Proof of the original Ward identity

Proof of original Ward identity

From
$$\left(S_F(p)\right)^{-1} = \not{p} - m \tag{13-20}$$
we find
$$\begin{aligned} 0 &= \frac{\partial(1)}{\partial p_\eta} = \frac{\partial}{\partial p_\eta}\left(\left(S_F(p)\right)\left(S_F(p)\right)^{-1}\right) = \frac{\partial}{\partial p_\eta}\left(\left(S_F(p)\right)\left(\not{p}-m\right)\right) \\ &= \frac{\partial S_F(p)}{\partial p_\eta}\left(\not{p}-m\right) + S_F(p)\frac{\partial}{\partial p_\eta}\left(\not{p}-m\right) = \frac{\partial S_F(p)}{\partial p_\eta}\left(S_F(p)\right)^{-1} + S_F(p)\gamma^\eta . \end{aligned} \tag{13-21}$$
Or
$$\frac{\partial S_F(p)}{\partial p_\eta}\left(S_F(p)\right)^{-1} = -S_F(p)\gamma^\eta \;\rightarrow\; \frac{\partial S_F(p)}{\partial p_\eta} = -S_F(p)\gamma^\eta S_F(p). \tag{13-22}$$
Taking $p_\eta \rightarrow p_\eta - k_\eta$, (13-22) becomes
$$\frac{\partial S_F(p-k)}{\partial\left(p_\eta - k_\eta\right)} = -S_F(p-k)\gamma^\eta S_F(p-k). \tag{13-23}$$

Then, with (13-23) used in the second line below, we have
$$\begin{aligned} \frac{\partial\Sigma(p)}{\partial p_\mu} &= \frac{\partial}{\partial p_\mu}\frac{i}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^\alpha iS_F(p-k)\gamma^\beta d^4k \\ &= \frac{i}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^\alpha i\frac{\partial S_F(p-k)}{\partial\left(p_\eta-k_\eta\right)}\underbrace{\frac{\partial\left(p_\eta-k_\eta\right)}{\partial p_\mu}}_{=\delta_\eta^\mu}\gamma^\beta d^4k = \frac{i}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^\alpha i\left(-S_F(p-k)\gamma^\mu S_F(p-k)\right)\gamma^\beta d^4k \\ &= \frac{-1}{(2\pi)^4}\int iD_{F\alpha\beta}(k)\gamma^\alpha iS_F(p-k)\gamma^\mu iS_F(p-k)\gamma^\beta d^4k = \Lambda^\mu(p,p). \end{aligned} \tag{13-24}$$
End of proof

13.2.4 The Ward Identities

Ward identities distinguished from original Ward identity

Ward's name is associated with an additional set of identities, which play a key role in renormalization, and also in scattering calculations. They are called Ward identities, but to distinguish them from (13-19), we called the earlier relation the "original Ward identity". We derive the Ward identities in this section, but before that, we need a bit of background information.

Gauge Invariance Means Amplitude Invariance

Local gauge invariance means our Lagrangian $\mathcal{L}$ is symmetric in form under the transformations
$$\psi \rightarrow \psi' = e^{-i\alpha(x)}\psi \qquad A_\nu \rightarrow A'_\nu = A_\nu - \tfrac{1}{e}\partial_\nu\alpha(x), \tag{13-25}$$
where the numeric (not operator) field $\alpha(x)$ is our gauge (and is not the QED coupling constant). Since $\mathcal{L}$ (= $\mathcal{L}_0 + \mathcal{L}_I$), is unchanged in form, then each of $\mathcal{L}_I$ and $\mathcal{L}_0$ retains the same functional form, as well. (See (11-36), pg. 294.) That is, under a symmetry transformation of the full $\mathcal{L}$, even though $\mathcal{L}_I$ alone is not symmetric in its own right, in combination with $\mathcal{L}_0$, the transformation yields two terms $\mathcal{L}_I$ and $\mathcal{L}_0$ that are identical in form to the pre-transformation terms $\mathcal{L}_I$ and $\mathcal{L}_0$.

Transition amplitude and probability are gauge invariant

And thus, our transition amplitude must also be the same in form, as depicted symbolically in

$\mathcal{L}$ sym $\rightarrow$ $\mathcal{L}_I$ unchanged $\rightarrow \mathcal{H}_I$ unchanged $\rightarrow$ S unchanged $\rightarrow$ S_{fi} unchanged $\rightarrow$ $|S_{fi}|^2$ unchanged.

Effectively, we can say that if $\mathcal{L}$ is symmetric under (13-25), then so is the amplitude S_{fi}. For the S operator,
$$S\left(\psi, A_\mu\right) = S\left(\psi', A'_\mu\right), \tag{13-25)+1}$$
i.e., it has the same functional form in terms of unprimed or primed (transformed) fields.

With $\mathcal{M}$, our Feynman amplitude, the transition amplitude, as we found in Chap. 8, is

$$S_{fi}=\delta_{fi}+\left((2\pi)^4\,\delta^{(4)}\left(P_f-P_i\right)\left(\prod^{\substack{\text{all external}\\ \text{bosons}}}\sqrt{\frac{1}{2V\omega}}\right)\left(\prod^{\substack{\text{all external}\\ \text{fermions}}}\sqrt{\frac{m}{VE}}\right)\right)\mathcal{M}\qquad \mathcal{M}=\sum_{n=1}^{\infty}\mathcal{M}^{(n)}\,. \tag{13-26}$$

Under the gauge transformation, the incoming and outgoing four-momenta P_i and P_f are unchanged, as are the volume V and the external particle energies, ω and E. Thus, $\mathcal{M}$ is gauge invariant if S_{fi} is.

