boosts and three for rotations. $M^{\mu\nu}$ is antisymmetric, so it has 6 independent parameters. Various values for these parameters determine the degree of rotation or boost the particle undergoes during transformation.

We know rotations do not commute, so it should be no surprise that the different components of $M^{\mu\nu}$ do not generally commute. Thus, unlike translation, each $M^{\mu\nu}$ (for given μ and ν) does not form an independent Lie group. The commutation relations between the $M^{\mu\nu}$ give rise to higher degree Lie groups. The rotation subgroup, for example, is $SO(3)$, which should be no surprise. $M^{\mu\nu}$ is Hermitian.

Note that for $i,j = 1,2,3$, $M^{ij} = x^i p^j - x^j p^i$ is angular momentum in the direction perpendicular to the i-j plane.

In the quantum realm, the commutator of $M^{\mu\nu}$ with 4D position vector generates the change in that vector under a Lorentz transformation (including rotations), as shown in the chart.

In Zwiebach (11.80), pg. 230, $\left[M^{\mu\nu}, M^{\rho\sigma}\right] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}$ (which can be proven via substitution). This commutator defines the Lorentz Lie algebra. Any quantum theory one poses must satisfy this commutation relation in order to be Lorentz covariant. The commutator is a constraint any potential theory must meet to be viable.

All of this is background for Zwiebach taking these results into the light-cone gauge and light-cone coordinates. See pg. 4 for the world sheet coordinates as a 4D field dependent on parameters τ,σ on the world sheet.

With specific regard to $M^{\mu\nu}$, in the covariant gauge with light-cone coordinates, the above commutation relation holds, as Zwiebach shows on pg.233. Since the commutator is a 4D covariant relationship, it should, of course, remain valid under a change of coordinates.

In the light -cone gauge, however, one must modify the $M^{\mu\nu}$ carefully in order to have it satisfy the commutator above and also, to be Hermitian (which is necessary for any generator of a Lie Algebra).

Showing Lorentz Transformation Generation via Poisson Bracket

$$\delta x^{\rho} = \varepsilon^{\rho\nu} x_{\nu} \stackrel{?}{=} \left\{ x^{\rho}, -\tfrac{1}{2}\varepsilon_{\mu\nu} M^{\mu\nu} \right\} \tag{1}$$

$$\left\{ x^{\rho}, -\tfrac{1}{2}\varepsilon_{\mu\nu} M^{\mu\nu} \right\} = -\tfrac{1}{2}\varepsilon_{\mu\nu} \left\{ x^{\rho}, \left(x^{\mu} p^{\nu} - x^{\nu} p^{\mu} \right) \right\} \tag{2}$$

$$\begin{aligned}
&= -\tfrac{1}{2}\varepsilon_{\mu\nu} \left(\left\{ x^{\rho}, x^{\mu} p^{\nu} \right\} - \left\{ x^{\rho}, x^{\nu} p^{\mu} \right\} \right) \\
&= -\tfrac{1}{2}\varepsilon_{\mu\nu} \left(\frac{\partial x^{\rho}}{\partial x^{\alpha}} \frac{\partial \left(x^{\mu} p^{\nu} \right)}{\partial p_{\alpha}} - \frac{\partial x^{\rho}}{\partial p_{\alpha}} \frac{\partial \left(x^{\mu} p^{\nu} \right)}{\partial x^{\alpha}} - \frac{\partial x^{\rho}}{\partial x^{\alpha}} \frac{\partial \left(x^{\nu} p^{\mu} \right)}{\partial p_{\alpha}} + \frac{\partial x^{\rho}}{\partial p_{\alpha}} \frac{\partial \left(x^{\nu} p^{\mu} \right)}{\partial x^{\alpha}} \right) \\
&= -\tfrac{1}{2}\varepsilon_{\mu\nu} \left(\delta^{\rho}_{\alpha} x^{\mu} g^{\alpha\nu} - (0) \frac{\partial \left(x^{\mu} p^{\nu} \right)}{\partial x^{\alpha}} - \delta^{\rho}_{\alpha} g^{\alpha\mu} x^{\nu} + (0) \frac{\partial \left(x^{\nu} p^{\mu} \right)}{\partial x^{\alpha}} \right) = -\tfrac{1}{2}\varepsilon_{\mu\nu} \left(x^{\mu} g^{\rho\nu} - g^{\rho\mu} x^{\nu} \right)
\end{aligned} \tag{3}$$

