Electroweak Symmetry Breaking

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Abstract

The electroweak symmetry breaking model of Glashow/Salam/Weinberg is presented in a pedagogic manner by leading up to it with the simpler to understand symmetry breaking models of Goldstone and Higgs. A wholeness (overview) chart summarizing and comparing all three models is included, as well as a separate wholeness chart for each particular model. Key results are deduced via detailed step-by-step derivations, many of which are not found in texts.

The notation used herein parallels that of Mandl and Shaw[1] and Klauber[2].

Caveats

Take caution that no one else has checked this material carefully for errata. So, if something doesn't seem right to you, there is a reasonable chance it is not and needs correcting. If you find such things, or have helpful suggestions to offer that may make the material easier for others to understand, please let me know via the email address in the "Feedback" link on the book website link above. Thank you.

1 Background for Electroweak Symmetry Breaking

There are a number of concepts related to symmetry breaking that one should have command of prior to delving deeply into the subject. We review (or introduce if you haven't seen them before) each such concept in a separate subsection below.

1.1 Mass Terms

1.1.1 Bosons

Note that in the usual Lagrangian for a free *complex* scalar boson field*,

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - \mu^2 \phi^\dagger \phi \qquad \left(\text{Lagrangian for free complex scalar field}\right), \tag{1}$$

the mass μ squared is the coefficient of the $-\phi^\dagger\phi$ term, and we can make that a general rule for similar such terms for any complex boson field.

For a *real* scalar boson field†, on the other hand, the mass term looks a little different, i.e., it has an additional factor of ½ in front (and $\phi^\dagger = \phi$), as in

$$\mathcal{L} = \tfrac{1}{2}\partial^\mu \phi \partial_\mu \phi - \tfrac{1}{2}\mu^2 \phi\phi \qquad \left(\text{Lagrangian for free real scalar field}\right) . \tag{2}$$

The additional factor of ½ comes in because we must substitute (2) into the Euler-Lagrange equation to get the boson field equation of motion wherein derivatives there are taken with respect to ϕ and $\phi_{,\mu}$ and said entities ϕ and $\phi_{,\mu}$ are squared in (2). In (1), to get the field equation of motion for ϕ, we take derivatives with respect to $\phi^\dagger$ and $\phi^\dagger_{,\mu}$, and said entities are not squared in (1). Handling it in this way leaves the same field equation for ϕ, whether it is real or complex. Otherwise the two would be off by a factor of 2.

As an example of the mass of a boson mass term we are already familiar with, the photon in QED has no term in the Lagrangian of form $A^\mu A_\mu$ (comparable to the term with $\phi\phi$ in (2)), so mass of the photon is zero.

Consequently, as a general principle, if we have a Lagrangian for any boson field, real or complex, we can simply read off the mass of the associated bosonic particle from the mass term, as in (1) and (2).

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* See Mandl and Shaw[1], pg. 40, equation (3.4) and Klauber[2], pg. 49, equation (3-32).
† See Mandl and Shaw[1], pg. 43, equation (3.22).

1.1.2 Leptons

For Dirac particles, the free Lagrangian has the form, where $\partial\!\!\!/ = \gamma^{\alpha}\partial_{\alpha}$,

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi \ . \tag{3}$$

So, given the Lagrangian for any spinor field, we can simply read off the associated particle mass as the coefficient of the $-\bar{\psi}\psi$ term.

1.2 Potential Energy Terms

In QFT, particularly in introductory courses, we don't usually consider potential energy in the Lagrangian. To derive classical potentials like the Coulomb potential, we consider QFT interactions via Møller scattering (without potential energy terms in the Lagrangian) to obtain the relevant Feynman amplitudes*, and from them, the effective potential. The characteristics of the virtual particles mediating the interaction give rise to the corresponding classical potential measured in experiments.

However, in classical field theory, and thus in QFT, one can work with a potential energy, to be precise, a potential energy *density* in the Lagrangian (density). (It is often simply called the "potential".) This will be done when we examine the Higgs field in symmetry breaking models. For now, simply note that we could have a scalar field Lagrangian of form (where $\mu = 0$ in this example)

$$\mathcal{L} = \partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi - \mathcal{V}\left(\phi,\phi^{\dagger}\right) \qquad \left(\text{massless complex scalar field}\right). \tag{4}$$

Note that potential energy density $\mathcal{V}$ is generally a function of the field (and its complex conjugate) and not derivatives of the field. This parallels what we know from classical particle theory†, where the parallel between particles and fields, i.e., $x(t)$ (particle) $\rightarrow \phi(t,x^i)$ (field), is commonly employed to extrapolate field theory from particle theory,

$$V(x)\ \left[\text{e.g., spring-mass system} = \tfrac{1}{2}kx^2\right] \quad \rightarrow \quad \mathcal{V}\left(\phi^{\dagger},\phi\right)\ \left[\text{e.g.,} = \tfrac{1}{2}\mu\phi^2\left(\text{real}\right) \text{ or } \mu\phi^2\left(\text{complex}\right)\right]. \tag{5}$$

And of course, just like we can have different systems in classical theory with different forms for potential energy (such as gravity around a planet, where $V = -GmM/r$), we can have different forms for $\mathcal{V}$ as a function of ϕ in field theory.

For a real field, $\mathcal{V} = \mathcal{V}(\phi)$.

Note that if we wished, at least mathematically, we could have considered (1) [or (2)] to represent a massless field with potential $\mu^2\phi^{\dagger}\phi$ [or $\frac{1}{2}\mu^2\phi\phi$ for a real field]. That is,

$$\mathcal{L} = \partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi - \mathcal{V} \qquad \mathcal{V} = \mu^2\phi^{\dagger}\phi = \mu^2\left|\phi\right|^2 \ . \tag{6}$$

Note that in (6), $\mathcal{V}$ must be positive (for real μ). Consider, on the other hand, a Lagrangian where μ is imaginary. That would make $\mathcal{V}$ negative. This case, with an effective imaginary mass, is a key part of electroweak symmetry breaking, as we shall see.

1.3 Different Ways to Represent ϕ

Note that we typically express a scalar field as

$$\phi(x) = \sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}(a(\mathbf{k})e^{-ikx} + b^{\dagger}(\mathbf{k})e^{ikx}) \qquad \phi^{\dagger}(x) = \sum_{\mathbf{k}'}\frac{1}{\sqrt{2V\omega_{\mathbf{k}'}}}(b(\mathbf{k}')e^{-ik'x} + a^{\dagger}(\mathbf{k}')e^{ik'x}) \ . \tag{7}$$

However, we will find it advantageous later on to express ϕ as a sum or its real part and its imaginary part. That is, where ϕ_1 and ϕ_2 are real,

$$\phi = \phi_1 + i\phi_2 \ . \tag{8}$$

* See Klauber[2], pgs. 404-410.

† See Klauber[2], Wholeness Chart 18-4, pgs. 506-507

From the LHS of (7), where for simplicity we only examine a single **k** value term,

$$\begin{aligned} Re\{\phi\} &= \phi_1 = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(\frac{a(\mathbf{k})e^{-ikx}+a^\dagger(\mathbf{k})e^{ikx}}{2}+\frac{b^\dagger(\mathbf{k})e^{ikx}+b(\mathbf{k})e^{-ikx}}{2}\right) \\ Im\{\phi\} &= i\phi_2 = \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(\frac{a(\mathbf{k})e^{-ikx}-a^\dagger(\mathbf{k})e^{ikx}}{2}+\frac{b^\dagger(\mathbf{k})e^{ikx}-b(\mathbf{k})e^{-ikx}}{2}\right) \end{aligned} \tag{9}$$

Thus, we can re-express ϕ_1 and ϕ_2 as

$$\begin{aligned} \phi_1 &= \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(\frac{a(\mathbf{k})e^{-ikx}+b(\mathbf{k})e^{-ikx}}{2}+\frac{a^\dagger(\mathbf{k})e^{ikx}+b^\dagger(\mathbf{k})e^{ikx}}{2}\right)=\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(\alpha(\mathbf{k})e^{-ikx}+\alpha^\dagger(\mathbf{k})e^{ikx}\right) \\ \phi_2 &= \frac{1}{i}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(\frac{a(\mathbf{k})e^{-ikx}-b(\mathbf{k})e^{-ikx}}{2}+\frac{-a^\dagger(\mathbf{k})e^{ikx}+b^\dagger(\mathbf{k})e^{ikx}}{2}\right)=\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(\beta(\mathbf{k})e^{-ikx}+\beta^\dagger(\mathbf{k})e^{ikx}\right), \end{aligned} \tag{10}$$

where $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$ are operators for the real and imaginary fields, respectively, whose definition in terms of $a(\mathbf{k})$, $b(\mathbf{k})$, and their complex conjugates should be obvious from (10). ϕ_1 and ϕ_2 each thus acts much like a photon field, which is real and has similar form to the RHS of each row of (10).

In electroweak symmetry breaking, one often refers to the fields ϕ_1 and ϕ_2 as real and imaginary components of ϕ, but their specific forms, as in (10), are not shown in any text I am aware of. The key thing to note is that given the usual postulates of QFT, one can deduce that the $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$ operators are destruction operators and their complex conjugates are construction operators. These create and destroy $|\phi_1\rangle$ and $|\phi_2\rangle$ particle states in a manner which we are used to.

1.4 Plotting $\mathcal{V}$

In electroweak symmetry breaking one commonly sees plots of $\mathcal{V}$ vs ϕ, as for example, for (6) using (8), where

$$\mathcal{V} = \mu^2\phi^\dagger\phi = \mu^2|\phi|^2 = \mu^2(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) = \mu^2(\phi_1^2+\phi_2^2) \tag{11}$$

as a paraboloid over the complex plane.

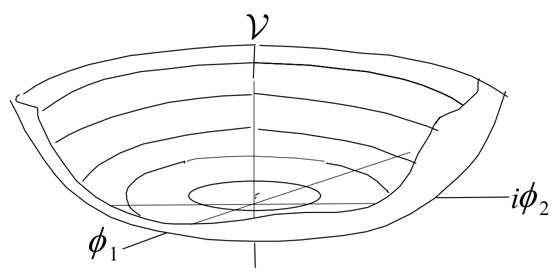

Figure 1. Plot of $\mathcal{V}$ vs ϕ for Real Mass μ

As a student I was perplexed by the use of such graphs in QFT where ϕ is not a simple variable, but an operator, as in (7), or (8) with (10). That is, ϕ in QFT does not vary. It does not have greater or lesser values extending outward from ϕ=0, as shown in Figure 1.

Unable to find any text that even addressed this question, I ruminated on it for a long time, and finally found the following resolution.

In the RHS of (6) we find $\mathcal{V}$ proportional to $\phi^\dagger\phi$. Thus, with (7),

$$\mathcal{V} = \mu^2\phi^\dagger\phi = \frac{\mu^2}{2V}\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\frac{1}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\begin{pmatrix} a(\mathbf{k})b(\mathbf{k}')e^{-ikx}e^{-ik'x}+a(\mathbf{k})a^\dagger(\mathbf{k}')e^{-ikx}e^{ik'x} \\ +b^\dagger(\mathbf{k})b(\mathbf{k}')e^{ikx}e^{-ik'x}+b^\dagger(\mathbf{k})a^\dagger(\mathbf{k}')e^{ikx}e^{ik'x} \end{pmatrix}, \tag{12}$$

$$V_\phi = \int \mathcal{V}\, d^3x \,, \tag{13}$$

and a substantial amount of algebra* (which would take us far afield of our present objective, so I hope you can take my word for it), we would find the effective total potential V_ϕ

$$V_\phi = \frac{\mu^2}{2}\sum_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}}}\left(a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})a^\dagger(\mathbf{k}) + 1\right) = \frac{\mu^2}{2}\sum_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}}}\left(N_a(\mathbf{k}) + N_b(\mathbf{k}) + 1\right), \tag{14}$$

where our old friends the number operators show up again. As is the usual (but yet to be fully justified by anyone) practice in QFT, we drop the constant term "1". This is the same as assuming (12) is normal ordered†, whereby the "1" term never shows up in (14).

Dividing (14) by the volume V, we get an effective (representing measurable) $\mathcal{V}$ operator,

$$\mathcal{V} = \frac{\mu^2}{2V}\sum_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}}}\left(N_a(\mathbf{k}) + N_b(\mathbf{k})\right) \,, \tag{15}$$

which operates on kets so we have expectation values (expected measured values)

$$\overline{\mathcal{V}} = \left\langle n_{\mathbf{k}}\phi_{\mathbf{k}}, n_{\mathbf{k}'}\phi_{\mathbf{k}'},..\right|\mathcal{V}\left|n_{\mathbf{k}}\phi_{\mathbf{k}}, n_{\mathbf{k}'}\phi_{\mathbf{k}'},..\right\rangle = \frac{\mu^2}{2V}\sum_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}}}\left(n_a(\mathbf{k}) + n_b(\mathbf{k})\right) \,. \tag{16}$$

The main point

The main point of all this is that when we plot $\mathcal{V}$ vs ϕ as in Figure 4, what we are really representing is $\overline{\mathcal{V}}$, the expectation value for potential energy density, vs the number of particles expected to be measured (per unit volume).

That is, in Figure 1, we really mean to plot

$$\overline{\mathcal{V}} \;\; vs \;\; \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}}}\left(n_a(\mathbf{k}) + n_b(\mathbf{k})\right) \;\; \text{where} \;\; \frac{1}{V}\sum_{\mathbf{k}} \frac{1}{\omega_{\mathbf{k}}}\left(n_a(\mathbf{k}) + n_b(\mathbf{k})\right) \text{ is radial distance from center} \,. \tag{17}$$

However, note that $\mathcal{V}$ in (6) is expressed as a function of the bilinear operator $\phi^\dagger\phi$ and $\overline{\mathcal{V}}$ of (17), which is derived from $\mathcal{V}$. If we were to plot $\overline{\mathcal{V}}$ vs n_a (and/or n_b), we would effectively be plotting $\overline{\mathcal{V}}$ vs $|\phi|^2$. So, if we plot $\overline{\mathcal{V}}$ vs ϕ, as it appears in Figure 1, then we essentially are plotting $\overline{\mathcal{V}}$ vs the square root of the particle number density $n_a(\mathbf{k})/V$ (and/or $n_b(\mathbf{k})/V$) [for all particles having the same **k**].

Bottom line of the main point

In plots of $\mathcal{V}$ vs ϕ, the distance from the centerline is related to the particle number density. Further out, more dense. Closer in, less dense. At the center, zero particle density.

1.5 Defining the Vacuum

We generally think of the vacuum as the state with zero real particles, i.e., zero particle density, which would make the centerline of Figure 1 the vacuum. This is also the lowest energy state (lowest $\mathcal{V}$).

When we get into electroweak symmetry breaking, however, we will find a situation where a state having finite particle density has a lower $\mathcal{V}$ than the state with zero particle density. Don't worry about this for now, as it will become clearer as we progress. (If you want to peek now at what is meant here see Figure 2 and Figure 3 on pg. 6.) In such cases, the "true" vacuum is often considered to be the lowest energy state, rather than the state with zero particles.

2 Overview of Three Symmetry Breaking Models

Wholeness Chart 1 on pg. 5 is an overview of the three symmetry breaking models, Goldstone, Higgs, and Weinberg/Salaam. Use it as a guide as you progress through the study of each separate model in subsequent sections.

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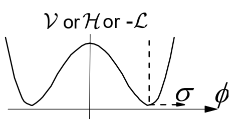
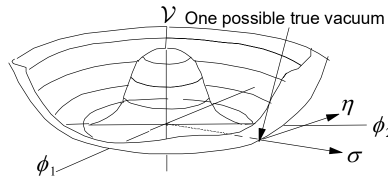
* See Klauber[2], pg. 53-54, for how this is done for the full Hamiltonian $H = \int \mathcal{H} d^3x$. Equation (3-53) therein effectively parallels (14) and (15) herein, and leads to the mass term expectation contribution of (16). The extra bilinear operator terms in (3-53) drop out when we take expectation values, so (14) herein is really an effective potential density operator giving the same result in (16) as the complete expression of (3-53) would have.

† See Klauber[2], pgs. 60, 203, 209.