Thus, Feynman amplitude also gauge invariant

Note that the gauge invariance applies to the total Feynman amplitude for *all* diagrams for given incoming and outgoing states. For example, in Bhabha scattering there are two ways for it to occur. (See Chap. 8, Fig. 8-2, pg. 221.) That is, for a given order in $e_0{}^n$, $\mathcal{M}^{(n)}=\mathcal{M}_{B1}^{(n)}+\mathcal{M}_{B2}^{(n)}$, where $\mathcal{M}_{B1}^{(n)}$ and $\mathcal{M}_{B2}^{(n)}$ each have many sub diagrams for $n\geq 2$. The point is that $\mathcal{M}^{(n)}$ is gauge invariant, but the individual $\mathcal{M}_{B1}^{(n)}$ and $\mathcal{M}_{B2}^{(n)}$ need not be.

But it must be the total Feynman amplitude from all diagrams (for given order n)

Recognize that if $\mathcal{L}$ is gauge invariant, then $\mathcal{H}_I$ remains the same under any such gauge, and each term in our S operator expansion (each term contains n factors of $\mathcal{H}_I$) does also. Thus, for each order of interaction n, $S^{(n)}$ is effectively gauge invariant. Hence, so are $S_{fi}^{(n)}$ and $\mathcal{M}^{(n)}$.

An Example

Consider the initial photon of the LHS of Fig. 13-1 to be a real photon (rather than virtual, i.e., rather than a photon propagator). The self energy Feynman amplitude of the real photon is

$$\mathcal{M}_{\gamma\,\text{self}}^{(2)}=\varepsilon_{r'\mu}\left(\mathbf{k}'\right)\underbrace{\left\{\frac{1}{(2\pi)^4}\text{Tr}\int S_F\left(p\right)ie_0\gamma^{\mu}S_F\left(p-k\right)ie_0\gamma^{\nu}d^4p\right\}}_{\mathcal{M}_{\gamma\,\text{self}}^{(2)\mu\nu}=ie_0^2\Pi^{\mu\nu}(k)}\varepsilon_{r\nu}\left(\mathbf{k}\right)=\varepsilon_{r'\mu}\left(\mathbf{k}'\right)\varepsilon_{r\nu}\left(\mathbf{k}\right)\mathcal{M}_{\gamma\,\text{self}}^{(2)\mu\nu}\,,\tag{13-27}$$

where we represent the part of the interaction that does not include the interaction photon contributions as $\mathcal{M}_{\gamma\,\text{self}}^{(2)\mu\nu}$. Of course, we know that $\mathbf{k}'=\mathbf{k}$ and $r'=r$, but that is not important for present purposes and we want to generalize, so we leave in the primes.

An example showing symbol for the part of the amplitude without external photon factors

The point is that for every interaction having one or more external photons, we can represent the Feynman amplitude in two factors, one for the photon polarization state(s) and one for the rest, where the latter has spacetime indices (which are summed with those on the polarization vectors).

Generalization

For any interaction having one or more photons as initial or final particle(s), we can represent the gauge invariant Feynman amplitude for any order n as

Generalizing the example to any amplitude

$$\mathcal{M}_{fi}^{(n)}=\varepsilon_{r_1\mu}\left(\mathbf{k}_1\right)\varepsilon_{r_2\nu}\left(\mathbf{k}_2\right)\varepsilon_{r_3\eta}\left(\mathbf{k}_3\right)\ldots\mathcal{M}_{fi}^{(n)\mu\nu\eta\ldots}\left(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\ldots\right),\tag{13-28}$$

where we again note that (13-28) is gauge invariant only when the amplitude includes the sub amplitudes for every diagram having the same incoming and outgoing states.

Ward Identities

As we prove below, gauge invariance leads to the Ward identities

Stating the Ward identities

$$k_{1\mu}\mathcal{M}_{fi}^{(n)\mu}\left(\mathbf{k}_1,\mathbf{k}_2,..\right)=k_{2\nu}\mathcal{M}_{fi}^{(n)\nu}\left(\mathbf{k}_1,\mathbf{k}_2,..\right)=k_{1\mu}k_{2\nu}\mathcal{M}_{fi}^{(n)\mu\nu}\left(\mathbf{k}_1,\mathbf{k}_2,..\right)=.....=0\tag{13-29}$$

Proof of Ward Identities

The gauge transformation of (13-25) means $\partial_\nu\alpha(x)$ must satisfy Maxwell's wave equation (where since A_ν is real, $\alpha(x)$ should be real), since, if our Maxwell equation has form

Proof of Ward identities

$$\partial^{\alpha}\partial_{\alpha}A_{\nu}\left(x\right)=0\qquad\text{(13-29)+1}$$

under (13-25), this becomes

$$\partial^{\alpha}\partial_{\alpha}\left(A'_{\nu}+\tfrac{1}{e}\partial_{\nu}\alpha\right)=0\quad\rightarrow\quad\partial^{\alpha}\partial_{\alpha}A'_{\nu}+\tfrac{1}{e}\partial^{\alpha}\partial_{\alpha}\partial_{\nu}\alpha=0\,.\qquad\text{(13-29)+2}$$

If we require (which we do, as our theory is built upon Maxwell's equation in this form)

$$\partial^{\alpha}\partial_{\alpha}A'_{\nu}=0\,,\qquad\text{(13-29)+3}$$

then from (13-29)+2, we see $\partial_\nu\alpha(x)$ must satisfy Maxwell's equation (LHS below).

$$\partial^{\alpha}\partial_{\alpha}\partial_{\nu}\alpha=0\qquad\text{or}\qquad\partial_{\nu}\partial^{\alpha}\partial_{\alpha}\alpha=0\xrightarrow[\text{solution solves}]{\text{one such}}\partial^{\alpha}\partial_{\alpha}\alpha=0\,,\qquad\text{(13-29)+4}$$

and $\alpha(x)$ can have essentially the same functional form as A^μ, the solution to (13-29)+1.[1]