In the row below, we make use of the anti-symmetry of $\varepsilon^{\rho\mu}$.

$$(3) = -\tfrac{1}{2}\varepsilon_{\mu}{}^{\rho} x^{\mu} + \tfrac{1}{2}\varepsilon^{\rho}{}_{\mu} x^{\mu} = -\tfrac{1}{2}\varepsilon^{\mu\rho} x_{\mu} + \tfrac{1}{2}\varepsilon^{\rho\mu} x_{\mu} = \varepsilon^{\rho\mu} x_{\mu} = \varepsilon^{\rho\nu} x_{\nu} \tag{4}$$

Continuation of Chart on Page 1

	Translation			**Lorentz Transformations (Rotations and Boosts)**			
	Particle – 1d	**Field – 4D**		**Particle – 4D**	**Field – 4D**	**World Sheet Coordinates X^μ as Field** **Covariant Fields, Light-Cone Coordinates**	**Light Cone gauge & Coordinates**
Dependent variable	See pg. 1 ↓	See pg. 1 ↓		See pg. 1 ↓	See pg. 1 ↓	$\phi^\mu(t,\mathbf{x})\to X^\mu(\tau,\sigma)$ $\pi^\mu(t,\mathbf{x})\to \mathcal{P}^\nu(\tau,\sigma)$	As at left, but light-cone gauge
Classical							
Poisson brackets						Like 1st & 3rd columns to left: $\{u,v\}=\left\{\frac{\partial u}{\partial X^r}\frac{\partial v}{\partial \mathcal{P}_r}-\frac{\partial u}{\partial \mathcal{P}_r}\frac{\partial v}{\partial X^r}\right\}\delta(\sigma'-\sigma)$	
Special case						$\mathcal{M}^{\mu\nu}=X^\mu\mathcal{P}^\nu-X^\nu\mathcal{P}^\mu$ $M^{\mu\nu}=\int\mathcal{M}^{\mu\nu}d\mathbf{x}$ X^μ and $\mathcal{P}^\nu$ satisfy $\{X^\mu,\mathcal{P}_\nu\}$ Poisson bracket relations	Must modify definition of Lorentz generators $M^{\mu\nu}$ to keep correct commutation relations for $M^{\mu\nu}$ (which are needed to keep Lorentz invariance)
Transformation						$X'^\mu(x'^\alpha)=\Lambda^\mu_\nu X^\nu\left(\Lambda^\alpha_\beta x^\beta\right)\to\delta X^\mu=\varepsilon^{\mu\nu}X_\nu$	The above restricts the theory to D=26 and leads to unstable tachyon scalars.
Via Poisson brackets						$\delta X^\mu=\int\left\{X^\mu,-\frac{1}{2}\varepsilon_{\alpha\beta}M^{\alpha\beta}\right\}d\mathbf{y}=\varepsilon^{\mu\nu}X_\nu$	See Zwiebach pgs. 260-262.
Transform operator						$-\frac{1}{2}M^{\alpha\beta}$ (via Poisson bracket)	
Quantum							
Commutators						Like 1st & 3rd columns to left: $\left[X^\mu(\tau,\sigma),\mathcal{P}_\nu(\tau,\sigma')\right]=i\delta^\mu_\nu\delta(\sigma-\sigma')$ $\to\left[X^\mu,\mathcal{P}^\nu\right]=ig^{\mu\nu}\delta(\sigma-\sigma')$	
Transformation						Similar to left for τ,σ indep variables, but τ & σ transfs not discussed	
Via commutators						N/A: τ,σ indep, not dep variable	
Transform operator						N/A for τ,σ	
ϕ (=X^μ here) transformtn						$X'^\mu(x'^\alpha)=\Lambda^\mu_\nu X^\nu\left(\Lambda^\alpha_\beta x^\beta\right)\to\delta X^\mu=\varepsilon^{\mu\nu}X_\nu$ $\delta X^\mu=\int\left[X^\mu,-\frac{1}{2}\varepsilon_{\alpha\beta}M^{\alpha\beta}\right]d\sigma'=\varepsilon^{\mu\nu}X_\nu$	
ϕ transform operator						$-\frac{1}{2}M^{\alpha\beta}$ (via field commutator)	
						N/A	