Wholeness Chart 1. Electroweak Symmetry Breaking

	Single Real Scalar Field	**Complex Scalar Field Singlet (Isoscalar)**		**Complex Scalar Field Doublet (Isospinor)**	
		Goldstone Model	**Higgs Model**	**(Not treated)**	**Glashow/Salam/Weinberg**
Higgs Field	$\phi(x)$ = real	$\phi(x)=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)$	as at left		$\Phi(x)=\begin{bmatrix}\phi_1+i\phi_2\\ \phi_3+i\phi_4\end{bmatrix}$
Symmetry Transform Type	Global reflection $\mathcal{L}(\phi)=\mathcal{L}(-\phi)$	Global U(1) (i.e., no interactions)	Local U(1) (i.e., interactions)	Global SU(2)XU(1)	Local SU(2) X U(1) (i.e., interactions)
$\mathcal{V}(\phi)$ [or $\mathcal{H}(\phi)$] (same symmetry as $\mathcal{L}$)			as at left		Similar in 5D (4 ϕ_i,1 $\mathcal{V}$) to Goldstone. Have true vacuum at min $\mathcal{V}$.
Symmetry Transform	$\phi(x)\rightarrow-\phi(x)$ (discrete)	$\delta\phi=i\alpha\phi$ (infinitesimal) $\left(\text{or } \phi'=e^{i\alpha}\phi \text{ finite}\right)$	$\delta\phi=i\alpha(x)\phi$ $\delta A^\mu=-\frac{1}{e}\partial^\mu\alpha(x)$ $[\& \not\partial\rightarrow\not D \text{ in } \mathcal{L}_0]$		$\delta\Phi=\frac{i}{2}\omega_i\tau_i\Phi+ig'Yf\Phi$ $\delta B^\mu=-\frac{1}{g'}\partial^\mu f$ $\delta W_i^\mu=-\frac{1}{g'}\partial^\mu\omega_i-\varepsilon_{ijk}\omega_j W_k^\mu$ $\delta\Psi_l^L=\frac{i}{2}\omega_i\tau_i\Psi_l^L+ig'Yf\Psi_l^L$
Degrees of Freedom, False Vacuum	1 (ϕ)	2 (ϕ_1, ϕ_2)	4 (ϕ_1, ϕ_2, 2 massless A^μ polariz states) $m_1=m_2$ $m_1^2<0$		12 (ϕ_1, ϕ_2, ϕ_3, ϕ_4, 2 massless B^μ, 2 each for 3 massless W_i^μ)
New Bosons = Lin Combins of Old					$B^\mu, W_i^\mu \rightarrow W_\pm^\mu, Z^\mu, A^\mu$ $\phi_{1,2,4}=\eta_{1,2,3}/\sqrt{2}$ $\phi_3=(\sigma+v)/\sqrt{2}$ Still 12 DOFs, still massless
Degrees of Freedom, True Vacuum	1 (σ)	2 (σ, η) $m_\eta=0$ $m_\sigma^2>0$	5 (σ, η, 3 from massive A^μ) 1 non physical field $m_\eta=0$ $m_\sigma^2>0$		15 (1 each from η_1, η_2, η_3, σ, 9 from massive $W_\pm^\mu, Z^\mu$, 2 from massless A^μ) 3 non physical fields
Unitary Gauge			Eliminate one DOF by choosing $\eta=0$		Get what is observed physically by choosing $\eta_1=\eta_2=\eta_3=0$. Eliminate 3 DOFs.
Result			1 massive Higgs σ 1 massive A^μ 4 DOFs		1 massive Higgs σ 3 massive $W_\pm^\mu, Z^\mu$ 1 massless A^μ 12 DOFs

3 Goldstone Model

3.1 Assumptions for the Goldstone Model

Note that Wholeness Chart 2 on pg. 12 summarizes the Goldstone model. That model is based on a particular form for potential energy density (see (18)),

1) a complex scalar Higgs field ϕ, and
2) no interactions (i.e., a Lagrangian density with global $U(1)$ symmetry [See Klauber[2], pg. 296]).

$$\begin{aligned}\mathcal{L} &= \partial^{\mu}\phi^{\dagger}\partial_{\mu}\phi \quad -\mu^2\phi^2 \quad -\lambda\phi^4 \qquad \text{Complex } \phi = \tfrac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right) \quad \phi_1,\phi_2 \text{ real} \\ &= \tfrac{1}{2}\left(\partial^{\mu}\phi_1\right)^2 + \tfrac{1}{2}\left(\partial^{\mu}\phi_2\right)^2 \underbrace{-\tfrac{\mu^2}{2}\left(\phi_1^2+\phi_2^2\right) - \tfrac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2}_{-\mathcal{V}}\end{aligned} \tag{18}$$

where we are using the symbolism $\phi^2 = \phi^{\dagger}\phi$. In this model, μ^2 is negative (μ is imaginary) and the potential density is

$$\begin{aligned}\mathcal{V} &= \mu^2\phi^2 + \lambda\phi^4 = \tfrac{\mu^2}{2}\left(\phi_1^2+\phi_2^2\right) + \tfrac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2 \qquad \mu^2 < 0 \quad \lambda > 0 \\ &= -\left|\mu^2\right|\phi^2 + \lambda\phi^4 = -\frac{\left|\mu^2\right|}{2}\left(\phi_1^2+\phi_2^2\right) + \tfrac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2.\end{aligned} \tag{19}$$

3.2 The True and False Vacuums

The potential of (19) is graphed in Figure 2 as what is commonly known as the "Mexican hat" potential. We have moved the horizontal axes down from $\mathcal{V} = 0$, as it will prove convenient in subsequent graphs. Note that for small ϕ_1 and ϕ_2, the λ term is negligible and the shape approaches an upside-down paraboloid (dependence on the squares of ϕ_1 and ϕ_2). For large ϕ_1 and ϕ_2, the λ term dominates and $\mathcal{V}$ approximates a 4th order dependence on ϕ_1 and ϕ_2. The $\phi_1 = \phi_2 = 0$ location (where $\mathcal{V} = 0$) is deemed the "false" vacuum, for reasons to be delineated.

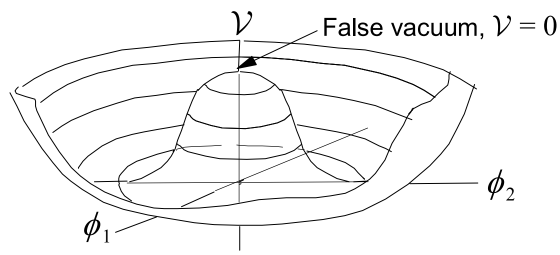


Figure 2. Higgs Field ϕ in Goldstone Model (Mexican Hat)

In classical field theory, the false vacuum is unstable and the same is true in QFT. The universe with a field ϕ having a potential like that shown in Figure 2 will be unstable and tend to move to lower $\mathcal{V}$. In the graph, it will tend to slide down the surface sloping away from the vertical centerline and settle somewhere in the circular trough surrounding the centerline. Which particular location circumferentially is random. See LHS of Figure 3.

We can now see why we called the top of the Mexican hat the false vacuum. It does not represent the lowest energy state. What we call the true vacuum (for which there are many possible locations in the trough of the hat) is the lowest energy state. Note our definition of vacuum for this determination is the lowest energy state, *not* the absence of real particles, since the true vacuum has a non-zero density of ϕ particles.

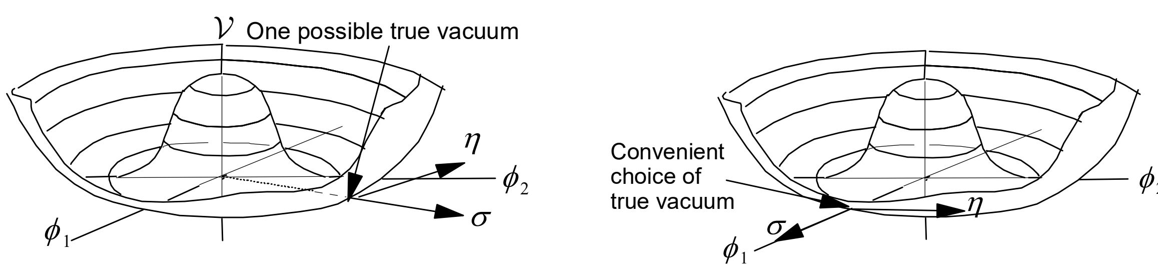


Figure 3. Possible and Preferred True Vacuum

There is no loss in generality in choosing as our true vacuum the location shown in the RHS of Figure 3, since due to symmetry, where we place our axes in the figure is arbitrary. It will turn out, in what follows, that the choice shown in the figure makes our mathematics easiest.

3.3 Characteristics of the Minimum Potential (True Vacuum)

3.3.1 Location of the Minimum

The minimum of $\mathcal{V}$ for $\phi_2 = 0$ is located at a real, positive value for ϕ_1 denoted by the symbol v. This allows us to find an expression relating μ and v, as follows.

$$\left.\frac{\partial \mathcal{V}}{\partial \phi_1}\right|_{\substack{\phi_1=v\\ \phi_2=0}} = 0 = \left(-\mu^2\phi_1 - \tfrac{\lambda}{4}2\left(\phi_1^2+\phi_2^2\right)2\phi_1\right)_{\substack{\phi_1=v\\ \phi_2=0}} \quad\rightarrow\quad \mu^2 = -\lambda\phi_1^2\Big|_{\phi_1=v} = -\lambda v^2 < 0\ , \tag{20}$$

Thus, the effective "mass" μ of ϕ (for λ positive) is imaginary. And the location of the minimum is

$$\phi_1\Big|_{min} = v = \sqrt{\frac{-\mu^2}{\lambda}}\ . \tag{21}$$

3.3.2 Defining New Fields at the Minimum

With an eye to future analyses, we define new fields

$$\sigma = \phi_1 - v \qquad \eta = \phi_2 \qquad \rightarrow \qquad \phi_1 = \sigma + v \qquad \phi_2 = \eta \tag{22}$$

and substitute the RHS of (22) into the potential $\mathcal{V}$ in (19) to get

$$\begin{aligned}\mathcal{V} &= \tfrac{\mu^2}{2}\left(\phi_1^2+\phi_2^2\right)+\tfrac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2 = \tfrac{\mu^2}{2}(\sigma+v)^2+\tfrac{\mu^2}{2}\eta^2+\tfrac{\lambda}{4}\left((\sigma+v)^2+\eta^2\right)^2\\ &= \tfrac{\mu^2}{2}\left(\sigma^2+\eta^2\right)+\mu^2\sigma v+\tfrac{\mu^2}{2}v^2+\tfrac{\lambda}{4}\left(\sigma^2+2\sigma v+v^2+\eta^2\right)^2 .\end{aligned} \tag{23}$$

Using the RHS of (20) for μ^2, this becomes

$$\mathcal{V} = -\tfrac{\lambda v^2}{2}\left(\sigma^2+\eta^2\right)-\lambda v^3\sigma-\tfrac{\lambda v^4}{2}+\tfrac{\lambda}{4}\left(\sigma^2+2\sigma v+v^2+\eta^2\right)^2 \tag{24}$$

Completing the square at the end of (24), we find

$$\begin{aligned}\mathcal{V} = \lambda\Bigg(& \overset{\boxed{1}}{-\frac{v^2\sigma^2}{2}} \ \overset{\boxed{2}}{-\frac{v^2\eta^2}{2}} \ \overset{\boxed{3}}{-v^3\sigma} \ -\frac{v^4}{2} \ +\tfrac{1}{4}\sigma^4 \ +\tfrac{1}{4}\sigma^2 2\sigma v \ \overset{\boxed{1a}}{+\tfrac{1}{4}\sigma^2 v^2} \ +\tfrac{1}{4}\sigma^2\eta^2\\ & +\tfrac{1}{4}\sigma^2 2\sigma v \ +\tfrac{1}{4}4\sigma^2 v^2 \ \overset{\boxed{3a}}{+\tfrac{1}{4}2\sigma v v^2} \ +\tfrac{1}{4}2\sigma v\eta^2\\ & \overset{\boxed{1b}}{+\tfrac{1}{4}\sigma^2 v^2} \ \overset{\boxed{3b}}{+\tfrac{1}{4}2\sigma v v^2} \ +\tfrac{1}{4}v^4 \ \overset{\boxed{2a}}{+\tfrac{1}{4}\eta^2 v^2}\\ & +\tfrac{1}{4}\sigma^2\eta^2 \ +\tfrac{1}{4}2\sigma v\eta^2 \ \overset{\boxed{2b}}{+\tfrac{1}{4}v^2\eta^2} \ +\tfrac{1}{4}\eta^4\Bigg).\end{aligned} \tag{25}$$

The terms with the same numbers (e.g., 1,2,3) in boxes above them cancel. So, we are left with (26) (where the terms with letters over them have the same form and can be combined)

$$\begin{aligned}\mathcal{V} &= \lambda\left(\begin{array}{l} \overset{\boxed{A}}{-\tfrac{1}{2}v^4} \ \overset{\boxed{B}}{+\tfrac{1}{4}\sigma^4} \ +\tfrac{1}{2}\sigma^3 v \ \overset{\boxed{C}}{+\tfrac{1}{4}\sigma^2\eta^2} \ \overset{\boxed{B}}{+\tfrac{1}{2}\sigma^3 v} \ +\sigma^2 v^2 \ \overset{\boxed{D}}{+\tfrac{1}{2}\sigma v\eta^2}\\ \qquad\qquad \overset{\boxed{A}}{+\tfrac{1}{4}v^4} \ \overset{\boxed{C}}{+\tfrac{1}{4}\sigma^2\eta^2} \ \overset{\boxed{D}}{+\tfrac{1}{2}\sigma v\eta^2} \ +\tfrac{1}{4}\eta^4\end{array}\right)\\ &= \lambda\left(\overset{\boxed{A}}{-\frac{v^4}{4}} + \frac{\sigma^4}{4} \ \overset{\boxed{B}}{+\ \sigma^3 v} \ \overset{\boxed{C}}{+\frac{\sigma^2\eta^2}{2}} \ \overset{\boxed{D}}{+\sigma v\eta^2} + \sigma^2 v^2 + \frac{\eta^4}{4}\right).\end{aligned} \tag{26}$$

Rearranging (26), we have

$$\mathcal{V} = \lambda\left(\overset{\boxed{A}}{-\frac{v^4}{4}} + \sigma^2 v^2 + \overset{\boxed{B}}{\sigma^3 v} + \overset{\boxed{D}}{\sigma v \eta^2} + \tfrac{1}{4}\left(\sigma^4 + \overset{\boxed{C}}{2\sigma^2\eta^2} + \eta^4\right)\right) \tag{27}$$
$$= -\tfrac{1}{4}\lambda v^4 + \tfrac{1}{2}\left(2\lambda v^2\right)\sigma^2 + \left(\lambda v\right)\sigma\left(\sigma^2 + \eta^2\right) + \tfrac{\lambda}{4}\left(\sigma^2 + \eta^2\right)^2 .$$

3.3.3 Interpretation of Terms in $\mathcal{V}$ in Light of the New Fields

The first term in (27) and the VEV of $\mathcal{V}$

The first term in the last row of (27) is a constant, that represents a contribution to the energy density of the vacuum at the true vacuum, i.e., after symmetry breaking, by the scalar field. Thus, the vacuum energy density remnant left in the vacuum after symmetry breaking one would expect to measure (the vacuum expectation value, VEV) is the negative value

$$\text{VEV of } \mathcal{V} = \left\langle 0_{true}\right| -\tfrac{1}{4}\lambda v^4 \left|0_{true}\right\rangle = -\tfrac{1}{4}\lambda v^4 \tag{28}$$

Aside on the VEV of ϕ_1

We can see, from (29) that v in the $-\frac{1}{4}\lambda v^4$ term is the VEV of the field ϕ_1 at the true vacuum.

$$\text{VEV of } \phi_1 = \left\langle 0_{true}\right|\phi_1\left|0_{true}\right\rangle = \left\langle 0_{true}\right|\sigma + v\left|0_{true}\right\rangle = \left\langle 0_{true}\right|v\left|0_{true}\right\rangle = v \tag{29}$$

Typically, in QFT, fields have zero VEVs, i.e., one cannot measure the field directly. This is because a typical quantum field contains only construction and destruction operators acting alone and no number operators [no bilinear combinations of operators], so

$$\text{VEV of typical field } \phi_{typical} = \left\langle 0\right|\phi_{typical}\left|0\right\rangle = 0 . \tag{30}$$

But in our case there is a constant term v in our field (along with a typical quantum field σ), so the Higgs field ϕ_3 acquires a VEV as a result of the symmetry breaking.

Back to the VEV of $\mathcal{V}$

We now can see that (28) is proportional to the fourth power of the VEV of the field ϕ_1 (i.e. of v) of (29), and linearly dependent on the positive constant λ. One can think of this as a constant value for the false vacuum Higgs field (v here) pervading all space. However, as we mentioned earlier, much like what is done in other parts of QFT, we ignore constant vacuum energy, no matter how large, and no matter whether it is negative or positive (negative in this case).

The other terms in (27)

The other terms in the bottom row of (27) have been arranged so constants are on the left and fields on the right. All of these terms will end up in the Lagrangian density (18), with opposite signs. There, the 2nd term in the bottom row of (27) acts like a mass term in the free Lagrangian for the σ field, where $2\lambda v^2$ is mass squared. The 3rd and 4th terms are not bilinear (as free terms in the Lagrangian are), but tri and quadrilinear and thus, represent interactions between fields (including the interaction of the σ with itself via σ^3 and σ^4 terms).

For bilinear terms of a certain field (such as σ here) in $\mathcal{L}$, the field equation is linear in that field (via the Euler-Lagrange equation). For cubic or quartic terms in $\mathcal{L}$, the field equation is nonlinear.

Note the mass of our new, *real* field σ, the true vacuum Higgs field, is $\sqrt{2\lambda v^2}$. The mass of the η field is zero, as there is no term having form η^2. We can think of the mass term as a term that is non-zero when small oscillations are made about the zero point ($\sigma = 0$ here) in the direction of the field in the Mexican hat diagram (in the σ axis direction here). Note that the σ field has real, not imaginary mass. Graphically, a real boson mass has an upward curvature (2nd derivative positive) and an imaginary mass (like ϕ if we interpret the ϕ^2 term as a mass term) has a downward curvature (2nd derivative negative).

The original field ϕ is also commonly called the Higgs field, but, more precisely, it is the false vacuum Higgs field. (We may be getting a little ahead of ourselves, as we need a more complicated symmetry breaking model than this in order to match reality, but the Higgs particle found at CERN in 2012 was the true vacuum Higgs, not the false vacuum Higgs.)

For the η field, small oscillations in the η axis direction produce no change, as the curve is essentially flat in that direction. Small oscillations are zero, and so is the η mass. The η field is known as a Goldstone boson field, or a Nambu-Goldstone boson field (after its discoverers). In general, any *massless* boson that results from a broken global symmetry is a Goldstone (or Nambu-Goldstone) boson.

3.3.4 An Aside on Vacuum Energy

Note we have two sources for large vacuum energy that we ignore, the negative (for $\mu^2 < 0$) term $\frac{\mu^2}{2}\sum_{\mathbf{k}}\frac{1}{\omega_{\mathbf{k}}}$ in (14) and the negative term $-\frac{1}{4}\lambda v^4$ in (28). In the usual QFT treatment for scalar fields, μ^2 is positive, and the former term is one part of the zero point energy $\sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}}{2}$ of the free field. (See footnote on pg. 4).

3.4 Comparing False and True Vacuums

3.4.1 Degeneracy

Note from Figure 2 and Figure 3 that there are many possible true vacuums, but only one false vacuum. One says the true vacuum state is degenerate, but the false vacuum state is not.