For a real photon field described in the Lorentz gauge by the eigen state plane wave

$$A_\mu = \sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_\mathbf{k}}}\left(\varepsilon_{\mu r}(\mathbf{k})a_r(\mathbf{k})e^{-ikx} + \varepsilon_{\mu r}(\mathbf{k})a_r^\dagger(\mathbf{k})e^{ikx}\right) \qquad (13\text{-}30)$$

a useful form (one particular gauge) for $\alpha(x)$ is

$$\alpha(x) = \sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_\mathbf{k}}}\left(\tilde{\alpha}_r(\mathbf{k})e^{-ikx} + \tilde{\alpha}_r^\dagger(\mathbf{k})e^{ikx}\right) \qquad \tilde{\alpha}_r(\mathbf{k}),\ \tilde{\alpha}_r^\dagger(\mathbf{k}) \text{ numbers, not operators}. \qquad (13\text{-}31)$$

Thus, for this case, the photon gauge transformation of (13-25) becomes

$$\begin{aligned}
A_\mu \to A'_\mu &= A_\mu - \tfrac{1}{e}\partial_\mu\alpha(x) = A_\mu - \tfrac{1}{e}\sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_\mathbf{k}}}\left(-ik_\mu\tilde{\alpha}_r(\mathbf{k})e^{-ikx} + ik_\mu\tilde{\alpha}_r^\dagger(\mathbf{k})e^{ikx}\right) \\
&= A_\mu - \tfrac{1}{e}\sum_{r,\mathbf{k}} k_\mu \underbrace{\frac{1}{\sqrt{2V\omega_\mathbf{k}}}\left(-i\tilde{\alpha}_r(\mathbf{k})e^{-ikx} + i\tilde{\alpha}_r^\dagger(\mathbf{k})e^{ikx}\right)}_{\text{call this } \tilde{\alpha}_{r,\mathbf{k}}(x)} = A_\mu - \underbrace{\tfrac{1}{e}\sum_{r,\mathbf{k}} k_\mu\tilde{\alpha}_{r,\mathbf{k}}(x)}_{=\partial_\mu\alpha(x)} \qquad (13\text{-}32) \\
&= \sum_{r,\mathbf{k}} \frac{1}{\sqrt{2V\omega_\mathbf{k}}}\left(\left(\varepsilon_{\mu r}(\mathbf{k})a_r(\mathbf{k}) + \tfrac{1}{e}ik_\mu\tilde{\alpha}_r(\mathbf{k})\right)e^{-ikx} + \left(\varepsilon_{\mu r}(\mathbf{k})a_r^\dagger(\mathbf{k}) - \tfrac{1}{e}ik_\mu\tilde{\alpha}_r^\dagger(\mathbf{k})\right)e^{ikx}\right).
\end{aligned}$$

Consider a typical term of S expressed in factors of $\mathcal{H}_I$, for example, the Compton scattering term for $n = 2$, (pg. 225), under the symmetry transformation (13-25), with (13-25)+1,

$$\begin{aligned}
&S_C^{(2)}(\psi, A_\mu) = S_C^{(2)}(\psi', A'_\mu) = -e^2\iint d^4x_1 d^4x_2 N\left\{(\bar{\psi}A\!\!\!/\psi)_{x_1}(\bar{\psi}A\!\!\!/\psi)_{x_2}\right\} \\
&\xrightarrow[\text{not positron}]{\text{for electron}} S_{C,e^-}^{(2)} = -e^2\iint d^4x_1 d^4x_2 N\left\{(\bar{\psi}A_\mu\gamma^\mu)_{x_1}\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ (A_\nu\gamma^\nu\psi)_{x_2}\right\} \xrightarrow[\text{transf}]{\text{under}} \\
&= -e^2\iint d^4x_1 d^4x_2 N\left\{(\bar{\psi}_{x_1}e^{i\alpha(x_1)}\left(A_{\mu x_1}\gamma^\mu - \tfrac{1}{e}\partial_\mu\alpha(x_1)\gamma^\mu\right)\left[e^{-i\alpha(x_1)}\psi_{x_1}^+, \bar{\psi}_{x_2}^- e^{i\alpha(x_2)}\right]_+ \left(A_\nu\gamma^\nu{}_{x_2} - \tfrac{1}{e}\partial_\nu\alpha(x_2)\gamma^\nu\right)e^{-i\alpha(x_2)}\psi_{x_2})\right\} \\
&= -e^2\iint d^4x_1 d^4x_2 N\left\{(\bar{\psi}_{x_1}\left(A_{\mu x_1}\gamma^\mu - \tfrac{1}{e}\partial_\mu\alpha(x_1)\gamma^\mu\right)\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ \left(A_{\nu x_2}\gamma^\nu - \tfrac{1}{e}\partial_\nu\alpha(x_2)\gamma^\nu\right)\psi_{x_2})\right\} \\
&= \underbrace{-e^2\iint d^4x_1 d^4x_2 N\left\{(\bar{\psi}A_\mu\gamma^\mu)_{x_1}\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ (A_\nu\gamma^\nu\psi)_{x_2}\right\}}_{S_{C,e^-}^{(2)}} \\
&\quad - e^2\iint d^4x_1 d^4x_2 N\left\{\left(\bar{\psi}A_\mu\gamma^\mu\right)_{x_1}\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ \left(-\tfrac{1}{e}\partial_\nu\alpha(x_2)\gamma^\nu\right)\psi_{x_2}\right\} \\
&\quad - e^2\iint d^4x_1 d^4x_2 N\left\{\bar{\psi}_{x_1}\left(-\tfrac{1}{e}\partial_\mu\alpha(x_1)\gamma^\mu\right)\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ \left(A_\nu\gamma^\nu\psi\right)_{x_2}\right\} \qquad (13\text{-}32)+1 \\
&\quad - e^2\iint d^4x_1 d^4x_2 N\left\{\bar{\psi}_{x_1}\left(-\tfrac{1}{e}\partial_\mu\alpha(x_1)\gamma^\mu\right)\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ \left(-\tfrac{1}{e}\partial_\nu\alpha(x_2)\gamma^\nu\right)\psi_{x_2}\right\}
\end{aligned}$$

Using our definition of $\tilde{\alpha}_{r,\mathbf{k}}$ from (13-32), this becomes

$$\begin{aligned}
S_{C,e^-}^{(2)}(\psi, A_\mu) &= S_{C,e^-}^{(2)}(\psi, A_\mu) + e\iint d^4x_1 d^4x_2 N\left\{\left(\bar{\psi}A_\mu\gamma^\mu\right)_{x_1}\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ \left(\sum_{r,\mathbf{k}} k_\nu\tilde{\alpha}_{r,\mathbf{k}}(x_2)\gamma^\nu\right)\psi_{x_2}\right\} \\
&\quad + e\iint d^4x_1 d^4x_2 N\left\{\bar{\psi}_{x_1}\left(\sum_{r,\mathbf{k}} k_\mu\tilde{\alpha}_{r,\mathbf{k}}(x_1)\gamma^\mu\right)\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ \left(A_\nu\gamma^\nu\psi\right)_{x_2}\right\} \qquad (13\text{-}32)+2 \\
&\quad - \iint d^4x_1 d^4x_2 N\left\{\bar{\psi}_{x_1}\left(\sum_{r',\mathbf{k}'} k'_\mu\tilde{\alpha}_{r',\mathbf{k}'}(x_1)\gamma^\mu\right)\left[\psi_{x_1}^+, \bar{\psi}_{x_2}^-\right]_+ \left(\sum_{r,\mathbf{k}} k_\nu\tilde{\alpha}_{r,\mathbf{k}}(x_2)\gamma^\nu\right)\psi_{x_2}\right\}.
\end{aligned}$$

We can see immediately, because identical terms appear on both sides of (13-32)+2, that the last three terms must sum to zero. More on that shortly.

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[1] Alternatively, our theory was developed in the Lorentz gauge (see pg. 141), i.e., $\partial^\mu A_\mu = 0$, so we need $\partial^\mu A'_\mu = 0$ also. Thus, $0 = \partial^\mu A'_\mu = \partial^\mu\left(A_\mu - \tfrac{1}{e}\partial_\mu\alpha\right) = \partial^\mu A_\mu - \tfrac{1}{e}\partial^\mu\partial_\mu\alpha = 0 - \tfrac{1}{e}\partial^\mu\partial_\mu\alpha$. To keep the Lorentz gauge under the transformation, $\partial^\mu\partial_\mu\alpha = 0$.

For now, using our symbol $iS^{+}\left(x_1-x_2\right)$ for the electron propagator, we find

$$\begin{aligned} S^{(2)}_{C,e^-} &= S^{(2)}_{C,e^-} + e\iint d^4x_1 d^4x_2 N\left\{\left(\bar{\psi}A_\mu\gamma^\mu\right)_{x_1} iS^{+}\left(x_1-x_2\right)\left(\sum_{r,\mathbf{k}} k_\nu \tilde{\alpha}_{r,\mathbf{k}}\left(x_2\right)\gamma^\nu\right)\psi_{x_2}\right\} \\ &+ e\iint d^4x_1 d^4x_2 N\left\{\bar{\psi}_{x_1}\left(\sum_{r,\mathbf{k}} k_\mu \tilde{\alpha}_{r,\mathbf{k}}\left(x_1\right)\gamma^\mu\right) iS^{+}\left(x_1-x_2\right)\left(A_\nu\gamma^\nu\psi\right)_{x_2}\right\} \qquad \text{(13-32)+3} \\ &- \iint d^4x_1 d^4x_2 N\left\{\bar{\psi}_{x_1}\left(\sum_{r',\mathbf{k}'} k'_\mu \tilde{\alpha}_{r',\mathbf{k}'}\left(x_1\right)\gamma^\mu\right) iS^{+}\left(x_1-x_2\right)\left(\sum_{r,\mathbf{k}} k_\nu \tilde{\alpha}_{r,\mathbf{k}}\left(x_2\right)\gamma^\nu\right)\psi_{x_2}\right\}. \end{aligned}$$

As long as we are restricting ourselves to electron Compton scattering, and not positron Compton scattering, we can express the electron propagator S^{+} as the full propagator S (where the last three terms below sum to zero)

$$\begin{aligned} S^{(2)}_{C} &= S^{(2)}_{C} + \sum_{r,\mathbf{k}} k_\nu \underbrace{e\iint d^4x_1 d^4x_2\, \tilde{\alpha}_{r,\mathbf{k}}\left(x_2\right) N\left\{\left(\bar{\psi}A_\mu\gamma^\mu\right)_{x_1} iS\left(x_1-x_2\right)\gamma^\nu\psi_{x_2}\right\}}_{S^{(2)\nu}_{C1}\text{ (1st way, no initial photon)}} \\ &+ \sum_{r,\mathbf{k}} k_\mu \underbrace{e\iint d^4x_1 d^4x_2\, \tilde{\alpha}_{r,\mathbf{k}}\left(x_1\right) N\left\{\bar{\psi}_{x_1}\gamma^\mu iS\left(x_1-x_2\right)\left(A_\nu\gamma^\nu\psi\right)_{x_2}\right\}}_{S^{(2)\mu}_{C2}\text{ (2nd way, no initial photon)}} \qquad \text{(13-32)+4} \\ &- \sum_{r',\mathbf{k}'} k'_\mu \sum_{r,\mathbf{k}} k_\nu \underbrace{\iint d^4x_1 d^4x_2\, \tilde{\alpha}_{r',\mathbf{k}'}\left(x_1\right)\tilde{\alpha}_{r,\mathbf{k}}\left(x_2\right) N\left\{\bar{\psi}_{x_1}\gamma^\mu iS\left(x_1-x_2\right)\gamma^\nu\psi_{x_2}\right\}}_{S^{(2)\mu\nu}_{C}\text{ (both ways, no initial or final photon)}}\ . \end{aligned}$$

The two terms in the sum $\Sigma k_\nu S^{(2)\nu}_{C1} + \Sigma k_\mu S^{(2)\mu}_{C2}$ have the same external particles, so that sum must equal zero independently of $\Sigma k_\nu \Sigma k'_\mu S^{(2)\mu\nu}_{C}$, which has different external particles.