3.4.2 Rotational Symmetry Transformation about False Vacuum (Vertical Axis in Figure 2)

With respect to false vacuum field ϕ

Complex representation

The U(1) transformation for the Goldstone model can be expressed in a complex representation of ϕ as shown in the first line of (18), i.e., where

$$\text{Complex } \phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right) \quad \phi_1, \phi_2 \text{ real}, \tag{31}$$

as (with α = constant for global symmetry)

$$\begin{aligned} \phi' &= U(1)\phi = e^{i\alpha}\phi = (1 + i\alpha + \ldots)\phi \quad \xrightarrow[\text{i.e., small } \alpha]{\text{infintesimal}} \quad \phi' \approx (1 + i\alpha)\phi = \phi + \delta\phi \\ \phi^{\dagger\prime} &= U^{\dagger}(1)\phi^{\dagger} = e^{-i\alpha}\phi = (1 - i\alpha + \ldots)\phi \quad \xrightarrow[\text{i.e., small } \alpha]{\text{infintesimal}} \quad \phi^{\dagger\prime} \approx (1 - i\alpha)\phi^{\dagger\prime} = \phi^{\dagger\prime} + \delta\phi^{\dagger\prime}. \end{aligned} \tag{32}$$

So, for infinitesimal transformations

$$\delta\phi = i\alpha\phi \qquad\qquad \delta\phi^{\dagger} = -i\alpha\phi^{\dagger}. \tag{33}$$

As an exercise, the reader should substitute the primed quantities in (32) for their respective unprimed quantities in the top row of (18). Do this for both the finite transformation and the infinitesimal transformation. You should get the first row of (18) back again without primes, i.e, you should show (18) is symmetric under the transformation (32).

Note that the quantity $i\alpha$ is known as the generator of the U(1) transformation. As an aside, for those versed in group theory, if we were to have an SU(2) transformation, the generator would be a 2X2 matrix. For an SU(3) transformation, a 3X3 matrix. And then, we would need a doublet form of the field Φ (two complex components) in SU(2) and a triplet form (three complex components) in SU(3).

For $\phi = 0$, there are no ϕ particles (zero expectation value for particles along the ϕ_1 an ϕ_2 axes as described in Section 1.4, pg. 3), so even though ϕ is a field (and not a state), $\phi = 0$ represents the zero ϕ particle state (i.e., the false vacuum) of $|0_{false}\rangle$. Thus, we must have $\alpha\phi = 0 = \delta\phi$. The transformation U(1) leaves $\phi = 0$ as $\phi = 0$, i.e., unchanged.

Real representation

Alternatively, we can represent the same field and the same transformation solely in terms of real entities, as follows.

$$\text{Real } \phi = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}, \tag{34}$$

$$\phi' = \begin{bmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\frac{\phi_1}{\sqrt{2}} \\ \frac{\phi_2}{\sqrt{2}}\end{bmatrix} \quad \xrightarrow[\text{i.e., small } \alpha]{\text{infintesimal}} \quad \phi' \approx \begin{bmatrix}1 & \alpha \\ -\alpha & 1\end{bmatrix}\begin{bmatrix}\frac{\phi_1}{\sqrt{2}} \\ \frac{\phi_2}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{\phi_1}{\sqrt{2}} + \alpha\frac{\phi_2}{\sqrt{2}} \\ \frac{\phi_2}{\sqrt{2}} - \alpha\frac{\phi_1}{\sqrt{2}}\end{bmatrix} = \phi + \delta\phi \tag{35}$$

where the matrix on the LHS of (35) is the simple rotation matrix in 2D. Thus, both forms of the transformation are effectively rotations about the vertical axis in Figure 2. Note,

$$\delta\phi = \frac{1}{\sqrt{2}}\begin{bmatrix}\alpha\phi_2 \\ -\alpha\phi_1\end{bmatrix}. \tag{36}$$

For a further exercise, the reader could check that the transformation (35), for both finite and infinitesimal cases, leaves the second row of the Lagrangian of (18) invariant ($\mathcal{L}$ is symmetric under (35)).

At different locations

Note that for the field ϕ, from either (33) or (36),

$$\left.\begin{array}{l}\text{at } \phi = 0,\ \delta\phi = 0 \rightarrow U(1)\phi = 0 \\ \text{at } \phi \neq 0,\ \delta\phi \neq 0 \rightarrow U(1)\phi \neq 0\end{array}\right\} \begin{array}{l}\text{And also, } \delta\mathcal{L} = 0 \text{ at all } \phi_1 \text{ and } \phi_2 \text{ under} \\ \text{the } U(1) \text{ tranformation about false vacuum}\end{array} \tag{37}$$

<u>With respect to true vacuum real fields σ and η</u>

Real representation only shown

The above results can be expressed in terms of the true vacuum fields σ and η, instead of ϕ (or equivalently, ϕ_1 and ϕ_2). We define the true vacuum fields by the symbol Σ, as in (38).

$$\text{At } \phi = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} \quad \left(\text{in terms of } \sigma \text{ and } \eta \rightarrow \Sigma = \frac{1}{\sqrt{2}}\begin{bmatrix}\sigma \\ \eta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1 - v \\ \phi_2\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-v \\ 0\end{bmatrix}\right) \tag{38}$$

$$\text{At } \phi = \frac{1}{\sqrt{2}}\begin{bmatrix}v \\ 0\end{bmatrix} \quad \left(\text{in terms of } \sigma \text{ and } \eta \rightarrow \Sigma = \frac{1}{\sqrt{2}}\begin{bmatrix}\sigma \\ \eta\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 \\ 0\end{bmatrix}\right) \tag{39}$$

From (38), (39), and (36), we have

$$\left.\begin{array}{l}\text{at } \Sigma = \frac{1}{\sqrt{2}}\begin{bmatrix}-v \\ 0\end{bmatrix},\quad \phi = \frac{1}{\sqrt{2}}\begin{bmatrix}0 \\ 0\end{bmatrix},\quad \delta\phi = 0 \rightarrow \delta\phi_1 = \delta\phi_2 = 0 \rightarrow \delta\Sigma = 0 \\ \text{at } \Sigma = \frac{1}{\sqrt{2}}\begin{bmatrix}0 \\ 0\end{bmatrix},\quad \phi = \frac{1}{\sqrt{2}}\begin{bmatrix}v \\ 0\end{bmatrix},\quad \delta\phi \neq 0 \rightarrow \delta\phi_1, \delta\phi_2 \neq 0 \rightarrow \delta\Sigma \neq 0\end{array}\right\} \begin{array}{l}\text{And also, } \delta\mathcal{L} = 0 \text{ at all } \sigma \text{ and } \eta \text{ under the} \\ U(1) \text{ tranformation about false vacuum}\end{array}. \tag{40}$$

3.4.3 Rotational Transformation about False Vacuum

Now imagine we consider a rotation about a vertical axis passing through the true vacuum of Figure 3 instead of the false vacuum. We will not do the math, but it should be fairly apparent (hopefully) that $\mathcal{V}$, and hence $\mathcal{L}$, will not be symmetric under this rotation, whether we express $\mathcal{L}$ in terms of σ and η or in terms of ϕ_1 and ϕ_2.

<u>Bottom line:</u> $\mathcal{L}$ is symmetric about the false vacuum, but not the true vacuum. This is true regardless of which fields we choose to express $\mathcal{L}$ in terms of.

3.5 Conclusions

<u>False vacuum</u>

Note that, at $\phi = 0$, i) the second derivative of $\mathcal{V}$ with respect to ϕ_1 or ϕ_2 is negative (for $\mu^2 < 0$), ii) $\delta\phi = 0$ means there is no degeneracy of ϕ, iii) symmetry is unbroken ($\mathcal{V}$ is symmetric in ϕ_1 and ϕ_2 and in terms of σ and η; rotate the axes in Figure 2 and all looks just the same), and iv) the VEV of the field ϕ ($= \phi_1 + i\phi_2$) is zero.

Note also that at $\phi = 0$, there are no particles (zero expectation value for particles along the ϕ_1 an ϕ_2 axes as described in Section 1.4 , so ϕ, even though a field (and not a state) $\phi = 0$ represents the zero ϕ particle state (i.e., the false vacuum of $|0\rangle_\phi$. Thus, we must have $\alpha\phi = 0$. This is often, somewhat confusingly in my experience, referred to via the jargon "the generator α destroys the vacuum".

True vacuum

At $\phi = \frac{1}{\sqrt{2}}v$, i) the second derivative of $\mathcal{V}$ with respect to σ is positive for $m_\sigma^2 > 0$ and with respect to η is zero, ii) $\delta\phi \neq$ 0 means there is degeneracy of ϕ, iii) symmetry is broken ($\mathcal{V}$ is unsymmetric in terms of ϕ_1 and ϕ_2 and in terms of σ and η), and iv) the VEV of the field ϕ is not zero.

Since $\delta\phi \neq 0$, then $i\alpha \neq 0$, and the generator α does not destroy the vacuum.

3.6 The Goldstone Theorem

The famous theorem associated with the Goldstone model is shown below and repeated in the last row of Wholeness Chart 2 (with explanations for some of the jargon letter referenced below and in parenthetical remarks in the chart). Note the symmetry in the Goldstone theorem is explicitly for *global*, not local, symmetry transformations, and those global transformations can be more general than what we have looked at here.

Here we have examined global U(1) symmetry, but the Goldstone theorem also hold for any other global symmetry. For those familiar with group theory, these could be SU(2), SU(3), etc. symmetries, which would have a matrix (2X2, 3X3, etc) instead of the scalar α as the generator of the transformation. In this Section 3, we have not examined more general cases than U(1) symmetry, but from working through the implications of that symmetry, we should be able to accept the Goldstone theorem implications for these other types of global symmetry, as well.

Goldstone Theorem (For general case of infinitesimal transformation $\varepsilon_i T_i$ (in above, $\varepsilon_i = i\alpha$; matrix T_i = the number 1)

For any global symmetry of $\mathcal{L}$[a] which is not realized[b] in the spectrum of physical states[c], i) there must exist a massless scalar particle (Goldstone boson), ii) there must be a degeneracy of physical states[d], iii) the corresponding generator of the transformation[e] does not annihilate the vacuum or more generally $T_i\Phi \neq 0$[f] for the true[g] vacuum.

a) i.e., $\delta\mathcal{L} = 0$ for $\Phi \rightarrow \Phi' = \Phi + \delta\Phi = \Phi + \varepsilon_i T_i \Phi$

b) i.e. is broken

c) i.e., range of particles we measure in our universe

d) i.e., of the vacuum

e) i.e. $i\alpha$ for the particular infinitesimal transformation shown earlier

f) i.e., $\delta\Phi \neq 0$

g) i.e, realized

3.7 Summary and Issues for the Goldstone Model

3.7.1 Problems with the Goldstone model are these.

1. It predicts a massless scalar (the η field) particle (a Goldstone boson), which is not observed in nature. A massless particle could not decay into other particles (as there are no lighter particles for it to decay into), so it should be readily found, if it existed.
2. It has no interactions in it, so it can't describe the real world.

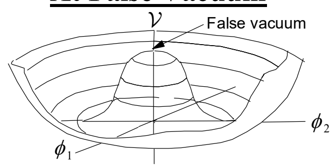
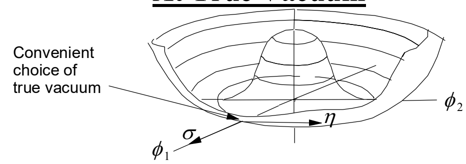
3.7.2 Usefulness of Goldstone model

The Goldstone model has the advantages of 1) being a good lead in to more realistic symmetry breaking models because it is easier to understand, and 2) using the same form for the potential $\mathcal{V}$ in more complicated models (including the one that is part of the standard model of elementary particle physics), so the same mathematical procedure used in it can be transferred directly to, and employed in, those other models.

3.7.3 Bottom line simplified language summary of the Goldstone model results

Any *global* symmetry (i.e., no interactions) in a scalar field potential that is unstable will spontaneously break and lead to a massless scalar field (Goldstone boson) that is not observed in the real world. It will also lead to a massive scalar field.

Wholeness Chart 2. Goldstone Model

Pictorially	**At False Vacuum** 		**At True Vacuum** 	
	Complex Rep	**Real Rep**	**Complex Rep**	**Real Rep**
Scalar Field (2 DOFs)	$\phi(x)=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)$	$\phi(x)=\frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$	$\Sigma=\phi-\frac{v}{\sqrt{2}}=\frac{1}{\sqrt{2}}(\phi_1-v+i\phi_2)$ $=\frac{1}{\sqrt{2}}(\sigma+i\eta)$	$\Sigma=\frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1-v\\ \phi_2\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sigma\\ \eta\end{bmatrix}$
$\mathcal{L}$	$\partial^\mu\phi^\dagger\partial_\mu\phi-\mu^2\lvert\phi\rvert^2-\lambda\lvert\phi\rvert^4$ $\mu^2<0,\ \lambda>0$	$\frac{1}{2}(\partial^\mu\phi_1)^2+\frac{1}{2}(\partial^\mu\phi_2)^2$ $-\frac{\mu^2}{2}(\phi_1^2+\phi_2^2)-\frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2$	Real rep more illuminating	$\left.\frac{1}{2}(\partial^\mu\sigma)^2-\frac{1}{2}(2\lambda v^2)\sigma^2+\frac{1}{2}(\partial^\mu\eta)^2\right\}\mathcal{L}_0$ $\left.-(\lambda v)\sigma(\sigma^2+\eta^2)-\frac{\lambda}{4}(\sigma^2+\eta^2)^2\right\}\mathcal{L}_I$
$\mathcal{V}$	$\mu^2\lvert\phi\rvert^2+\lambda\lvert\phi\rvert^4$	$\frac{\mu^2}{2}(\phi_1^2+\phi_2^2)+\frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2$		$\frac{1}{2}(2\lambda v^2)\sigma^2+(\lambda v)\sigma(\sigma^2+\eta^2)+\frac{\lambda}{4}(\sigma^2+\eta^2)^2$
$\mathcal{H}$	$\partial^0\phi^\dagger\partial_0\phi+\nabla\phi^\dagger\bullet\nabla\phi+\mathcal{V}$ $=\mathcal{V}$ for min $\mathcal{H}$	(form not so important) $=\mathcal{V}$ for min $\mathcal{H}$		$=\mathcal{V}$ for min $\mathcal{H}$
Note	$\mathcal{V}$ ($\mathcal{H}$ & $\mathcal{L}$) sym functions of ϕ_1,ϕ_2 about false vac		$\mathcal{V}$ ($\mathcal{H}$ & $\mathcal{L}$) not sym functs of σ,η about true vac	
Classical Interp	Unstable at $\phi=0$	As at left	Stable in σ at $\sigma=0$. (Not unstable in η)	
QFT Interpret	Unstable at $\phi=0$ & VEV ${}_{false}\langle 0\vert\phi\vert 0\rangle_{false}=0$	As at left	Not unstable at $\sigma=0$. Small changes $\to$ σ particles created. ${}_{true}\langle 0\vert\phi\vert 0\rangle_{true}={}_{true}\langle 0\vert\frac{1}{\sqrt{2}}(\sigma+v)\vert 0\rangle_{true}=\frac{v}{\sqrt{2}}\neq 0$	
Particle Masses	For small ϕ, coeff of ϕ^2 $m_\phi^2=\mu^2<0\to m_\phi$ imag	For small ϕ_1, ϕ_2, as at left	small σ, η, as at right	For small σ, η, coeffs of $\sigma^2/2$, $\eta^2/2$ $m_\sigma^2=2\lambda v^2>0$ $\quad m_\eta^2=0$
Slope of $\mathcal{V}$	Curved downward $\to$ imaginary effective mass		Curved upward $\to$ real mass. Flat $\to$ zero mass	
Degeneracy	For $\mathcal{V}$ (& $\mathcal{L}$ & $\mathcal{H}$) stationary, a unique $\phi=0$. No degeneracy (of false vacuum).		For $\mathcal{V}$ (& $\mathcal{H}$) min ($\mathcal{L}$ max), many possible Σ, i.e., degeneracy of (true) vacuum. (But once universe settles in a vacuum state, it stays there.)	
U(1) Global Symmetry Transformation (α = constant & $\delta\mathcal{L}=\delta\mathcal{V}=0$)	$\phi'=U(1)\phi=e^{i\alpha}\phi$ (finite) Complex plane rotation $\left.\begin{array}{l}\phi=0,\ U(1)\phi=0\\ \phi\neq 0,\ U(1)\phi\neq 0\end{array}\right\}\delta\mathcal{L}=0$ $\delta\phi=i\alpha\phi$ (infinitesimal)	$\phi'=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\frac{\phi_1}{\sqrt{2}}\\ \frac{\phi_2}{\sqrt{2}}\end{bmatrix}$ Real plane rotation As at left for $\delta\mathcal{L}$ $\delta\phi=\frac{1}{\sqrt{2}}\begin{bmatrix}\alpha\phi_2\\ -\alpha\phi_1\end{bmatrix}$ (infinitesimal	Same transformation about false vacuum, i.e., as at $\phi=\frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}\left(\text{or }\Sigma=\frac{1}{\sqrt{2}}\begin{bmatrix}\sigma\\ \eta\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}-v\\ 0\end{bmatrix}\right)$. But at $\phi=\frac{1}{\sqrt{2}}\begin{bmatrix}v\\ 0\end{bmatrix}\left(\text{or }\Sigma=\frac{1}{\sqrt{2}}\begin{bmatrix}\sigma\\ \eta\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ 0\end{bmatrix}\right)$, then $\delta\phi=i\alpha\phi\left(=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\ -\alpha v\end{bmatrix}\right)\neq 0$ (but still $\delta\mathcal{L}=0$)	
Conclusions for $\alpha\phi$=0 & $\delta\mathcal{L}$=0	At $\phi=0$, $\delta\phi=0$ means 1. 2nd deriv of $\mathcal{V}$ wrspt ϕ_1 or ϕ_2 is neg $\to$ imag "mass" 2. No degeneracy of ϕ 3. Sym unbroken about false vac ($\mathcal{V}$ sym in ϕ_1 & ϕ_2) 4. VEV of field ϕ is zero. 5. Generator α annihilates vacuum ($\alpha\phi=0$ at $\phi=0$)		At $\phi=v/\sqrt{2}$, $\delta\phi\neq 0$ means 1. 2nd deriv of $\mathcal{V}$ wrspt η is 0 $\to$ zero mass for η 2. Degeneracy of ϕ 3. Sym broken about true vac ($\mathcal{V}$ not sym in σ & η) 4. VEV of field ϕ is not zero. (Field gets a VEV.) 5. Generator α does not annihilate vacuum ($\alpha\phi\neq 0$)	
Goldstone Theorem	For general case of infinitesimal transformation $\varepsilon_i T_i$ (in above, $\varepsilon_i=i\alpha$; matrix T_i = number1): For any global symmetry of $\mathcal{L}$ (i.e., $\delta\mathcal{L}=0$ for $\Phi\to\Phi+\varepsilon_i T_i\Phi$) which is not realized (i.e. is broken) in the spectrum of physical states (i.e., range of particles we measure in our universe), i) there must exist a massless scalar particle (Goldstone Boson), ii) there must be degeneracy of physical states (i.e., of the vacuum), iii) the corresponding generator of the transformation (i.e. $i\alpha$ for infinitesimal transformation above) does not annihilate the vacuum or more generally $T_i\Phi\neq 0$ (i.e., $\delta\Phi\neq 0$) for the true (realized) vacuum.			