Recall that to find the amplitude $S_{fi} = S_{Compton}$ for Compton scattering (to 2nd order on the RHS below), we carry out steps, as we did in Chap. 8, to evaluate

$$S_{Compton} = \langle f|S|i\rangle = \left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right|\sum_n S^{(n)}\left|e^-_{\mathbf{p},s},\gamma_{\mathbf{k},r}\right\rangle \rightarrow S^{(2)}_{Compton} = \left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right|S^{(2)}_{C}\left|e^-_{\mathbf{p},s},\gamma_{\mathbf{k},r}\right\rangle . \qquad \text{(13-32)+5}$$

When we did that, we found

$$S^{(2)}_{Compton} = \sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\left(2\pi\right)^4\delta^{(4)}\left(p'+k'-p-k\right)\mathcal{M}^{(2)}_{Compton}$$

$$\begin{aligned} \mathcal{M}^{(2)}_{Compton} = \mathcal{M}^{(2)}_{C1} + \mathcal{M}^{(2)}_{C1} \qquad \mathcal{M}^{(2)}_{C1} &= -e^2\bar{u}_{s'}(\mathbf{p}')\varepsilon_{\mu,r'}(\mathbf{k}')\gamma^\mu iS_F(q=p+k)\varepsilon_{\nu,r}(\mathbf{k})\gamma^\nu u_s(\mathbf{p}) \qquad \text{(13-32)+6} \\ \mathcal{M}^{(2)}_{C2} &= -e^2\bar{u}_{s'}(\mathbf{p}')\varepsilon_{\mu,r}(\mathbf{k})\gamma^\mu iS_F(q=p-k')\varepsilon_{\nu,r'}(\mathbf{k}')\gamma^\nu u_s(\mathbf{p}), \end{aligned}$$

which results from the $S^{(2)}_{C}$ term in (13-32)+4. Doing a similar thing with the 2nd and 3rd terms on the RHS of (13-32)+4, where our initial state lacks the photon of (13-32)+6, we get (see Appendix)

$$\left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right|\underbrace{\sum_{r,\mathbf{k}} k_\nu S^{(2)\nu}_{C1} + \sum_{r,\mathbf{k}} k_\mu S^{(2)\mu}_{C2}}_{=0}\left|e^-_{\mathbf{p},s}\right\rangle \rightarrow$$

$$0 = \frac{1}{e}\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\left(2\pi\right)^4\delta^{(4)}\left(p'+k'-p-k\right) i\underbrace{k_\nu\mathcal{M}^{(2)\nu}_{Compton}}_{\text{must = 0}}\left(\tilde{a}_r(\mathbf{k})+\tilde{a}^\dagger_r(-\mathbf{k})\right) \qquad \text{(13-32)+7}$$

$$\begin{aligned} \mathcal{M}^{(2)\nu}_{Compton} = \mathcal{M}^{(2)\nu}_{C1} + \mathcal{M}^{(2)\nu}_{C2} \qquad \mathcal{M}^{(2)\nu}_{C1} &= -e^2\,\varepsilon_{\mu,r'}(\mathbf{k}')\bar{u}_{s'}(\mathbf{p}')\gamma^\mu iS_F(p'+k')\gamma^\nu u_s(\mathbf{p}) \\ \mathcal{M}^{(2)\nu}_{C2} &= -e^2\,\varepsilon_{\mu,r'}(\mathbf{k}')\bar{u}_{s'}(\mathbf{p}')\gamma^\nu iS_F(p'+k')\gamma^\mu u_s(\mathbf{p}) \end{aligned}$$

Thus (where the RHS of (13-33) follows from similar analysis of the last term in (13-32)+4),

$$k_\nu\left(\mathcal{M}^{(2)\nu}_{C1} + \mathcal{M}^{(2)\nu}_{C2}\right) = k_\nu\mathcal{M}^{(2)\nu}_{Compton} = 0 \qquad\qquad k'_\mu k_\nu\mathcal{M}^{(2)\mu\nu}_{Compton} = 0\ . \qquad \text{(13-33)}$$

If we take $\mathbf{k} \rightarrow \mathbf{k}_1$ and $\mathbf{k'} \rightarrow \mathbf{k}_2$, then (13-32)+8 and (13-33) above equal (13-29) for Compton scattering.

One should be able to visualize a similar result from any amplitude with fermion propagators and external fermions and photons. For the $\psi \rightarrow \psi'$ of (13-25), the $e^{-i\alpha(x)}$ and $e^{i\alpha(x)}$ factors will always cancel. The external $A_\nu \rightarrow A'_\nu$, due to the α part, will always leave a series of terms in the S operator expansion of form similar to those in (13-32)+5 (with appropriately more such terms when there are more factors of A_ν.) And each of these terms must equal zero because a term like $S^{(2)}_{Ce^-}\left(\psi, A_\mu\right)$ in (13-32)+2 occurs on each side the relationship resulting from the transformation.

For a photon propagator, we have (with similar results for $iD^{\mu\nu-}\left(x_1 - x_2\right)$)

$$iD^{\mu\nu+}\left(x_1 - x_2\right) = \left[A^{\mu\,+}_{x_1}, A^{\nu\,-}_{x_2}\right] \rightarrow \left[\left(A'^{\mu\,+}_{x_1} + \underbrace{\tfrac{1}{e}\partial^\mu \alpha^+\left(x_1\right)}_{\text{a number}}\right), \left(A'^{\nu\,-}_{x_2} + \underbrace{\tfrac{1}{e}\partial^\nu \alpha^-\left(x_2\right)}_{\text{a number}}\right)\right] = \left[A'^{\mu\,+}_{x_1}, A'^{\nu\,-}_{x_2}\right] \quad (13\text{-}34)$$

Thus, any photon propagator in any amplitude keeps the same form under the transformation, so we get no extra terms from it, as in (13-32)+4, that must equal zero.

And so, we have proven the Ward identities (13-29) using local gauge invariance (which manifested in (13-25)+1, the starting point of our proof).

End of Ward identities proof

End of proof

Note that (13-19) is a relation for n =2 order between the photon loop and the vertex loop, whereas (13-29) is good at any order for any amplitude involving at least one external photon.

Additional Identities

Ward identities a special case of Ward-Takahashi identities

There are yet other identities called Ward-Takahashi identities, of which (13-29) is a special case, but we will not treat those here. In Ward-Takahashi identities, the $k_{i\mu}$ are not restricted to represent external photons, but can be off shell (propagators), and the RHS of (13-29) is, for internal photons, not zero. The Ward identities are the Ward-Takahashi identities for real photons.

The Process

How to apply Ward identities

For any amplitude relation of form on the LHS of (13-35) below, the RHS, representing the Ward identities, is true. That is, we simply replace the polarization vector by the associated four-momentum and the result equals zero.

$$\boxed{\mathcal{M}^{(n)}_{fi}\left(\mathbf{k}_1,...,\mathbf{k}_j,...\right) = \varepsilon_{r_j\,\mu}\mathcal{M}^{(n)\mu}_{fi}\left(\mathbf{k}_1,...,\mathbf{k}_j,...\right) \quad \rightarrow \quad k_{j\,\mu}\mathcal{M}^{(n)\mu}_{fi}\left(\mathbf{k}_1,...,\mathbf{k}_j,...\right) = 0} \,. \quad (13\text{-}35)$$

The Message

Gauge invariance & Ward identities the same thing

Local gauge invariance leads to both charge conservation and the Ward identities. All three are different ways of saying the same thing. Each implies the other two.

charge conservation $\leftrightarrow$ local gauge invariance $\leftrightarrow$ Ward identities.

13.3 Ward Identities, Renormalization, and Gauge Invariance

Application of Ward identities in renormalization for photon-photon scattering case

Consider the scattering of light by light shown in Fig. 13-2. Two incoming photons scatter via fermion virtual particles to yield two outgoing photons. This is called photon-photon scattering, or light-by-light scattering, or less commonly, Delbrück scattering. Occasionally, it is referred to as a "four photon vertex", but this is misleading as there are really four vertices, not a single one with four photons connected directly to it.

Light-by-light scattering does not occur in classical electromagnetism, but does so in QFT due to higher order corrections. Classical electromagnetism contains only terms linear in the photon field A^μ and corresponds to our tree level diagrams. However, via the Dyson-Wicks expansion in QFT, we have terms contributing to the scattering amplitude beyond tree level, at second and higher order, which effectively make non-linear contributions.

Fig. 13-2 represents one way four external photons can scatter at second order. There are other ways the same states can scatter at second order, and Prob. 2 asks you to draw the Feynman diagrams for at least three other possibilities. Note that in Fig. 13-2 we have depicted a certain time order (from left to right) for the vertices in order to make the internal line four-momenta easy to determine. Depicting a different vertex time order (such as the upper left vertex before, rather than

Corrections to 2nd order

$$iD_{F\mu\nu}\left(k\right)\Rightarrow iD_{F\mu\nu}^{2nd}\left(k\right)=\left(iD_{F\mu\nu}\left(k\right)\right)\left(1-e_0^2A'\left(k,\Lambda\right)-e_0^2\Pi_c(k^2)\right). \qquad (13\text{-}73)$$

$$iS_F\left(p\right)\Rightarrow iS_F^{2nd}\left(p\right)=\left(iS_F\left(p\right)\right)\left(1-e_0^2B\left(\Lambda\right)-e_0^2\Sigma_c\left(\not p-m\right)\right) \qquad (13\text{-}74)$$

$$\begin{aligned}
&u_r\left(\mathbf{p}\right)\Rightarrow u_r^{2nd}\left(\mathbf{p}\right)=\left(1-\tfrac{1}{2}e_0^2B\left(\Lambda\right)\right)u_r\left(\mathbf{p}\right) \qquad \bar{u}_r\left(\mathbf{p}\right)\Rightarrow \bar{u}_r^{2nd}\left(\mathbf{p}\right)=\left(1-\tfrac{1}{2}e_0^2B\left(\Lambda\right)\right)\bar{u}_r\left(\mathbf{p}\right)\\
&v_r\left(\mathbf{p}\right)\Rightarrow v_r^{2nd}\left(\mathbf{p}\right)=\left(1-\tfrac{1}{2}e_0^2B\left(\Lambda\right)\right)v_r\left(\mathbf{p}\right) \qquad \bar{v}_r\left(\mathbf{p}\right)\Rightarrow \bar{v}_r^{2nd}\left(\mathbf{p}\right)=\left(1-\tfrac{1}{2}e_0^2B\left(\Lambda\right)\right)\bar{v}_r\left(\mathbf{p}\right) \qquad (13\text{-}75)\\
&\varepsilon_\mu\left(\mathbf{k}\right)\Rightarrow \varepsilon_\mu^{2nd}\left(\mathbf{k}\right)=\left(1-\tfrac{1}{2}e_0^2A'\left(\Lambda\right)\right)\varepsilon_\mu\left(\mathbf{k}\right)
\end{aligned}$$

$$ie_0\gamma^\mu\Rightarrow ie_0\gamma_{2nd}^\mu\left(p,p'\right)=\underbrace{ie_0\left\{\gamma^\mu\right.}_{ie_0\gamma^\mu}\left(1+e_0^2L\left(\Lambda\right)\right)+e_0^2\Lambda_c^\mu\left(p,p'\right)\left.\right\} \qquad (13\text{-}76)$$