4 Higgs Model

The difference between the Goldstone and Higgs models is simply that the latter includes interactions, as shown in Wholeness Chart 3 below. Note the potential $\mathcal{V}$ is the same.

	Goldstone Model	Higgs Model
Higgs fields	complex scalar ϕ	same as at left
Interactions?	No (global $U(1)$ sym of $\mathcal{L}$)	Yes (local $U(1)$ sym of $\mathcal{L}$)

Wholeness Chart 3. Difference Between Goldstone and Higgs Models

$$\begin{aligned}\mathcal{L} &= \left|D_\mu\phi\right|^2 \quad -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad -\mu^2\phi^2 \quad -\lambda\phi^4 \qquad \phi=\tfrac{1}{\sqrt{2}}\left(\phi_1+i\phi_2\right) \quad \phi_1,\phi_2 \text{ real}\\ &=\left(D^\mu\phi\right)^\dagger\left(D^\mu\phi\right)-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\underbrace{-\tfrac{\mu^2}{2}\left(\phi_1^2+\phi_2^2\right)-\tfrac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2}_{-\mathcal{V}}\end{aligned} \tag{41}$$

where
$$D^\mu=\partial^\mu+iqA^\mu \qquad F^{\mu\nu}=\partial^\nu A^\mu-\partial^\mu A^\nu\,, \tag{42}$$

and where we are using the symbolism $\phi^2=\phi^\dagger\phi$. The Higgs model is summarized in Wholeness Chart 4, pg. 16.

As with the Goldstone model, the minimum of $\mathcal{V}$ for $\phi_2 = 0$ is located at a positive value for ϕ_1 denoted by the symbol v. This allows us to find an expression relating μ and v, as follows.

$$\left.\frac{\partial\mathcal{V}}{\partial\phi_1}\right|_{\substack{\phi_1=v\\ \phi_2=0}}=0=\left(-\mu^2\phi_1-\tfrac{\lambda}{4}2\left(\phi_1^2+\phi_2^2\right)2\phi_1\right)_{\substack{\phi_1=v\\ \phi_2=0}} \quad\rightarrow\quad \mu^2=-\lambda\phi_1^2\Big|_{\phi_1=v}=-\lambda v^2<0\;, \tag{43}$$

And thus, as before, for $\lambda > 0$, the "mass" μ of ϕ is imaginary.

4.1 $\mathcal{V}$ of $\mathcal{L}$ expressed via new fields σ and η

So, just as we did in the Goldstone model in (22), we define new fields

$$\sigma=\phi_1-v \qquad \eta=\phi_2 \qquad \rightarrow \qquad \phi_1=\sigma+v \qquad \phi_2=\eta \tag{44}$$

and with (44) (the same as (22)) into the potential $\mathcal{V}$ of (41) (the same as in (18)), we get the same $\mathcal{V}$ we had in the Goldstone case (the same as (27)),

$$\mathcal{V}=-\tfrac{1}{4}\lambda v^4 \;+\tfrac{1}{2}\left(2\lambda v^2\right)\sigma^2 \;+(\lambda v)\sigma\left(\sigma^2+\eta^2\right) \;+\tfrac{\lambda}{4}\left(\sigma^2+\eta^2\right)^2. \tag{45}$$

4.2 Kinetic terms in $\mathcal{L}$ expressed via new fields σ and η

With the gauge covariant derivative definition of (42), we can calculate the derivative terms in $\mathcal{L}$ of (41) in terms of ϕ_1 and ϕ_2.

$$\begin{aligned}\left|D_\mu\phi\right|^2&=\left(\partial^\mu\phi+iqA^\mu\phi\right)^\dagger\left(\partial_\mu\phi+iqA_\mu\phi\right)=\left(\partial^\mu\phi^\dagger-iqA^\mu\phi^\dagger\right)\left(\partial_\mu\phi+iqA_\mu\phi\right)\\ &=\left(\partial^\mu\phi^\dagger\right)\partial_\mu\phi+iqA_\mu\left(\partial^\mu\phi^\dagger\right)\phi-iqA^\mu\phi^\dagger\partial_\mu\phi+q^2A^\mu A_\mu\phi^\dagger\phi\\ &=\frac{1}{2}\left[\begin{array}{l}\left(\partial^\mu\left(\phi_1-i\phi_2\right)\right)\partial_\mu\left(\phi_1+i\phi_2\right)+iqA_\mu\left(\partial^\mu\left(\phi_1-i\phi_2\right)\right)\left(\phi_1+i\phi_2\right)\\ \qquad -iqA^\mu\left(\phi_1-i\phi_2\right)\partial_\mu\left(\phi_1+i\phi_2\right)+q^2A^\mu A_\mu\left(\phi_1-i\phi_2\right)\left(\phi_1+i\phi_2\right)\end{array}\right]\end{aligned} \tag{46}$$

$$= \frac{1}{2}\left[\begin{array}{l}\left(\partial^\mu\phi_1\right)^2+\left(\partial^\mu\phi_2\right)^2 \overset{\boxed{1}}{+iqA_\mu\left(\partial^\mu\phi_1\right)\phi_1} \overset{\boxed{A}}{-qA_\mu\left(\partial^\mu\phi_1\right)\phi_2} \overset{\boxed{B}}{+qA_\mu\left(\partial^\mu\phi_2\right)\phi_1} \overset{\boxed{2}}{+iqA_\mu\left(\partial^\mu\phi_2\right)\phi_2} \\ \qquad \overset{\boxed{1}}{-iqA_\mu\phi_1\partial^\mu\phi_1} \overset{\boxed{B}}{+qA_\mu\phi_1\partial^\mu\phi_2} \overset{\boxed{A}}{-qA_\mu\phi_2\partial^\mu\phi_1} \overset{\boxed{2}}{-iqA_\mu\phi_2\partial^\mu\phi_2} + q^2A^\mu A_\mu\phi_1^2+q^2A^\mu A_\mu\phi_2^2\end{array}\right]. \tag{47}$$

The terms with numbers over them cancel. The terms with letters over them are equal if ϕ_1 and ϕ_2 commute, and they do because they are different fields (and also because they are real, so for any construction/destruction operator $a(\mathbf{k})$ in them, $a(\mathbf{k})=a^\dagger(\mathbf{k})$, so $\left[a(\mathbf{k}),a^\dagger(\mathbf{k})\right]=\left[a(\mathbf{k}),a(\mathbf{k})\right]=0$. This leaves us with

$$\left|D_\mu\phi\right|^2=\frac{1}{2}\left[\left(\partial^\mu\phi_1\right)^2+\left(\partial^\mu\phi_2\right)^2-2qA_\mu\left(\partial^\mu\phi_1\right)\phi_2+2qA_\mu\left(\partial^\mu\phi_2\right)\phi_1+q^2A^\mu A_\mu\phi_1^2+q^2A^\mu A_\mu\phi_2^2\right]. \tag{48}$$

With the RHS of (44), and realizing v is a constant. (48) becomes

$$\begin{aligned}\frac{1}{2}\left|D^\mu\phi\right|^2&=\frac{1}{2}\left(\partial^\mu\sigma\right)\left(\partial^\mu\sigma\right)+\frac{1}{2}\left(\partial^\mu\eta\right)\left(\partial_\mu\eta\right)-qA_\mu\left(\partial^\mu\sigma\right)\eta+qA_\mu\left(\partial^\mu\eta\right)(\sigma+v)\\&\qquad+\frac{q^2}{2}A^\mu A_\mu(\sigma+v)^2+\frac{q^2}{2}A^\mu A_\mu\eta^2\\&=\frac{1}{2}\left(\partial^\mu\sigma\right)\left(\partial^\mu\sigma\right)+\frac{1}{2}\left(\partial^\mu\eta\right)\left(\partial_\mu\eta\right)-qA_\mu\left(\partial^\mu\sigma\right)\eta+qA_\mu\left(\partial^\mu\eta\right)\sigma+qvA_\mu\left(\partial^\mu\eta\right)\\&\qquad+\frac{q^2}{2}A^\mu A_\mu\sigma^2+q^2vA^\mu A_\mu\sigma+\frac{q^2}{2}v^2A^\mu A_\mu+\frac{q^2}{2}A^\mu A_\mu\eta^2.\end{aligned} \tag{49}$$

Re-arranging so the bilinear (quadratic in fields) terms come first, we have

$$\begin{aligned}\frac{1}{2}\left|D^\mu\phi\right|^2&=\frac{1}{2}\left(\partial^\mu\sigma\right)\left(\partial^\mu\sigma\right)+\frac{1}{2}\left(\partial^\mu\eta\right)\left(\partial_\mu\eta\right)+qvA_\mu\left(\partial^\mu\eta\right)\\&\quad-qA_\mu\left(\partial^\mu\sigma\right)\eta+qA_\mu\left(\partial^\mu\eta\right)\sigma+\frac{q^2}{2}A^\mu A_\mu\sigma^2+q^2vA^\mu A_\mu\sigma+\frac{q^2}{2}v^2A^\mu A_\mu+\frac{q^2}{2}A^\mu A_\mu\eta^2.\end{aligned} \tag{50}$$

4.3 $\mathcal{L}$ expressed via new fields σ and η

With (45) and (50), the last row of (41) becomes

$$\begin{aligned}\mathcal{L}&=\frac{1}{2}\left(\partial^\mu\sigma\right)\left(\partial^\mu\sigma\right)+\frac{1}{2}\left(\partial^\mu\eta\right)\left(\partial_\mu\eta\right)+qvA_\mu\left(\partial^\mu\eta\right)-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\&\quad-qA_\mu\left(\partial^\mu\sigma\right)\eta+qA_\mu\left(\partial^\mu\eta\right)\sigma+\frac{q^2}{2}A^\mu A_\mu\sigma^2+q^2vA^\mu A_\mu\sigma+\frac{q^2}{2}v^2A^\mu A_\mu+\frac{q^2}{2}A^\mu A_\mu\eta^2\\&\quad\underbrace{+\frac{1}{4}\lambda v^4-\frac{1}{2}\left(2\lambda v^2\right)\sigma^2-(\lambda v)\sigma\left(\sigma^2+\eta^2\right)-\frac{\lambda}{4}\left(\sigma^2+\eta^2\right)^2}_{-\mathcal{V}}.\end{aligned} \tag{51}$$

Dropping the constant term $\frac{1}{4}\lambda v^4$ once again, and re-arranging, (51) becomes

$$\begin{aligned}\mathcal{L}&=\underbrace{\frac{1}{2}\left(\partial^\mu\sigma\right)\left(\partial^\mu\sigma\right)-\frac{1}{2}\left(2\lambda v^2\right)\sigma^2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{q^2v^2}{2}A^\mu A_\mu+\frac{1}{2}\left(\partial^\mu\eta\right)\left(\partial_\mu\eta\right)}_{=\mathcal{L}_0\ \text{(bilinear terms, with no mixed products, represent free fields)}}\ \underbrace{+qvA_\mu\left(\partial^\mu\eta\right)}_{\text{interaction of } A_\mu \text{ and } \eta}\\&\quad\underbrace{-qA_\mu\left(\partial^\mu\sigma\right)\eta+qA_\mu\left(\partial^\mu\eta\right)\sigma-(\lambda v)\sigma\left(\sigma^2+\eta^2\right)-\frac{\lambda}{4}\left(\sigma^2+\eta^2\right)^2+\frac{q^2}{2}A^\mu A_\mu\sigma^2+q^2vA^\mu A_\mu\sigma+\frac{q^2}{2}A^\mu A_\mu\eta^2}_{\text{cubic and quartic terms in the fields, represent interactions}}.\end{aligned} \tag{52}$$

Note that the last term in the top row of (52) couples the A^μ and η fields, so they are no longer independent normal fields (at lowest order). Hence, we cannot simply surmise that the 4th term describes a field A^μ of mass q^2v^2.

4.4 Degrees of Freedom and the Unitary Gauge

4.4.1 Degrees of Freedom

Note that the photon field, which is a massless vector field, for a given **k**, has two degrees of freedom (DOF). Any general photon can be considered composed of two independent fields, each with a polarization aligned at a right angle to the other and also at a right angle to the direction of **k**.[*] In 4 D, one might naively expect a 4D vector to have four degrees of freedom, since there would be four independent components, but as shown in the reference just cited, the longitudinal and time-like components for photons precisely cancel one another. As a result, neither is ever measured. The bottom line: massless vector fields effectively have two independent components, two DOF.

A real scalar field, such as ϕ_1 or ϕ_2, has only one DOF. A scalar is a single number classically. (A complex scalar field would have two DOF, the real part being one and the imaginary part being the other.)

A massive vector field, it turns out, has three degrees of freedom. As a result of having mass, it cannot travel at the speed of light, and thus all polarization type components do not have to be perpendicular to **k**, as was the case for photons. This frees up one degree of freedom compared to massless vector fields/particles. Its independent components can align with 3 independent directions in 3D.

4.4.2 Symmetry Breaking Appears to Change the Number of Degrees of Freedom

Note that before symmetry breaking, we have four DOF: one for ϕ_1; one for ϕ_2; and two for the massless A^{μ}.

However, after symmetry breaking, we have five: one for σ; one for η; and three for the massive A^{μ}. The symmetry breaking seems to have added a degree of freedom. But that cannot be a physical reality. We cannot simply redefine our fields and change the nature of reality, the total number of degrees of freedom expressed by the fields.

The answer lies at the heart of gauge theory. There are certain things we can measure, resulting from behaviors of underlying fields, but the fields themselves are not measurable. They are gauge fields. So, as long as we develop a theory where the measurables are unchanged, we can manipulate the underlying (unmeasurable) gauge field in any convenient way. This is what we did in classical e/m and QED. We choose the Lorenz gauge because it made our math easiest.

In our case, we do a similar thing, keeping in mind that nature can be described with only four degrees of freedom in its fields (using fields defined relative to the false vacuum), then any theory with five has an extra, superfluous, one.

4.4.3 The Unitary Gauge

Since, using the fields defined relative to the true vacuum, we have an extra degree of freedom, we can constrain our DOFs in some way, with some constraint equation between them, that will simplify our work.

The simplest such gauge (constraint) choice turns out to be the constraint equation $\underline{\eta = 0}$. This is known as the <u>unitary gauge</u>. With it, (52) takes the much simpler form, with A^{μ} no longer coupled in bilinear fashion with a non-zero field η,

$$\begin{aligned}\mathcal{L} = \tfrac{1}{2}\left(\partial^{\mu}\sigma\right)\left(\partial^{\mu}\sigma\right) - \tfrac{1}{2}\left(2\lambda v^{2}\right)\sigma^{2} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{(qv)^{2}}{2}A^{\mu}A_{\mu} \quad &\Big\}\,\mathcal{L}_{0} \\ -\left(\lambda v\right)\sigma^{3} - \tfrac{\lambda}{4}\sigma^{4} + \tfrac{q^{2}}{2}A^{\mu}A_{\mu}\sigma^{2} + q^{2}vA^{\mu}A_{\mu}\sigma \quad &\Big\}\,\mathcal{L}_{I}\,.\end{aligned} \tag{53}$$

The free field terms are in the top row of (53) and the interaction terms on the bottom row. A^{μ} is now independent at lowest order from other fields, so we can read its mass off directly (4th term). From the 2nd term, $m_{\sigma} = \sqrt{2\lambda v^{2}}$. The A^{μ} field now has a mass whereas it did not have one before symmetry breaking. As an aside, recall[†] that if the photon had mass in QED, we would not have symmetry of $\mathcal{L}$. Thus, the A^{μ} mass term breaks symmetry in the present Higgs model.

4.5 Summary and Issues for the Higgs Model

4.5.1 Problems with the Higgs Model

The Higgs model yields a massive photon and so cannot be correct. Further, it doesn't include any weak force terms, which, of course, are part of our universe.

4.5.2 Advantages of the Higgs Model

The Higgs model advances what we learned in the Goldstone model to fields with interactions, and thus helps us when we proceed to the Weinberg/Glashow/Salam model, which does a pretty good job of describing our world.

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[*] See Klauber[2], Chap. 5, pgs. 141-144, 150-155.