13.8 Appendix: Finding Ward Identities for Compton Scattering

Since we worked through every step of finding quite a number of amplitudes the long way in Chap. 8, we will be briefer here. If you have trouble at any point, please refer to the detailed derivation of the Compton scattering transition amplitude on pgs. 225-228, which closely parallels the following.

$$\left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right|\sum_{r,\mathbf{k}}k_\nu S_{C1}^{(2)\nu}+\sum_{r,\mathbf{k}}k_\mu S_{C2}^{(2)\mu}\left|e^-_{\mathbf{p},s}\right\rangle=0 \quad \text{because} \quad \sum_{r,\mathbf{k}}k_\nu S_{C1}^{(2)\nu}+\sum_{r,\mathbf{k}}k_\mu S_{C2}^{(2)\mu}=0 \qquad (13\text{-}77)$$

For Compton scattering the first way (LH of Fig. 8-3, pg. 225) without the incoming photon, the part of (13-77) with $S_{C1}^{(2)\nu}$ becomes

$$\begin{aligned}
&\left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right|\sum_{r,\mathbf{k}}k_\nu S_{C1}^{(2)\nu}\left|e^-_{\mathbf{p},s}\right\rangle\\
&=\left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right|e\iint d^4x_1d^4x_2\,N\left\{\left(\bar\psi A_\mu\gamma^\mu\right)_{x_1}iS\left(x_1-x_2\right)\left(\sum_{r,\mathbf{k}}k_\nu\tilde\alpha_{r,\mathbf{k}}\left(x_2\right)\right)\gamma^\nu\psi_{x_2}\right\}\left|e^-_{\mathbf{p},s}\right\rangle\\
&=\left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right|e\iint d^4x_1d^4x_2\left(\sum_{s'',\mathbf{p}''}\sqrt{\frac{m}{VE_{\mathbf{p}''}}}c^\dagger_{s''}(\mathbf{p}'')\bar{u}_{s''}(\mathbf{p}'')e^{ip''x_1}\right)\times \qquad (13\text{-}78)\\
&\qquad\left(\sum_{r'',\mathbf{k}''}\sqrt{\frac{1}{2V\omega_{\mathbf{k}''}}}a^\dagger_{r''}\varepsilon_{\mu,r''}(\mathbf{k}'')e^{ik''x_1}\gamma^\mu\right)\frac{1}{(2\pi)^4}\int d^4q\,iS_F(q)e^{-iq(x_1-x_2)}\times\\
&\qquad\left(i\sum_{r,\mathbf{k}}k_\nu\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\left(-\tilde a_r(\mathbf{k})e^{-ikx_2}+\tilde a^\dagger_r(\mathbf{k})e^{ikx_2}\right)\right)\gamma^\mu\left(\sum_{s''',\mathbf{p}'''}\sqrt{\frac{m}{VE_{\mathbf{p}'''}}}c_{s'''}(\mathbf{p}''')u_{s'''}(\mathbf{p}''')e^{-ip'''x_2}\right)\left|e^-_{\mathbf{p},s}\right\rangle
\end{aligned}$$

$$\begin{aligned}
&=\sum_{s'',\mathbf{p}''}\sum_{r'',\mathbf{k}''}\underbrace{\left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\middle|\middle|e^-_{\mathbf{p}'',s''},\gamma_{\mathbf{k}'',r''}\right\rangle}_{\delta_{\mathbf{p}'\mathbf{p}''}\delta_{s's''}\delta_{\mathbf{k}'\mathbf{k}''}\delta_{r'r''}}e\iint d^4x_1d^4x_2\left(\sqrt{\frac{m}{VE_{\mathbf{p}''}}}\bar{u}_{s''}(\mathbf{p}'')e^{ip''x_1}\right)\left(\sqrt{\frac{1}{2V\omega_{\mathbf{k}''}}}\varepsilon_{\mu,r''}(\mathbf{k}'')e^{ik''x_1}\gamma^\mu\right)\\
&\times\frac{1}{(2\pi)^4}\int d^4q\,iS_F(q)e^{-iq(x_1-x_2)}\left(i\sum_{r,\mathbf{k}}k_\nu\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\left(-\tilde a_r(\mathbf{k})e^{-ikx_2}+\tilde a^\dagger_r(\mathbf{k})e^{ikx_2}\right)\right)\gamma^\nu\left(\sqrt{\frac{m}{VE_{\mathbf{p}}}}u_s(\mathbf{p})e^{-ipx_2}\right) \qquad (13\text{-}79)
\end{aligned}$$

$$\begin{aligned}
&=e\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\varepsilon_{\mu,r'}(\mathbf{k}')\bar{u}_{s'}(\mathbf{p}')\gamma^\mu\frac{1}{(2\pi)^4}\int d^4q\,iS_F(q)\gamma^\nu u_s(\mathbf{p})\times\\
&\qquad\left(i\sum_{r,\mathbf{k}}k_\nu\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\left(-\tilde a_r(\mathbf{k})e^{-ikx_2}+\tilde a^\dagger_r(\mathbf{k})e^{ikx_2}\right)\right)\left\{\int d^4x_1e^{-iqx_1}e^{ip'x_1}e^{ik'x_1}\int d^4x_2e^{iqx_2}e^{-ipx_2}\right\} \qquad (13\text{-}80)
\end{aligned}$$

$$\begin{aligned}
&=e\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\varepsilon_{\mu,r'}(\mathbf{k}')\bar{u}_{s'}(\mathbf{p}')\gamma^\mu\frac{1}{(2\pi)^4}\int d^4q\,iS_F(q)\gamma^\nu u_s(\mathbf{p})\times\\
&\left(\sum_{r,\mathbf{k}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\int d^4x_1e^{-iqx_1}e^{ip'x_1}e^{ik'x_1}\,ik_\nu\left(-\tilde a_r(\mathbf{k})\int d^4x_2e^{iqx_2}e^{-ipx_2}e^{-ikx_2}+\tilde a^\dagger_r(\mathbf{k})\int d^4x_2e^{iqx_2}e^{-ipx_2}e^{+ikx_2}\right)\right) \qquad (13\text{-}81)
\end{aligned}$$