[†] Klauber[2], pg. 296

Wholeness Chart 4. Higgs Model

	At False Vacuum		At True Vacuum	
Pictorially	Same as Goldstone figure		Same as Goldstone figure	
	Complex Rep	**Real Rep**	**Complex Rep**	**Real Rep**
Higgs Field (same as Goldstone)	$\phi(x)=\frac{1}{\sqrt{2}}(\phi_1+i\phi_2)$	$\phi(x)=\frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$	$\Sigma=\phi-\frac{v}{\sqrt{2}}=\frac{1}{\sqrt{2}}(\phi_1-v+i\phi_2)$ $=\frac{1}{\sqrt{2}}(\sigma+i\eta)$	$\Sigma=\frac{1}{\sqrt{2}}\begin{bmatrix}\phi_1-v\\ \phi_2\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}\sigma\\ \eta\end{bmatrix}$
$\mathcal{L}$	$(D^\mu\phi)^\dagger D_\mu\phi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ $-\mu^2\lvert\phi\rvert^2-\lambda\lvert\phi\rvert^4$ $D^\mu=\partial^\mu+iqA^\mu$	$\frac{1}{2}\lvert D^\mu\phi_1\rvert^2+\frac{1}{2}\lvert D^\mu\phi_2\rvert^2-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ $-\frac{\mu^2}{2}(\phi_1^2+\phi_2^2)-\frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2$	Real rep more illumina-ting	For transformation to σ,η coordinates $\left.\begin{array}{l}\frac{1}{2}(\partial^\mu\sigma)^2-\frac{1}{2}(2\lambda v^2)\sigma^2+\frac{1}{2}(\partial^\mu\eta)^2\\ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}(qv)^2A_\mu A^\mu\end{array}\right\}\mathcal{L}_0$ $\left.+qvA_\mu(\partial^\mu\eta)+\text{cubic \& quartic terms}\right\}\mathcal{L}_I$
$\mathcal{V}$	Same as Goldstone model	Same as Goldstone model		Same as Goldstone model
U(1) Local Symmetry Transformation	Infinitesimal form: $\delta\phi=i\alpha(x)\phi$ $\delta A_\mu=-\partial_\mu\alpha(x)$ $\mathcal{L}_{false}$ sym fn of ϕ & A_μ about false vacuum	2D rotation in ϕ_1, ϕ_2 space, different at each x^μ $\mathcal{L}_{false}$ sym fn of ϕ_1, ϕ_2, A_μ about false vacuum		$\mathcal{L}_{true}$ not symmetric function of σ, η, A_μ about true vacuum
Difficulties with Form for $\mathcal{L}$				Product term in A_μ & $\partial^\mu\eta$ means A_μ & η not indep normal modes & cannot equate factors of terms in $\mathcal{L}_0$ with mass. $\mathcal{L}_{false}$ has 4 DOF. $\mathcal{L}_{true}$ has 5 (1 σ, 1 η, 3 from masssive A_μ). Change of variable cannot alter DOF. A non-physical field has arisen.
Fixing Up with Unitary Gauge				Choose unitary gauge $\eta=0$ to restore 4 DOF and make σ & A_μ independent normal modes.
$\mathcal{L}$ Under Unitary Gauge				$\left.\begin{array}{l}\frac{1}{2}(\partial^\mu\sigma)^2-\frac{1}{2}(2\lambda v^2)\sigma^2\\ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}(qv)^2A_\mu A^\mu\end{array}\right\}\mathcal{L}_0$ $\left.\begin{array}{l}-(\lambda v)\sigma^3-\frac{\lambda}{4}\sigma^4\\ +\frac{q^2}{2}A^\mu A_\mu\sigma^2+q^2vA^\mu A_\mu\sigma\end{array}\right\}\mathcal{L}_I$
Masses				For small σ & A_μ, $m_\sigma=\sqrt{2\lambda v^2}\quad m_\gamma=\lvert qv\rvert$
Result				At true vacuum: 1 real, massive scalar field σ (Higgs) 1 real, massive vector field A_μ (photon)
Summary	Global symmetry (Goldstone model) $\rightarrow$ massless Goldstone boson η (not observed) Local symmetry (Higgs model) $\rightarrow$ massive A_μ (not what we observe) and no η.			
Conclusion	Neither the Goldstone nor Higgs model matches reality.			

5 Glashow/Salam/Weinberg Model

5.1 Background

The Glashow/Weinberg/Salam (GSW) model is summarized in Wholeness Chart 6, pg. 29. It should help to follow that as we progress through the development of the model.

5.1.1 GSW Model vs Goldstone and Higgs Models

The GSW model is based on

1) two complex scalar Higgs fields ϕ_a and ϕ_b (instead of one ϕ), which we will find can be expressed in what is known as a doublet (more on this below) $\Phi(x) = \begin{bmatrix} \phi_a \\ \phi_b \end{bmatrix} = \begin{bmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{bmatrix} \quad \phi_1, \phi_2, \phi_3, \phi_4 \text{ real}$ (54)

2) a Lagrangian density with local SU(2) X U(1) symmetry (more on this below)

The difference between the Goldstone, Higgs, and GSW models is shown in Wholeness Chart 5 below.

	Goldstone Model	Higgs Model	Glashow/Salam/Weinberg Model
Higgs fields	complex scalar singlet ϕ	same as at left	complex scalar doublet Φ
Interactions?	No (global U(1) sym of $\mathcal{L}$)	Yes (local U(1) sym of $\mathcal{L}$)	Yes (local U(2) X U(1) sym of $\mathcal{L}$)
Normalization?	$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$	Same as at left	$1/\sqrt{2}$ factor left out of Φ definition (see (54))

Wholeness Chart 5. Difference Between Goldstone, Higgs, and Glashow/Salam/Weinberg Models

5.1.2 Higgs Potential in GSW Model

The Higgs potential in the GSW model parallels that of the Goldstone and Higgs models except the one complex scalar field ϕ used there becomes two complex scalar fields $\phi_a = \phi_1 + i\phi_2$ and $\phi_b = \phi_3 + i\phi_4$ here. So instead of (19), we get (note the $1/\sqrt{2}$ normalization factor is not used in the GSW model)

$$\mathcal{V} = \mu^2\left(\phi_a^2 + \phi_b^2\right) + \lambda\left(\phi_a^2 + \phi_b^2\right)^2 = \mu^2\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right) + \lambda\left(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2\right)^2 \qquad \mu^2 < 0,\ \lambda > 0 \quad (55)$$

Note we can re-write (55) in a more streamlined way, using the doublet form of (54), as

$$\begin{aligned} \mathcal{V} &= \mu^2 \begin{bmatrix} \phi_1 - i\phi_2 & \phi_3 - i\phi_4 \end{bmatrix} \begin{bmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{bmatrix} + \lambda\left(\begin{bmatrix} \phi_1 - i\phi_2 & \phi_3 - i\phi_4 \end{bmatrix} \begin{bmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{bmatrix}\right)^2 \\ &= \mu^2 \Phi^\dagger \Phi + \lambda\left(\Phi^\dagger\Phi\right)^2 \qquad \mu^2 < 0,\ \lambda > 0\ . \end{aligned} \quad (56)$$

So, we can imagine a 5D plot of (55), similar to the 3D plots of Figure 3, with $\mathcal{V}$ on the vertical axis plotted against four mutually perpendicular axes for the four ϕ_i. Except for the number of dimensions, the behavior will be similar. The universe will tend to "fall off" the peak and into the valley, the region of lowest potential.

5.1.3 Ignoring Quarks for Now

To keep things as simple as possible at the beginning, we will ignore quark interactions, both electroweak and with the Higgs field. Once one has an understanding of the lepton behavior under symmetry breaking, incorporation of quarks becomes much easier.

5.1.4 The General Approach

Our Lagrangian at high energy (false vacuum, just after Big Bang) needs to have massless particles (needed for local symmetry to hold and thus, for a viable interaction theory*). At the true vacuum, leptons, vector bosons, and the Higgs field acquire mass, much like we saw for bosons in the Goldstone and Higgs models, and as we will find in the GSW model. The key, we will see, is that these fields are not quite the same fields after symmetry breaking as they were before. Recall, the Higgs σ field in earlier models was related to, but not quite the same as, the Higgs ϕ field. $\mathcal{L}$ was symmetric at the false vacuum in terms of the ϕ field, but not at the true vacuum in terms of the σ field. This general theme will be repeated in the GSW model.

We will first investigate the free Lagrangian (subscript "0") at the high energy of the false vacuum. Then, we will extrapolate from methods used in QED to go from the free to the interaction Lagrangian, still at high energy. Finally, we will explore the GSW theory for how that Lagrangian changes as we go from high to low energies. See (57).

$$\underbrace{\mathcal{L}_0}_{\substack{\text{at false vacuum} \\ \text{(high energy)}}} \quad \rightarrow \quad \underbrace{\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I}_{\substack{\text{at false vacuum} \\ \text{(high energy)}}} \quad \rightarrow \quad \underbrace{\mathcal{L} = \mathcal{L}_0 + \mathcal{L}}_{\substack{\text{at true vacuum} \\ \text{(present epoch energy)}}} \tag{57}$$

Lagrangians We Will Examine (in this order)

5.2 Weak Interactions at High Energy (False Vacuum)

The weak interaction is simpler at high energy, just after the big bang, than it is at low energy (shortly thereafter up to the present) when broken symmetry makes things messier. So, we will first briefly summarize it at high energy, as follows.

5.2.1 Chirality and Weak Interactions

By some seemingly quirky design of nature, the high energy weak interaction only affects certain types of leptons, ones that possess what is called <u>left hand chirality</u>. Chirality is often confused for helicity†, but they are different. The differences between helicity and chirality are explained in Klauber[3] [scroll down to link on chirality vs helicity].

While helicity can be clearly related to the right/left hand rule (thumb in direction of **p**, fingers in direction of spin), there is no such clear relation in chirality, even though chirality is talked about as being left handed or right handed. The nomenclature is unfortunate and it would probably have been better to designate the two types of chirality as black and white, or inside and outside, or something with otherwise opposite senses.

For particles traveling at the speed of light (massless leptons, as neutrinos were once thought to be), those particles must have either left handed or right handed helicity. It just so happens (see Klauber[3] ref above for details) that any such speed-of-light particle having left hand helicity also has left hand chirality. And any such particle having right hand helicity also has right hand chirality. For speeds less than c, this one-to-one correspondence between helicity and chirality does not hold, but this limiting speed characteristic is probably the reason the left hand/right hand nomenclature was adopted for chirality.

Chirality is defined in terms of a Dirac matrix, designated γ^5 ‡.

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 , \tag{58}$$

In the standard (i.e., the Dirac-Pauli) representation,

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} . \tag{59}$$

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* For example, in QED, a massive photon would make $\mathcal{L}$ non-symmetric. See Klauber[2], pg. 295-296. As described therein, a guiding principle in development of quantum field theories with interactions is local symmetry. Nature appears to be set up such that only Lagrangians that are symmetric (invariant) under local transformations correctly describe real world interactions.

† See Klauber[2], pgs. 96, 100, 115.

‡ For the γ^j expressed in the standard (i.e., the Dirac-Pauli) representation, see Klauber[2], pg. 87, equation (4-10).

As an exercise, one can derive the following relations using (59) and the definition of γ^0 in the reference cited.

$$\gamma^{5\dagger} = \gamma^5 \qquad \gamma^5\gamma^0 = -\gamma^0\gamma^5 \qquad \gamma^5\gamma^5 = 1\,. \tag{60}$$

Left and right handed chirality leptons are defined as

$$\psi^L = \tfrac{1}{2}\left(1-\gamma^5\right)\psi \qquad\qquad \psi^R = \tfrac{1}{2}\left(1+\gamma^5\right)\psi\ . \tag{61}$$

As noted, it turns out that at high energy, only the left hand variety ψ^L feels the weak force. The right hand ψ^R does not. No one knows why. We simply know that it does work this way.

As an exercise, using (61) and (60), you can show (answer in Appendix A, pg. 43)

$$\bar{\psi}^L = \psi^{L\dagger}\gamma^0 = \tfrac{1}{2}\bar{\psi}\left(1+\gamma^5\right) \qquad\qquad \bar{\psi}^R = \psi^{R\dagger}\gamma^0 = \tfrac{1}{2}\bar{\psi}\left(1-\gamma^5\right). \tag{62}$$

At high energy, there are three real vector fields designated W_i^μ (i = 1,2,3) that act as the gauge bosons transmitting the weak force. They couple to ψ^L, but not ψ^R. As with any interaction type, there is a type of charge associated with the weak force, appropriately called the weak charge, or more precisely, weak isospin charge, where we elaborate on the nomenclature chosen of "isospin" in Section 8.9, pg. 36 (though it is probably best to wait until you learn a bit more before looking at that). All right hand chiral fields ψ^R, whether they be electron, muon, tau, or their associated neutrinos, have zero weak isospin charge. They don't interact weakly. We remind the reader that the prior statements are made for high energy. Things are more complicated at low energy.

We summarize the operator effect of the right and left fields below. A helpful mnemonic is the following: L and R always refers to particle chirality, never to antiparticle chirality. No bar means particles destroyed; bar means particles created (as in QFT in general). For a given chiral field (e.g., ψ^L), think of what it does to a particle (in this example, destroys LH particle). For the other effect of the same (ψ^L) field, reverse everything (L $\rightarrow$ R; creates; antiparticle).

ψ^L	destroys LH particle, creates RH antiparticle	ψ^R	destroys RH particle, creates LH antiparticle
$\bar{\psi}^L$	creates LH particle, destroys RH antiparticle	$\bar{\psi}^R$	creates RH particle, destroys LH antiparticle

Wholeness Chart Summary of RH/LH Chiral Fields Creation & Destruction Properties

5.2.2 Hypercharge Interactions

As noted, at high energy, things are a little different than at the low energy of our present universe. What we call the electromagnetic field charge in our present day is really a mix of the high energy weak field and another high energy field, similar to but different from, the electromagnetic field. This other field is a single vector field designated herein by B^μ. (Some authors use other symbols, such as X^μ.) The charge associated with it is called the weak hypercharge, whose fundamental unit of charge is designated by g', parallel to the fundamental unit of electric charge being designated by e. Whereas in QED, a particle has fundamental charge $Q = eN_Q$, (for example, an electron has $N_Q = -1$), in weak interaction theory, a given particle has weak hypercharge $g'Y$, where different particles have different values for Y.

Weak hypercharge values of Y for left and right hand spinor fields are shown below where l = 1,2,3 refers to electron, muon, tau, and ν_l where l = 1,2,3 refers to electron neutrino, muon neutrino, tau neutrino.

$$\text{weak hypercharge } \ Y = -\tfrac{1}{2} \text{ for } \psi_l^L \text{ and } \psi_{\nu_l}^L \qquad -1 \text{ for } \psi_l^R \qquad 0 \text{ for } \psi_{\nu_l}^R\,. \tag{63}$$

It is common practice to refer to weak hypercharge as simply Y, rather than the strictly correct $g'Y$. And again, no one knows why the values in (63) are what they are. It is simply that by using these, we eventually get a low energy theory that matches experiment. (Note that many places in the literature define Y as twice that of (63). It can be confusing that different conventions are used by different authors, but it is the unfortunate reality.)

5.2.3 Summary of Weak Isospin, Weak Hypercharge, and Electric Charge

We will discuss weak isospin charge and weak hypercharge in greater depth later on in Section 8.8, which contains a wholeness chart of all leptons and bosons showing their weak isospin charge, weak hypercharge, and electric charge.

5.2.4 The High Energy Theory

So, at the false vacuum, at the beginning of the universe, we have two fields, the weak high energy field and the weak hypercharge field mediated, respectively, by vector bosons W_i^μ and B^μ. We will see that these fields mix with one another to yield somewhat different fields, the low energy weak and electromagnetic fields that we observe today.

It turns out that the math for the one B^μ gauge field is similar to that of the familiar photon field A^μ. That is, it can be described by a theory based on U(1) type transformations (symmetry under $e^{i\alpha}$ gauge transformations on fields ψ^L and ψ^R). In fact, it is analogous in all regards. In e/m theory, particles had e/m charges of +1, –1, and 0. In weak hypercharge theory, they have charges as in (63) and Section 8.8, pg. 33.

The math for the three W_i^μ is a little more complicated. It turns out there that certain combinations of spinor fields naturally group together. In particular, it has been found that grouping the electron and electron neutrino fields together as components in a doublet leads to meaningful symmetries (and thus relevant interactions that match experiment) as below. (Note, as with the Higgs doublet symbolized by the capital letter Φ, we use capital Ψ for lepton doublets.)

$$\Psi_e^L = \begin{pmatrix} \psi_{\nu_e}^L \\ \psi_e^L \end{pmatrix} \qquad \bar{\Psi}_e^L = \left(\bar{\psi}_{\nu_e}^L, \bar{\psi}_e^L\right) \qquad \text{electron doublet} \tag{64}$$

Thus, we will find a term in the free part of the Lagrangian like the LHS in (65) that can be more compactly expressed, using (64), as the RHS of (65).

$$\bar{\psi}_{\nu_e}^L \gamma^\mu \partial_\mu \psi_{\nu_e}^L + \bar{\psi}_e^L \gamma^\mu \partial_\mu \psi_e^L = \bar{\Psi}_e^L \gamma^\mu \partial_\mu \Psi_e^L = \bar{\Psi}_e^L \not{\partial} \Psi_e^L \ . \tag{65}$$

Similarly, it turns out that the interaction terms in the Lagrangian with the W_3^μ field in the LHS of (66) take the compact form of the RHS. (Recall the interaction term in QED had the form $\bar{\psi}_e \not{A} \psi_e$ multiplied by the coupling constant $-e$.)

$$\bar{\psi}_{\nu_e}^L \not{W}_3 \psi_{\nu_e}^L - \bar{\psi}_e^L \not{W}_3 \psi_e^L = \left(\bar{\psi}_{\nu_e}^L, \bar{\psi}_e^L\right)\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \not{W} \begin{pmatrix} \psi_{\nu_e}^L \\ \psi_e^L \end{pmatrix}_3 = \bar{\Psi}_e^L \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \not{W}_3 \Psi_e^L = \bar{\Psi}_e^L \tau_3 \not{W}_3 \Psi_e^L \ . \tag{66}$$

Note the τ_3 matrix is the third Pauli matrix. It turns out that (66) can be generalized to incorporate all three W_i^μ fields into the Lagrangian using all the Pauli matrices, as follows, where repeated i indices means summation. Although we are simply postulating the interaction form for the fields of (67), that form has been validated empirically.

$$\text{Weak interaction terms in Lagrangian of form} \quad \bar{\Psi}_e^L \tau_i \not{W}_i \Psi_e^L \qquad \tau_i = \text{Pauli matrix } i = 1,2,3 \ . \tag{67}$$

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{68}$$

We note now the following identities for Pauli matrices (which can be proven by simple substitution of (68) into (69)), which will be useful in the future. ε_{ijk} is the Levi-Civita permutation tensor. The last row in (69) is merely a generalization of the 2nd and 3rd terms in the first row.

$$\begin{aligned} &\tau_i^\dagger = \tau_i \qquad \tau_1\tau_1 = \tau_2\tau_2 = \tau_3\tau_3 = I \qquad \tau_i\tau_j = i\varepsilon_{ijk}\tau_k \quad (\text{for } i \neq j) \\ &\tau_i\tau_j = \mathrm{I}\delta_{ij} + i\varepsilon_{ijk}\tau_k \qquad \left[\tau_i, \tau_j\right] = 2i\varepsilon_{ijk}\tau_k \qquad (\text{for any } i,j,k) \end{aligned} \tag{69}$$

(66), with τ_3, provided interactions between electron fields and other electron fields, and between electron type neutrinos and other electron type neutrinos, via the W_3 boson. Interactions between electron fields and electron type neutrino fields are provided via the other two Pauli matrices in (67). For example,

$$\bar{\Psi}_e^L \tau_1 \not{W}_1 \Psi_e^L = \left(\bar{\psi}_{\nu_e}^L, \bar{\psi}_e^L\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \not{W} \begin{pmatrix} \psi_{\nu_e}^L \\ \psi_e^L \end{pmatrix}_1 = \left(\bar{\psi}_{\nu_e}^L \not{W}_1 \psi_e^L + \bar{\psi}_e^L \not{W}_1 \psi_{\nu_e}^L\right) = \bar{\psi}_{\nu_e}^L \not{W}_1 \psi_e^L + \bar{\psi}_e^L \not{W}_1 \psi_{\nu_e}^L \ . \tag{70}$$

We can further generalize (67) (again, this is a postulate that ends up working) to include all three lepton families as

All lepton weak interaction terms in $\mathcal{L}$ $\qquad \bar{\Psi}_l^L \tau_i \not{W}_i \Psi_{\nu_l}^L \qquad$ (sum over l and i). $\qquad$ (71)

We will of course (and we do below) have to multiply above relations such as (71) by the appropriate coupling constant for weak interactions (at high energy).

Note we have transitioned into the use of a two dimensional space, with the column matrix doublet of (64) acting as a 2D vector and the 2X2 Pauli matrices acting as operators in that 2D space. We have moved into the arena of group theory, but we won't delve too much more deeply into it here. We do note that just as any n dimensional space (2D in this case) has vectors (column matrices) of n components, it also has scalars of 1 component. In group theory, the equivalent of a 2D vectors is a doublet. The scalar equivalent is a singlet. The group we will be working with is designated SU(2) for "special unitary" in 2D space, for reasons we won't get into here. As an aside, in strong interactions we work in SU(3) theory, where the column vectors have components of fields with the three different color charges and are called triplets.

5.2.5 SU(2) X U(1)

The weak interaction terms in the Lagrangian comprise fields arranged as doublets and singlets that behave as such entities do in standard SU(2) group theory. A transformation in the associated 2D space acts like a 2X2 matrix multiplication on doublets (like a rotation transformation in 2D Euclidean space can be represented as a 2X2 matrix that operates on a two component column vector). The Pauli matrices, along with identity matrix, are, in fact, a basis in this space for all possible such ("rotation-like") transformations. (Although the identity matrix represents zero rotation.)

Note that just as a scalar in 2D Euclidean space is unchanged under a 2D rotation, so is a singlet in SU(2) under an SU(2) transformation.

So, we have fields that are players in the SU(2) domain of weak interactions. And we have fields that are unaffected by the weak interaction and are not players in that field. Likewise, some fields feel the hypercharge force of the U(1) domain, and others do not.

Hence, each field can be considered as made up of combinations of different parts, somewhat like the wave function in non-relativistic quantum mechanics (NRQM) had a spacetime related part (of form e^{-ikx}) and a spin related part (of forms $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$). An operator like 3-momentum ($-i\frac{\partial}{\partial x^i}$) only acted on the space part of the wave function, not the spin part. Conversely, the spin operator acted only on the spin part of the wave function, and not the spacetime part. The two parts of the wave function are in a sense, in different spaces, and each part behaves on its own as an entity in its particular space. We can think of the wave function heuristically as a kind of outer product of the two parts. They don't affect/interact with each other, as an inner product of them would do.

In a similar way, the left hand chirality electron field ψ^L has a part of it that responds to the SU(2) weak force, and a part that responds to the U(1) hypercharge force. An SU(2) operator affects the SU(2) part; a U(1) operator affects the U(1) part. We can think of the two parts as being multiplied together in the field via a, sort of, outer product. This is where the "X" comes from in SU(2) X U(1).

Now some fields don't have one of these parts. For example, the right hand chirality electron field ψ^R has no SU(2) part, as it does not live in the SU(2) space. It doesn't respond to the weak force. It has zero weak charge. It does however have a hypercharge part, since it has weak hypercharge (of – ½ , see (63)). It is a player in U(1) type interactions.

When we construct a Lagrangian (at high energy) below, it will reflect these characteristics.

5.2.6 Minimal Substitution

In Klauber[2], pg. 297, we showed that in QED, inserting the gauge covariant derivative (where $-e$ is the charge on the electron, which is also the coupling constant for QED reflecting the strength of the electromagnetic interaction)

$$D^{\mu} = \partial^{\mu} - ieA^{\mu} \qquad (72)$$

in place of ∂^{μ} in the free Lagrangian gave rise to the correct interaction terms. QED is a U(1) theory. The QED full (free plus interaction) Lagrangian was then also symmetric under the appropriate transformation set. (See reference just cited.)

In developing QFT, researchers postulated that the same type procedure, deemed minimal substitution, should work in SU(2) theories, and thus SU(2) X U(1) theories, as well. And they were right.

Thus, it turns out, as we will soon see, that substituting analogous covariant derivatives (see (73)) into the free field Lagrangian yields the correct interactions for the high energy SU(2) X U(1) electroweak theory. Note that, from group theory (see Ticciati[4]), the vector boson covariant derivatives have somewhat different form from that for fermions and scalars. (It turns out they must be of this form to get a correct theory.) The relations shown are for high energy, where g and g' represent fundamental units of charge, which are also the coupling constants reflecting the strength of the high energy weak and hypercharge interactions, like $-e$ for electromagnetism in (72). The factor of ½ in front of g is conventional. Note that some texts use a different convention without the ½ factor.

$$\begin{gathered}\partial^\mu \to D^\mu = \partial^\mu + \tfrac{i}{2} g \tau_j W_j^\mu + ig'YB^\mu \quad \text{for fermions and scalars} \\ \partial^\mu W_i^\nu \to D^\mu W_i^\nu = \partial^\mu W_i^\nu - \tfrac{g}{2}\varepsilon_{ijk} W_j^\mu W_k^\nu \qquad \partial^\mu B^\nu \to D^\mu B^\nu = \partial^\mu B^\nu \quad \text{for vector bosons}\end{gathered} \tag{73}$$

The presence of the Pauli matrices τ_i in one term of (73), implies that the other terms have a 2X2 identity matrix in them that is not shown explicitly.

5.3 Lagrangian at False Vacuum (Excluding Quarks)

5.3.1 The Free Lagrangian at High Energy

The free Lagrangian at the false vacuum, ignoring quarks for now, has a form one might intuitively expect. The leptons look familiar from QED, the vector gauge boson terms are similar to what we saw for photons*, and the scalar Higgs has the usual form for scalars, except for the λ term, which we have seen before in earlier symmetry breaking models. We do use the streamlined, and meaningful in terms of group theory, doublet notation for the leptons and Higgs.

$$\begin{aligned}\mathcal{L}_0 &= \mathcal{L}_0^L + \mathcal{L}_0^B + \mathcal{L}_0^H \\ \mathcal{L}_0^L &= i\left(\bar{\Psi}_l^L \partial\!\!\!/ \Psi_l^L + \bar{\psi}_l^R \partial\!\!\!/ \psi_l^R + \bar{\psi}_{\nu_l}^R \partial\!\!\!/ \psi_{\nu_l}^R\right) && \text{(Leptons, massless)} \\ \mathcal{L}_0^B &= -\tfrac{1}{4} W_i^{\mu\nu} W_{i\,\mu\nu} - \tfrac{1}{4} B^{\mu\nu} B_{\mu\nu} && \text{(Gauge bosons, false vac)} \\ \mathcal{L}_0^H &= \left(\partial^\mu \Phi\right)^\dagger \left(\partial_\mu \Phi\right) - \mu^2 \Phi^\dagger \Phi - \lambda\left(\Phi^\dagger \Phi\right)^2 && \text{(Higgs boson, false vac)} \\ & \qquad W_i^{\mu\nu} = \partial^\nu W_i^\mu - \partial^\mu W_i^\nu \qquad B^{\mu\nu} = \partial^\nu B^\nu - \partial^\mu B^\nu\end{aligned} \tag{74}$$

In (74) we use slightly different (easier to remember in my opinion) notation from Mandl and Shaw[1] where our $W_i^{\mu\nu}$ equals their $F_i^{\mu\nu}$.

5.3.2 The Interaction Lagrangian at High Energy

To get electroweak interaction theory at high energy, we do two things.

1) Use minimal substitution of (73) into (74), and
2) postulate interaction (coupling) terms between the Higgs and the lepton fields.

Doing this, we find the full Lagrangian at false vacuum of (75) (where g_l and g_{l_ν} are coupling constants between the Higgs and the leptons and $\tilde{\Phi}$ is defined in (82)). (75) has a border around it because, in my opinion, it is the easiest way to remember the high energy $\mathcal{L}$. We elaborate on each term in (75) further on below.

$$\boxed{\begin{aligned}\mathcal{L} &= \mathcal{L}^L + \mathcal{L}^B + \mathcal{L}^H + \mathcal{L}^{LH} \\ \mathcal{L}^L &= i\left(\bar{\Psi}_l^L D\!\!\!\!/ \Psi_l^L + \bar{\psi}_l^R D\!\!\!\!/ \psi_l^R + \bar{\psi}_{\nu_l}^R D\!\!\!\!/ \psi_{\nu_l}^R\right) && \text{(Leptons, massless)} \\ \mathcal{L}^B &= -\tfrac{1}{4} G_i^{\mu\nu} G_{i\,\mu\nu} - \tfrac{1}{4} B^{\mu\nu} B_{\mu\nu} && \text{(Gauge bosons, false vac)} \\ \mathcal{L}^H &= \left(D^\mu \Phi\right)^\dagger \left(D_\mu \Phi\right) - \mu^2 \Phi^\dagger \Phi - \lambda\left(\Phi^\dagger \Phi\right)^2 && \text{(Higgs boson, false vac)} \\ \mathcal{L}^{LH} &= -g_l\left(\bar{\Psi}_l^L \psi_l^R \Phi + \Phi^\dagger \bar{\psi}_l^R \Psi_l^L\right) - g_{\nu_l}\left(\bar{\Psi}_{\nu_l}^L \psi_{\nu_l}^R \tilde{\Phi} + \tilde{\Phi}^\dagger \bar{\psi}_{\nu_l}^R \Psi_{\nu_l}^L\right) && \text{(Lepton-Higgs coupled)} \\ & \text{where } G_i^{\mu\nu} = \partial^\nu W_i^\nu - \partial^\mu W_i^\nu - g\varepsilon_{ijk} W_j^\nu W_k^\mu\end{aligned}} \tag{75}$$

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* Klauber[2], pg. 288, eq (11-6) and (11-7). $F^{\mu\nu} = \partial^\nu A^\nu - \partial^\mu A^\nu$.

For $\mathcal{L}^L$:

There are three lepton families (l = 1,2,3 for e, μ, τ). g is the weak coupling constant between each of these leptons and the W_i^μ; g' is the hypercharge coupling constant between the leptons and B^μ.

$$\Psi_l^L = \begin{pmatrix} \psi_{\nu_l}^L \\ \psi_l^L \end{pmatrix} \qquad \bar{\Psi}_l^L = \left(\bar{\psi}_{\nu_l}^L, \bar{\psi}_l^L\right) \tag{76}$$

Since the high energy right hand chirality leptons do not feel the weak force, we drop the term in W_j^μ in the second and third lines of (77) so we will not get terms in the Lagrangian coupling ψ_l^R and $\psi_{\nu_l}^R$ with the W_j^μ.

$$\begin{aligned} \not{D}\Psi_l^L &= \gamma_\mu D^\mu \Psi_l^L = \gamma_\mu \left(\partial^\mu + \tfrac{i}{2} g\tau_j W_j^\mu + ig'YB^\mu\right)\Psi_l^L \\ \not{D}\psi_l^R &= \gamma_\mu D^\mu \psi_l^R = \gamma_\mu \left(\partial^\mu + ig'YB^\mu\right)\psi_l^R \\ \not{D}\psi_{\nu_l}^R &= \gamma_\mu D^\mu \psi_{\nu_l}^R = \gamma_\mu \left(\partial^\mu + ig'YB^\mu\right)\psi_{\nu_l}^R \end{aligned} \tag{77}$$

$$\text{weak hypercharge } \; Y = -\tfrac{1}{2} \text{ for } \Psi_l^L \qquad -1 \text{ for } \psi_l^R \qquad 0 \text{ for } \psi_{\nu_l}^R \tag{78}$$

For $\mathcal{L}^B$:

We assume the W_i^μ and B^μ fields are not coupled to one another.

$$\begin{aligned} W_i^{\mu\nu} &= \partial^\nu W_i^\mu - \partial^\mu W_i^\nu \rightarrow G_i^{\mu\nu} \\ G_i^{\mu\nu} &= D^\nu W_i^\mu - D^\mu W_i^\nu = \partial^\nu W_i^\mu - \tfrac{g}{2}\varepsilon_{ijk} W_j^\nu W_k^\mu - \partial^\mu W_i^\nu + \tfrac{g}{2}\varepsilon_{ijk} W_j^\mu W_k^\nu \\ &= \partial^\nu W_i^\mu - \partial^\mu W_i^\nu - \tfrac{g}{2}\varepsilon_{ijk} W_j^\nu W_k^\mu - \tfrac{g}{2}\varepsilon_{ikj} W_k^\nu W_j^\mu = \partial^\nu W_i^\mu - \partial^\mu W_i^\nu - \tfrac{g}{2}\varepsilon_{ijk} W_j^\nu W_k^\mu - \tfrac{g}{2}\varepsilon_{ijk} W_j^\nu W_k^\mu \, . \\ &= \partial^\nu W_i^\mu - \partial^\mu W_i^\nu - \tfrac{g}{2}\varepsilon_{ijk} W_j^\nu W_k^\mu - \tfrac{g}{2}\varepsilon_{ijk} W_j^\nu W_k^\mu = \underbrace{\partial^\nu W_i^\mu - \partial^\mu W_i^\nu}_{W_i^{\mu\nu}} - g\varepsilon_{ijk} W_j^\nu W_k^\mu \, . \end{aligned} \tag{79}$$

For $D^\nu B^\mu$, one might expect a similar relation in (73) as for W_i^μ, but we only have one field B^μ, not three (as with the W_i^μ), so the last term in (73), for $D^\mu B^\nu$, becomes zero. Thus,

$$B^{\mu\nu} \quad \rightarrow \quad D^\nu B^\nu - D^\mu B^\nu = \partial^\nu B^\nu - \partial^\mu B^\nu = B^{\mu\nu} \, . \tag{80}$$

As an aside, in QED, we naively used the same minimal substitution in the free photon term of the Lagrangian as for the free fermion term. (See Klauber[2], pg. 297). It worked there because, for a U(1) field like A^μ (similar to B^μ here), we get the same thing either way. That is, we got a Lagrangian photon term having a factor like (80) with A^μ in place of B^μ.

Note that non-Abelian (non-commuting) transformation groups, such as SU(2) here for the W_i^μ fields, give rise to extra terms in the interaction Lagrangian, like that on the RHS of the bottom row of (79). Abelian groups, such as U(1) here for the B^μ, have only a single field associated with them, so do not give rise to these extra terms, as shown in (80).

For $\mathcal{L}^H$:

$$D^\mu \Phi = \left(\partial^\mu + \tfrac{i}{2} g\tau_j W_j^\mu + ig'YB^\mu\right)\Phi \qquad\qquad Y = \tfrac{1}{2} \text{ for } \Phi \tag{81}$$

For $\mathcal{L}^{LH}$:

The symbol $\tilde{\Phi}$ used in (75) is defined as

$$\tilde{\Phi} = \begin{pmatrix} \phi_b^* \\ -\phi_a^* \end{pmatrix} = \begin{bmatrix} \phi_3 - i\phi_4 \\ -\phi_1 + i\phi_2 \end{bmatrix} . \tag{82}$$

There is nothing really to derive here from the free Lagrangian, as we are simply postulating that this term be inserted as part of the interaction Lagrangian. The coupling in lepton-Higgs terms is sometimes called Yukawa coupling.

5.4 Why This Form for $\mathcal{L}$?

The Lagrangian takes the form (75) because that form is symmetric. Recall from QED, that if the Lagrangian is symmetric under some set of local transformations of its fields, then the interactions that show up in that particular Lagrangian mirror those in the real world. See Klauber[2], pgs. 293-298. Further, Gerardus 't Hooft has shown that the electroweak Lagrangian must be symmetric to be renormalizable. We need a symmetric $\mathcal{L}$.

The transformation set under which (75) is symmetric (invariant) is shown below, where the symbol x is shorthand for x^{μ}. Demonstrating explicitly how this invariance is so is lengthy. We do it in Appendix B, pg. 45.