$$= e\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}\,\varepsilon_{\mu,r'}(\mathbf{k}')\bar{u}_{s'}(\mathbf{p}')\gamma^{\mu}\frac{1}{(2\pi)^4}\int d^4q\, iS_F(q)\gamma^{\nu}u_s(\mathbf{p})\times$$
$$(2\pi)^4\delta^{(4)}(q-p'-k')\,i\left(\sum_{r,\mathbf{k}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}k_{\nu}\left(-\tilde{a}_r(\mathbf{k})(2\pi)^4\delta^{(4)}(q-p-k)+\tilde{a}_r^{\dagger}(\mathbf{k})(2\pi)^4\delta^{(4)}(q-p+k)\right)\right)$$
$$= \frac{1}{e}\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}(2\pi)^4\underbrace{e^2\varepsilon_{\mu,r'}(\mathbf{k}')\bar{u}_{s'}(\mathbf{p}')\gamma^{\mu}iS_F(p'+k')\gamma^{\nu}u_s(\mathbf{p})}_{=-\mathcal{M}_{C1}^{(2)\nu}}\times \tag{13-82}$$
$$i\left(\sum_{r,\mathbf{k}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}k_{\nu}\left(-\underbrace{\tilde{a}_r(\mathbf{k})\delta^{(4)}(p'+k'-p-k)}_{\substack{=0\text{ except when }k=p'+k'-p,\\ \text{the value in full Compton scattering}}}+\underbrace{\tilde{a}_r^{\dagger}(\mathbf{k})\delta^{(4)}(p'+k'-p+k)}_{\substack{=0\text{ except when }k=-p'-k'+p,\text{ negative}\\ \text{of the value in full Compton scattering}}}\right)\right)$$

$$\begin{aligned}&= -\frac{1}{e}\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}(2\pi)^4\delta^{(4)}(p'+k'-p-k)\,i\mathcal{M}_{C1}^{(2)\nu}\left(-k_{\nu}\tilde{a}_r(\mathbf{k})-k_{\nu}\tilde{a}_r^{\dagger}(-\mathbf{k})\right)\\ &= \frac{1}{e}\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}(2\pi)^4\delta^{(4)}(p'+k'-p-k)\,ik_{\nu}\mathcal{M}_{C1}^{(2)\nu}\left(\tilde{a}_r(\mathbf{k})+\tilde{a}_r^{\dagger}(-\mathbf{k})\right)\end{aligned} \tag{13-83}$$

For Compton scattering the second way (RH of Fig. 8-3), the part of (13-77) with $S_{C2}^{(2)\mu}$, after similar evaluation, yields, with the sub amplitude $\mathcal{M}_{C2}^{(2)\nu}$ as shown in (13-32)+7,

$$\begin{aligned}&\left\langle e^-_{\mathbf{p}',s'},\gamma_{\mathbf{k}',r'}\right|\sum_{r,\mathbf{k}}k_{\mu}S_{C\,2nd}^{(2)\mu}\left|e^-_{\mathbf{p},s}\right\rangle\\ &= \frac{1}{e}\sqrt{\frac{m}{VE_{\mathbf{p}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}'}}}\sqrt{\frac{1}{2V\omega_{\mathbf{k}}}}\sqrt{\frac{m}{VE_{\mathbf{p}}}}(2\pi)^4\delta^{(4)}(p'+k'-p-k)\,ik_{\nu}\mathcal{M}_{C2}^{(2)\nu}\left(\tilde{a}_r(\mathbf{k})+\tilde{a}_r^{\dagger}(-\mathbf{k})\right).\end{aligned} \tag{13-84}$$

(13-83) and (13-84) summed equal the LHS of (13-77), so their sum equals zero. To do this, the coefficient of $\tilde{a}_r(\mathbf{k})$, which is arbitrary, in that sum must vanish (as must the coefficient of $\tilde{a}_r^{\dagger}(-\mathbf{k})$). The only way this can happen is if

$$k_{\nu}\left(\mathcal{M}_{C1}^{(2)\nu}+\mathcal{M}_{C2}^{(2)\nu}\right)=k_{\nu}\mathcal{M}_{Compton}^{(2)\nu}=0. \tag{13-85}$$

13.9 Problems

1. Show that $(\not{p}+m)v_r(\mathbf{p})=0$. (Hint: Follow steps like we did to get (13-10).)

2. Draw at least three ways, other than that shown in Fig. 13-2, for which the incoming same two photon state scatters at second order into the same outgoing two photon state.

3. Re-draw the Feynman diagram of Fig. 13-2 with the upper left vertex occurring before the lower left vertex. Label the internal line four-momenta. Show by writing out the Feynman amplitude for this diagram using Feynman rules, that the amplitude you get is the same as we got in (13-36) for Fig. 13-2. (Hint: re-express (13-36) with p_2, p_3, and p_4 in terms of p_1. Then express your new diagram where all propagator factors are in terms of p_1. Remember that for anti-particle internal lines, the four-momentum has opposite sign from physical reality. See Wholeness Chart 8-1, pg. 234.) Realize that the diagram you drew for this problem is not one of the answers for Prob. 2.

4. Show that by using part b) of Fig. 13-5 for Feynman diagrams to 2nd order of fermion self energy, you obtain (13-55). Hint: In (13-54) take $m_0 \rightarrow m$ and $ie_0^2\Sigma(p)\rightarrow ie_0^2\Sigma(p)+i\delta m$.

5. Show (13-62) using similar logic to what we used for (13-60). Note that in (13-52), the $e_0^2\Pi_c(k^2)$ term is an expansion with terms in k^2 to various powers, but that for a real photon k^2 = 0.