5.4.1 Local Finite Transformations

SU(2)

$$\begin{aligned}
&\Psi_l^L(x) \rightarrow \Psi_l^{L\prime} = SU(2)\Psi_l^L = e^{i\omega_i(x)\tau_i/2}\Psi_l^L = \left(1+\tfrac{i}{2}\omega_i(x)\tau_i+....\right)\Psi_l^L = \Psi_l^L + \delta\Psi_l^L \\
&\bar{\Psi}_l^L(x) \rightarrow \bar{\Psi}_l^{L\prime} = \bar{\Psi}_l^L\left(SU(2)\right)^{\dagger} = \bar{\Psi}_l^L e^{-i\omega_i(x)\tau_i/2} = \bar{\Psi}_l^L\left(1-\tfrac{i}{2}\omega_i(x)\tau_i+....\right) = \bar{\Psi}_l^L + \delta\bar{\Psi}_l^L \\
&\psi^R_{l\,\&\,\nu_l}(x) \rightarrow \psi^R_{l\,\&\,\nu_l}{}' = \psi^R_{l\,\&\,\nu_l} \\
&\bar{\psi}^R_{l\,\&\,\nu_l}(x) \rightarrow \bar{\psi}^R_{l\,\&\,\nu_l}{}' = \bar{\psi}^R_{l\,\&\,\nu_l}
\end{aligned} \tag{83}$$

$$W_i^{\mu}(x) \rightarrow W_i^{\mu\prime}\tau_i = e^{i\omega_i(x)\tau_i/2}\left(W_j^{\mu}\tau_j\right)e^{-i\omega_k(x)\tau_k/2} = \left(W_i^{\mu}+\delta W_i^{\mu}\right)\tau_i \tag{84}$$

$$B^{\mu} \rightarrow B^{\mu\prime} = B^{\mu} \tag{85}$$

$$\begin{aligned}
&\Phi(x) \rightarrow \Phi' = e^{i\omega_i(x)\tau_i/2}\Phi = \left(1+\tfrac{i}{2}\omega_i(x)\tau_i+...\right)\Phi = \Phi + \delta\Phi \\
&\tilde{\Phi}(x) \rightarrow \tilde{\Phi}' = e^{i\omega_i(x)\tau_i/2}\tilde{\Phi} = \left(1+\tfrac{i}{2}\omega_i(x)\tau_i+...\right)\tilde{\Phi} = \tilde{\Phi} + \delta\tilde{\Phi}
\end{aligned} \tag{86}$$

U(1)

$$\begin{aligned}
&\Psi_l^L(x) \rightarrow \Psi_l^{L\prime} = U(1)\Psi_l^L = e^{iYf(x)}\Psi_l^L = \left(1+iYf(x)+....\right)\Psi_l^L = \Psi_l^L + \delta\Psi_l^L \\
&\bar{\Psi}_l^L(x) \rightarrow \bar{\Psi}_l^{L\prime} = \bar{\Psi}_l^L\left(U(1)\right)^{\dagger} = \bar{\Psi}_l^L e^{-iYf(x)} = \bar{\Psi}_l^L\left(1-iYf(x)+....\right) = \bar{\Psi}_l^L + \delta\bar{\Psi}_l^L \\
&\psi_l^R(x) \rightarrow \psi_l^{R\prime} = U(1)\psi_l^R = e^{iYf(x)}\psi_l^R = \left(1+iYf(x)+....\right)\psi_l^R = \psi_l^R + \delta\psi_l^R \\
&\bar{\psi}_l^R(x) \rightarrow \bar{\psi}_l^{R\prime} = \bar{\psi}_l^R\left(U(1)\right)^{\dagger} = \bar{\psi}_l^R e^{-iYf(x)} = \bar{\psi}_l^R\left(1-iYf(x)+....\right) = \bar{\psi}_l^R + \delta\bar{\psi}_l^R \\
&\psi_{\nu_l}^R(x) \rightarrow \psi_l^{R\prime} = \psi_{\nu_l}^R \qquad\qquad \bar{\psi}_{\nu_l}^R(x) \rightarrow \bar{\psi}_l^{R\prime} = \bar{\psi}_{\nu_l}^R
\end{aligned} \tag{87}$$

$$W_i^{\mu}(x) \rightarrow W_i^{\mu\prime} = W_i^{\mu} - \frac{1}{g}\partial^{\mu}\omega_i(x) = W_i^{\mu} + \delta W_i^{\mu} \tag{88}$$

$$B^{\mu} \rightarrow B^{\mu\prime} = B^{\mu} - \frac{1}{g'}\partial^{\mu}f(x) = B^{\mu} + \delta B^{\mu} \tag{89}$$

$$\begin{aligned}
&\Phi(x) \rightarrow \Phi' = e^{iYf(x)}\Phi = \left(1+iYf(x)+...\right)\Phi = \Phi + \delta\Phi \\
&\tilde{\Phi}(x) \rightarrow \tilde{\Phi}' = e^{-iYf(x)}\tilde{\Phi} = \left(1-iYf(x)+...\right)\tilde{\Phi} = \tilde{\Phi} + \delta\tilde{\Phi}
\end{aligned} \tag{90}$$

5.4.2 Local Infinitesimal Transformations

It is far easier to examine the invariance of $\mathcal{L}$ for infinitesimal transformations, i.e., for the above where ω_i and f are very small. One can then, in principle, integrate to obtain the finite transformation case. But, if $\mathcal{L}$ is symmetric under the infinitesimal transformation, then it should also be so under the finite transformation. So, we take the easy (most efficient) way out, and examine the Lagrangian symmetry under the infinitesimal transformation set corresponding to the finite set (83) through (90). The relevant relations are then as follows.

As an exercise, derive (92) below from (84) using the first relation in the second row of (69). Answer is in Appendix A, pg. 43.

SU(2)

$$
\begin{array}{lll}
\Psi_l^L(x)\rightarrow\Psi_l^{L\prime}\approx\left(1+\frac{i}{2}\omega_i(x)\tau_i\right)\Psi_l^L=\Psi_l^L+\delta\Psi_l^L & \delta\Psi_l^L=\frac{i}{2}\omega_i(x)\tau_i\Psi_l^L & \\
\bar{\Psi}_l^L(x)\rightarrow\bar{\Psi}_l^{L\prime}\approx\bar{\Psi}_l^L\left(1-\frac{i}{2}\omega_i(x)\tau_i\right)=\bar{\Psi}_l^L+\delta\bar{\Psi}_l^L & \delta\bar{\Psi}_l^L=-\frac{i}{2}\omega_i(x)\bar{\Psi}_l^L\tau_i & (91)\\
\psi_{l\,\&\nu_l}^R(x)\rightarrow\psi_{l\,\&\nu_l}^{R}{}'=\psi_l^R & \delta\psi_{l\,\&\nu_l}^R=0 & \\
\bar{\psi}_{l\,\&\nu_l}^R(x)\rightarrow\bar{\psi}_{l\,\&\nu_l}^{R}{}'=\bar{\psi}_{l\,\&\nu_l}^R & \delta\bar{\psi}_{l\,\&\nu_l}^R=0 & \\
W_i^\mu(x)\rightarrow W_i^{\mu\prime}\approx W_i^\mu-\varepsilon_{ijk}\omega_j(x)W_k^\mu=W_i^\mu+\delta W_i^\mu & \delta W_i^\mu=-\varepsilon_{ijk}\omega_j(x)W_k^\mu & (92)\\
B^\mu\rightarrow B^{\mu\prime}=B^\mu & \delta B^\mu=0 & (93)\\
\Phi(x)\rightarrow\Phi'\approx\left(1+\frac{i}{2}\omega_i(x)\tau_i\right)\Phi=\Phi+\delta\Phi & \delta\Phi=\frac{i}{2}\omega_i(x)\tau_i\Phi & \\
\tilde{\Phi}(x)\rightarrow\tilde{\Phi}'\approx\left(1+\frac{i}{2}\omega_i(x)\tau_i\right)\tilde{\Phi}=\tilde{\Phi}+\delta\tilde{\Phi} & \delta\tilde{\Phi}=\frac{i}{2}\omega_i(x)\tau_i\tilde{\Phi} & (94)
\end{array}
$$

U(1)

$$
\begin{array}{lll}
\Psi_l^L=0(x)\rightarrow\Psi_l^{L\prime}\approx\left(1+iYf(x)\right)\Psi_l^L=\Psi_l^L+\delta\Psi_l^L & \delta\Psi_l^L=iYf(x)\Psi_l^L & \\
\bar{\Psi}_l^L(x)\rightarrow\bar{\Psi}_l^{L\prime}\approx\bar{\Psi}_l^L\left(1-iYf(x)\right)=\bar{\Psi}_l^L+\delta\bar{\Psi}_l^L & \delta\bar{\Psi}_l^L=-iYf(x)\bar{\Psi}_l^L & \\
\psi_l^R(x)\rightarrow\psi_l^{R\prime}\approx\left(1+iYf(x)\right)\psi_l^R=\psi_l^R+\delta\psi_l^R & \delta\psi_l^R=iYf(x)\psi_l^R & (95)\\
\bar{\psi}_l^R(x)\rightarrow\bar{\psi}_l^{R\prime}\approx\bar{\psi}_l^R\left(1-iYf(x)\right)=\bar{\psi}_l^R+\delta\bar{\psi}_l^R & \delta\bar{\psi}_l^R=-iYf(x)\bar{\psi}_l^R & \\
\psi_{\nu_l}^R(x)\rightarrow\psi_l^{R\prime}=\psi_{\nu_l}^R\qquad\bar{\psi}_{\nu_l}^R(x)\rightarrow\bar{\psi}_l^{R\prime}=\bar{\psi}_{\nu_l}^R & \delta\psi_{\nu_l}^R=\delta\bar{\psi}_{\nu_l}^R=0 & \\
W_i^\mu(x)\rightarrow W_i^{\mu\prime}\approx W_i^\mu-\frac{1}{g}\partial^\mu\omega_i(x)=W_i^\mu+\delta W_i^\mu & \delta W_i^\mu=-\frac{1}{g}\partial^\mu\omega_i(x) & (96)\\
B^\mu\rightarrow B^{\mu\prime}=B^\mu-\frac{1}{g'}\partial^\mu f(x)=B^\mu+\delta B^\mu & \delta B^\mu=-\frac{1}{g'}\partial^\mu f(x) & (97)\\
\Phi(x)\rightarrow\Phi'\approx\left(1+iYf(x)\right)\Phi=\Phi+\delta\Phi & \delta\Phi=iYf(x)\Phi & \\
\tilde{\Phi}(x)\rightarrow\tilde{\Phi}'\approx\left(1-iYf(x)\right)\tilde{\Phi}=\tilde{\Phi}+\delta\tilde{\Phi} & \delta\tilde{\Phi}=-iYf(x)\tilde{\Phi} & (98)
\end{array}
$$

5.4.3 Local SU(2)XU(1) Infinitesimal Transformations

Under SU(2)XU(1) transformation combining the local, infinitesimal SU(2) and U(1) transformations, the differences in the fields (the terms with δ in front of them) add. Thus,

$$
\begin{array}{llll}
\Psi_l^L\rightarrow\Psi_l^{L\prime}\approx\left(1+\frac{i}{2}\omega_i(x)\tau_i+iYf(x)\right)\Psi_l^L=\Psi_l^L+\delta\Psi_l^L & \delta\Psi_l^L=\frac{i}{2}\omega_i(x)\tau_i\Psi_l^L+iYf(x)\Psi_l^L & Y=-\frac{1}{2} & \\
\bar{\Psi}_l^L\rightarrow\bar{\Psi}_l^{L\prime}\approx\bar{\Psi}_l^L\left(1-\frac{i}{2}\omega_i(x)\tau_i-iYf(x)\right)=\bar{\Psi}_l^L+\delta\bar{\Psi}_l^L & \delta\bar{\Psi}_l^L=-\frac{i}{2}\omega_i(x)\bar{\Psi}_l^L\tau_i-iYf(x)\bar{\Psi}_l^L & & \\
\psi_l^R\rightarrow\psi_l^{R\prime}=\psi_l^R+iYf(x)\psi_l^R & \delta\psi_l^R=iYf(x)\psi_l^R & Y=-1 & (99)\\
\bar{\psi}_l^R\rightarrow\bar{\psi}_l^{R\prime}=\bar{\psi}_l^R-iYf(x) & \delta\bar{\psi}_l^R=-iYf(x)\bar{\psi}_l^R & & \\
\psi_{\nu_l}^R(x)\rightarrow\psi_l^{R\prime}=\psi_{\nu_l}^R\qquad\bar{\psi}_{\nu_l}^R(x)\rightarrow\bar{\psi}_l^{R\prime}=\bar{\psi}_{\nu_l}^R & \delta\psi_{\nu_l}^R=\delta\bar{\psi}_{\nu_l}^R=0 & & \\
W_i^\mu\rightarrow W_i^{\mu\prime}\approx W_i^\mu-\frac{1}{g}\partial^\mu\omega_i(x)-\varepsilon_{ijk}\omega_j(x)W_k^\mu=W_i^\mu+\delta W_i^\mu & \delta W_i^\mu=-\frac{1}{g}\partial^\mu\omega_i(x)-\varepsilon_{ijk}\omega_j(x)W_k^\mu & & (100)\\
B^\mu\rightarrow B^{\mu\prime}=B^\mu-\frac{1}{g'}\partial^\mu f(x) & \delta B^\mu=-\frac{1}{g'}\partial^\mu f(x) & & (101)\\
\Phi\rightarrow\Phi'\approx\left(1+\frac{i}{2}\omega_i(x)\tau_i+iYf(x)\right)\Phi=\Phi+\delta\Phi & \delta\Phi=\frac{i}{2}\omega_i(x)\tau_i\Phi+iYf(x)\Phi & Y=\frac{1}{2} & \\
\tilde{\Phi}\rightarrow\tilde{\Phi}'\approx\left(1+\frac{i}{2}\omega_i(x)\tau_i-iYf(x)\right)\tilde{\Phi}=\tilde{\Phi}+\delta\tilde{\Phi} & \delta\tilde{\Phi}=\frac{i}{2}\omega_i(x)\tau_i\tilde{\Phi}-iYf(x)\Phi & & (102)
\end{array}
$$

5.4.4 Showing $\mathcal{L}$ is Symmetric

As shown in Appendix B, pg. 45, plugging the primed fields of (99) to (102) in for the unprimed fields in (75) gives us (75) back again in terms of unprimed fields. In other words, $\mathcal{L}$ of (75) is invariant under the set of transformations (99) to (102)

5.5 Returning to Breaking of Higgs Field Symmetry

We now turn back to our original focus (before justifying the form of $\mathcal{L}$ for the false vacuum as we did in Sect. 5.4) of breaking the Higgs field (in our present case represented by Φ) symmetry.

5.5.1 Higgs Fields in Terms of Other Real Fields

The Higgs field doublet Φ at false vacuum is represented in (54) (and (82)) by the real fields $\phi_1, \phi_2, \phi_3, \phi_4$. Just as we did earlier in the Goldstone and Higgs models, we examine only the potential part of the free Φ field of (18), i.e.,

$$\mathcal{V} = \mu^2 \Phi^\dagger \Phi + \lambda \left(\Phi^\dagger \Phi\right)^2 , \tag{103}$$

which is symmetric in Φ (and in $\phi_1, \phi_2, \phi_3, \phi_4$). We then, again similar to earlier procedures, designate new fields that are functions of the old ones. That is, where η_1, η_2, σ, and η_3, are those new (real) fields, normalized by convention with a $\sqrt{2}$ factor, and v is a real constant.

$$\phi_1 = \frac{1}{\sqrt{2}}\eta_1;\;\; \phi_2 = \frac{1}{\sqrt{2}}\eta_2;\;\; \phi_3 = \frac{1}{\sqrt{2}}(\sigma + v);\;\; \phi_4 = \frac{1}{\sqrt{2}}\eta_3 \quad \rightarrow \quad \Phi(x) = \begin{bmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \eta_1 + i\eta_2 \\ (\sigma + v) + i\eta_3 \end{bmatrix} . \tag{104}$$

It turns out that when we substitute (104) into (75),and move to the minimum of $\mathcal{V}$ at the true vacuum, we end up with 1) $\mathcal{L}$ being not symmetric in η_1, η_2, σ, and η_3, and 2) additional degrees of freedom (independent components of fields) in $\mathcal{L}$ from what we had in $\mathcal{L}$ for $\phi_1, \phi_2, \phi_3, \phi_4$. This parallels the Higgs model case, where we had one extra degree of freedom at the true vacuum with the new fields versus the false vacuum with the original fields. Recall that the additional degree of freedom in the Higgs model arose from the transformation of two massless boson at false vacuum to one massless boson and one massive boson at the true vacuum. The change in massive bosons by one meant one additional degree of freedom. Here we start with four real massless fields instead of two, and like in Higgs model, we end up with only one massless boson at the true vacuum. Because here in the GSW model we have three more massive particles than originally, we pick up three additional degrees of freedom.

We have not shown this gaining of mass by bosons behavior explicitly for the GSW model, as it is cumbersome to do so. We can gain some confidence in our presumption of this behavior by recognizing that our presuming of it will (as we will see) lead to a valid theory, supported by experiment.

Recall that we cannot measure fields directly, and our measurables cannot be affected by how we designate our fields. That is, our fields are gauge fields. See Klauber[2], pgs 177-178. So, our having unneeded extra degrees means we are free to constrain those 3 extra degrees of freedom in any way we like. That is, we can designate a gauge condition for each, as is convenient, without affecting any measurables.

So, of course, we want to pick the gauge conditions that give us the simplest way to analyze the case at hand. That turns out to be what is called the <u>unitary gauge</u> (the same term as used in the Higgs model), i.e.,

$$\eta_1 = 0 \qquad \eta_2 = 0 \qquad \eta_3 = 0 \qquad\qquad \text{unitary gauge} . \tag{105}$$

and thus,

$$\Phi(x) = \begin{bmatrix} 0 \\ \phi_3 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \sigma + v \end{bmatrix} \quad \text{in unitary gauge} . \tag{106}$$

5.5.2 Higgs Field at True Vacuum

To find the true vacuum ($\mathcal{V}$ is a minimum), we use (106) in (103) and take the derivative of (103) with respect to ϕ_3.

$$0=\frac{\partial\mathcal{V}}{\partial\phi_3}=\frac{\partial}{\partial\phi_3}\left(\mu^2\Phi^\dagger\Phi+\lambda\left(\Phi^\dagger\Phi\right)^2\right)=\frac{\partial}{\partial\phi_3}\left(\mu^2\phi_3^2+\lambda\phi_3^4\right)=2\mu^2\phi_3+4\lambda\phi_3^3\quad\text{(at true vacuum)}$$
$$\rightarrow -\mu^2=2\lambda\phi_{3,truevac}^2\qquad\rightarrow\quad\phi_{3,truevac}=\sqrt{\frac{-\mu^2}{2\lambda}}.\tag{107}$$

The last expression in (107) is the location on the ϕ_3 axis where the Higgs field potential is a minimum.

With an eye to what comes later, we want to define σ as zero at the Higgs potential minimum. Thus in (106), we take

$$v=\sqrt{\frac{-\mu^2}{2\lambda}}\ .\tag{108}$$

Note from (108), that if v is real (which is assumed, since we took ϕ_1 as positive at that location), then μ is imaginary.

5.5.3 One More Wrinkle in the Glashow/Salam/Weinberg Model

One might now naively think (and early researchers probably did) that we simply need to plug (106) and (108) into (75) to get our electroweak theory at the true vacuum where we reside today. Not. When one does that, the result has different fields than we find in experiment. For one example, we get one vector field, which one might expect to be a photon, to have mass.

The answer was found by Steven Weinberg, and it is this. The B^μ and W_i^μ fields are really linear combinations of the photon field A^μ and intermediate vector boson (weak gauge) fields Z^μ, W_μ, and $W_\mu^\dagger$ (which are associated with the weak force particles found at CERN in the 1980s). That is, the former are transformations of the latter. The relations between the two sets of fields are given by

$$\begin{bmatrix}B_\mu\\ W_{3\mu}\end{bmatrix}=\begin{bmatrix}\cos\theta_W & -\sin\theta_W\\ \sin\theta_W & \cos\theta_W\end{bmatrix}\begin{bmatrix}A_\mu\\ Z_\mu\end{bmatrix}\qquad W_{1\mu}=\frac{W_\mu+W_\mu^\dagger}{\sqrt{2}}\qquad W_{2\mu}=i\frac{W_\mu-W_\mu^\dagger}{\sqrt{2}},\tag{109}$$

where Weinberg modestly calls θ_W the weak mixing angle, but others call it the Weinberg mixing angle. Note that the LHS of (109) is much like a rotation in a real 2D space. Since the determinant of the matrix there is unity, from matrix theory, we know the "lengths" of the "vectors" on each side of the equal sign remain unchanged. Since, in quantum mechanics, lengths of such "vectors" reflect probabilities, the total probability of measuring a component (any component at all, not a particular given component) in either vector is unchanged. This is the hallmark of a unitary transformation.

For future reference, we note, from the second two relations in (109),

$$W_\mu=\frac{W_{1\mu}-iW_2^\mu}{\sqrt{2}}\ .\tag{110}$$

Note further, from the LHS of (109), that because B_μ and $W_{3\mu}$ are real fields, A_μ and Z_μ are real fields. (Recall* that real fields create and destroy chargeless particles [particles that are their own antiparticles]; and complex fields create and destroy charged particles [and their antiparticles].) The photon and the Z particle are thus chargeless and their own antiparticles. The charges here include both electromagnetic charge and weak charge.

However, even though $W_{1\mu}$ and $W_{2\mu}$ are real fields, from (110), W_μ is complex and thus gives rise to a charged W particle and its antiparticle (which are the two W particles of the present day weak interaction found at CERN in the 1980s), which we represent as W^+ and W^-. That is, in the spirit of Section 1.3, pg. 2, (110) can alternatively be expressed in terms of destruction and creation operator terms that destroy W^+ and create W^-; whereas $W_\mu^\dagger$ destroys W^- and creates W^+.

For future reference, we also define

$$W^{\mu\nu}=\partial^\nu W^\mu-\partial^\mu W^\nu\qquad\qquad Z^{\mu\nu}=\partial^\nu Z^\mu-\partial^\mu Z^\nu\ ,\tag{111}$$

where our notation differs from Mandl and Shaw[1], as their $F_W^{\mu\nu}$ equals our $W^{\mu\nu}$.

---

* Klauber[2], pg. 65.

5.5.4 Finally, the Lagrangian at True Vacuum

Now if we substitute (106), (108), *and* (109) into (75), we will get the present day electro-weak Lagrangian.

I do not have time to do this step-by-step right now (but hopefully will one day), but the final result is

$$\boxed{\begin{aligned}
&\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I\\
&\mathcal{L}_0=\bar{\psi}_l\Big(i\partial\!\!\!/-\underbrace{\frac{vg_l}{\sqrt{2}}}_{m_l}\Big)\psi_l+\bar{\psi}_{\nu_l}\Big(i\partial\!\!\!/-\underbrace{\frac{vg_{\nu_l}}{\sqrt{2}}}_{m_{\nu_l}}\Big)\psi_{\nu_l} && (\text{leptons})\\
&\qquad-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} && (\text{photons})\\
&\qquad-\tfrac{1}{2}W^\dagger_{\mu\nu}W^{\mu\nu}+\underbrace{\left(\tfrac{1}{2}vg\right)^2}_{m_W^2}W^\dagger_\mu W^\mu && (W\ \text{fields})\\
&\qquad-\tfrac{1}{4}Z_{\mu\nu}Z^{\mu\nu}+\tfrac{1}{2}\underbrace{\left(\tfrac{1}{2}\frac{vg}{\cos\theta_W}\right)^2}_{m_Z^2}Z_\mu Z^\mu && (Z\ \text{field})\\
&\qquad+\tfrac{1}{2}\left(\partial_\mu\sigma\right)\left(\partial^\mu\sigma\right)-\tfrac{1}{2}\underbrace{\left(-2\mu^2\right)}_{m_H^2}\sigma^2 && (\text{True vac Higgs}=\sigma)\\
&\mathcal{L}_I=\mathcal{L}^{LB}+\mathcal{L}^{BB}+\mathcal{L}^{HH}+\mathcal{L}^{HB}+\mathcal{L}^{HL} && (\text{interaction terms}), \qquad (112)
\end{aligned}}$$

where we will wait to another day to write down the interaction terms. (Shown in Mandl & Shaw[1], pgs. 421-422.)

Note that for a *complex* boson field (W_μ here) mass is, as has always been the case before, the square root of the factor in front of the bilinear expression in the Lagrangian of that single type of complex boson field. For a *real* boson field (Z^μ and σ here), mass is the square root of that same factor aside from a factor of ½. The photon field has no such term, meaning its mass is zero, one indication that the methodology used by GSW does indeed give us a valid theory.

The mass of the contemporary Higgs can be expressed either in terms of μ or v and λ, via (108).

$$m_H=\sqrt{2}\left|\mu\right|=\sqrt{2}v\sqrt{2\lambda}=2v\sqrt{\lambda}\ . \qquad (113)$$

Prior to the discovery of the Higgs particle in 2012, v, g, and θ_W had been determined from scattering experiments, so it was possible to correctly predict the correct masses of the W's and the Z particles. But λ was still unknown, so it was not possible to get a handle on m_H. Note that m_H is the tree level mass for the Higgs, but that is modified by higher order corrections (higher order Feynman diagrams/amplitudes). See Sections 8.18 and 8.21. The masses for the W and Z particles also change due to higher order corrections, but theorists took those into account in predicting the experimentally measured values.

Note that we started in (75) with all massless fields (because massive fields mean the Lagrangian cannot be symmetric), and after symmetry breaking (of the Φ field, as easily seen in the Mexican hat figure, but also other massless fields), we now have fields with mass terms. The symmetry breaking bore the fruit of massive particles, corresponding to those in our present-day universe.

The couplings shown in the last row of Wholeness Chart 6 are found from the interaction terms in (112). For example, one of those terms is the LHS of (114), so knowing what this term must be in QED, we can deduce e as the RHS of (114).

$$g\sin\theta_W\bar{\psi}\gamma^\mu\psi A_\mu \qquad \rightarrow \qquad e=g\sin\theta_W \qquad (114)$$

5.5.5 Notes for Deriving the True Vacuum Lagrangian of (112)

As an exercise, you can derive the following, which is needed to get (112). Or you can just find the derivation done for you in Appendix A, pg. 43.

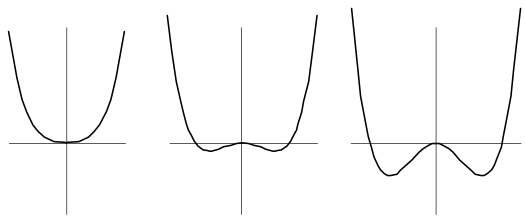
$$\bar{\psi}_l\psi_l=\bar{\psi}_l^L\psi_l^R+\bar{\psi}_l^R\psi_l^L \qquad (\text{no sum on } l)\ . \qquad (115)$$

Wholeness Chart 6. Glashow/Salam/Weinberg Model

	At False Vacuum **Complex Rep**	**Real Rep**	**At True Vacuum** **Complex Rep**	**Real Rep**
Higgs Field (isospin doublet)	$\Phi(x)=\begin{bmatrix}\phi_a\\ \phi_b\end{bmatrix}=\begin{bmatrix}\phi_1+i\phi_2\\ \phi_3+i\phi_4\end{bmatrix}$	$\Phi(x)=\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{bmatrix}$	$\Sigma=\begin{bmatrix}\eta_1+i\eta_2\\ \sigma+i\eta_3\end{bmatrix}$	$\Sigma(x)=\begin{bmatrix}\eta_1\\ \eta_2\\ \sigma\\ \eta_3\end{bmatrix}=\begin{bmatrix}\sqrt{2}\phi_1\\ \sqrt{2}\phi_2\\ \sqrt{2}\phi_3-v\\ \sqrt{2}\phi_4\end{bmatrix}$ $\phi_3=v/\sqrt{2}$ is min of $\mathcal{V}$. $\mathcal{V}_{min}=\sqrt{\frac{-\mu^2}{2\lambda}}$
$\mathcal{L}$	$\mathcal{L}_{false}=\mathcal{L}^L+\mathcal{L}^B+\mathcal{L}^H+\mathcal{L}^{LH}$ (massless) $\mathcal{L}^L=i\left(\bar{\Psi}^L_l \not{D}\Psi^L_l+\bar{\psi}^R_l\not{D}\psi^R_l+\bar{\psi}^R_{\nu_l}\not{D}\psi^R_{\nu_l}\right)$ $\mathcal{L}^B=-\frac{1}{4}G_i^{\mu\nu\dagger}G_{i\mu\nu}-\frac{1}{4}B^{\mu\nu}B$ $\mathcal{L}^H=\left(D^\mu\Phi\right)^\dagger\left(D_\mu\Phi\right)-\mu^2\Phi^\dagger\Phi-\lambda\left(\Phi^\dagger\Phi\right)^2$ $\mathcal{L}^{LH}=-g_l\bar{\Psi}^L_l\psi^R_l\Phi-g_{\nu_l}\bar{\Psi}^L_{\nu_l}\psi^L_{\nu_l}\tilde{\Phi}+h.c.$ $D^\mu=\partial^\mu+\frac{i}{2}g\tau_jW^\mu_j+ig'YB^\mu$	Use algebra on $\mathcal{L}$ relations at left to obtain theory in terms of real ϕ_i	Real rep more illuminating	For $\mathcal{L}_{true}$: 1. Subst in $\mathcal{L}_{false}$ (translates axis to $\mathcal{V}_{min}$) $\phi_{1,2,4}=\eta_{1,2,4}/\sqrt{2}\quad \phi_3=(\sigma+v)/\sqrt{2}$ 2. Substitute $B_\mu=\cos\theta_W A_\mu-\sin\theta_W Z_\mu$ $W_{3\mu}=\sin\theta_W A_\mu+\cos\theta_W Z_\mu$ $W_{1\mu}=\frac{W_\mu+W^\dagger_\mu}{\sqrt{2}}\quad W_{2\mu}=i\frac{W_\mu-W^\dagger_\mu}{\sqrt{2}}$ 3. Result: Complicated $\mathcal{L}_{true}$ with a) non independent normal modes b) 3 more DOF than $\mathcal{L}_{false}$ 4. Resolve using unitary gauge below.
Local Symmetry Transformation **SU(2)XU(1)**	$\delta\Psi^L_l=\frac{i}{2}\omega_i\tau_i\Psi^L_l+ig'Yf\Psi^L_l\quad Y=-\frac{1}{2}$ $\delta\psi^R_l=ig'Yf\psi^R_l\quad Y=-1$ $\delta W^\mu_i=-\frac{1}{g}\partial^\mu\omega_i-\varepsilon_{ijk}\omega_jW^\mu_k$ $\delta B^\mu=-\frac{1}{g'}\partial^\mu f$ $\delta\Phi=\frac{i}{2}\omega_i\tau_i\Phi+ig'Yf\Phi\quad Y=\frac{1}{2}$			$\mathcal{L}_{true}$ not symmetric in η_1, η_2, σ, η_3 at true vacuum (broken symmetry)
Unitary Gauge				$\eta_1=\eta_2=\eta_3=0$ (3 DOF less, norm modes)
$\mathcal{L}$ **in Unitary Gauge**			Via steps 1, 2, 4 above $\mathcal{L}_{true}=\mathcal{L}_0+\mathcal{L}_I$ becomes $\mathcal{L}_0=\bar{\psi}_l\left(i\not{\partial}-\frac{vg_l}{\sqrt{2}}\right)\psi_l+\bar{\psi}_{\nu_l}\left(i\not{\partial}-\frac{vg_{\nu_l}}{\sqrt{2}}\right)\psi_{\nu_l}$ $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}W^\dagger_{\mu\nu}W^{\mu\nu}+\left(\frac{1}{2}vg\right)^2W^\dagger_\mu W^\mu$ $-\frac{1}{4}Z_{\mu\nu}Z^{\mu\nu}+\frac{1}{2}\left(\frac{1}{2}\frac{vg}{\cos\theta_W}\right)^2Z_\mu Z^\mu$ $+\frac{1}{2}\left(\partial_\mu\sigma\right)\left(\partial^\mu\sigma\right)-\frac{1}{2}\left(-2\mu^2\right)\sigma^2$ $\mathcal{L}_I=\mathcal{L}^{LB}+\mathcal{L}^{BB}+\mathcal{L}^{HH}+\mathcal{L}^{HB}+\mathcal{L}^{HL}$	
Determining Masses			Inspection of above: $m_l=\frac{vg_l}{\sqrt{2}}\quad m_{\nu_l}=\frac{vg_{\nu_l}}{\sqrt{2}}\quad m_\gamma=0$ $m_W=\frac{1}{2}vg\quad m_Z=\frac{1}{2}\frac{vg}{\cos\theta_W}\quad m_H=-2\mu^2=\sqrt{2\lambda v^2}$	
Note			1. Zero photon mass ($m_\gamma=0$ as no $A^\mu A_\mu$ term) 2. Know g, θ_W, v in 1970s via exper$\rightarrow$ predict $m_{Z,W}$ 3. λ only in $\mathcal{L}^{HH}$. Not predict m_H. Need to measure.	
Determining Couplings			Interaction terms: e^- with $A^\mu\rightarrow e=g\sin\theta_W$; ν w A^μ $\rightarrow$ 0; e^-& ν w $W\rightarrow g/2\sqrt{2}$, w $Z\rightarrow g/4\cos\theta_W$	

6 Possible Changes and Differences in Mexican Hat Shape

It is possible that the Mexican hat shape shown repeatedly herein for the Higgs field potential energy density is not static but changes over time, i.e., in (19), $\mu = \mu(t)$, or $\mu = \mu(t)$ and $\lambda = \lambda(t)$. One example, illustrated in 2D only for simplicity, could be like that shown in Figure 4.



t_1 $\quad t_2 > t_1 \quad$ $t_3 > t_2$

Figure 4. Possible Change in Shape of Higgs Potential with Time (Only 2D shown)

In the case of Figure 4, at t_1 the field would be in a stable configuration. However, at later times, the false vacuum would be unstable.

It is also possible, that the true vacuum might be slightly, but not strongly stable, as in Figure 5. In such case, we would expect the universe, at some point, to tunnel out of the small indention in the middle of the curve and fall into the stable true vacuum state.

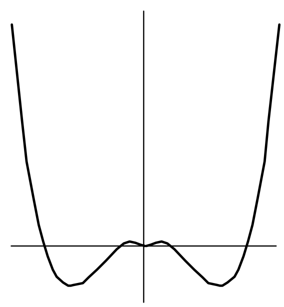


Figure 5. Possible Slightly Stable False Vacuum

7 Summary of Postulates for Electroweak Theory

The basic postulates from which we derive electroweak theory are these.

1. Free fields are represented by terms similar in from to those in QED.
2. The potential in $\mathcal{L}$ has form $\mathcal{V} = \mu^2\Phi^\dagger\Phi + \lambda\left(\Phi^\dagger\Phi\right)^2 \quad \mu^2 < 0,\ \lambda > 0$ and the universe spontaneously seeks lowest $\mathcal{V}$.
3. Interaction theory is obtained via minimal substitution in the free field $\mathcal{L}$ plus 4 below.
4. Extra terms are added to $\mathcal{L}$ to couple the Higgs field with lepton fields.
5. Fields associated with observed gauge bosons are linear combinations of the fields used in minimal substitution.

With these postulates, one finds a symmetric theory whose interactions match experiment.

8 Other Things to Note

8.1 Initiating Symmetry Breaking

In Figure 2 and Figure 3 on pg. 6, the universe would be in equilibrium at the false vacuum, but not be stable. A slight fluctuation in the ϕ particle density would displace the universe off of the false vacuum (ϕ particle density not precisely zero) and begin the universe's descent into the true vacuum, which is in equilibrium, but not unstable. For the case of Figure 5, larger fluctuations in the underlying particle density would be necessary to initiate symmetry breaking. Fluctuations in the horizontal direction (i.e., in the density of ϕ particles) would need to exceed the local maxima to begin the "roll down" into the valley.

8.2 Particle Creation

As the universe slides down the slope of the Mexican hat potential from the false vacuum to the true vacuum, it moves radially outward from the central axis of Figure 2 and Figure 3. As noted in Section 1.4, this means the ϕ particle density one would expect to measure increases. Hence, the reduction in potential energy is accompanied by an increase in particles, i.e., an increase in mass-energy.

Typically, these $|\phi\rangle$ particles decay (via the coupling with other particles) into other standard model particles. Thus, the process of spontaneous symmetry breaking populates the universe with more and more particles (until it stabilizes in the true vacuum.)

8.3 GSW Model and Unification of E/M with Weak force

Before I first saw the GSW model some decades ago, I had heard a lot of talk about unifying the two forces, electromagnetic and weak, into a single electroweak force. This seemed so elegant and profound to me.

However, when I actually studied and learned it, I realized, with some disappointment, that there was more hype than substance in all the talk. At the true vacuum, there are two different types of interactions, electromagnetic and weak, mediated by two different types of gauge boson fields, the e/m photon A^{μ} and the weak intermediate vector bosons Z^{μ}, W^{μ}, and $W^{\mu\dagger}$. In a true unification, we would expect these two to merge at the false vacuum (high energy) into a single field. However, at the GSW false vacuum we still have two kinds of interactions, mediated by the two types of gauge boson fields, the hypercharge field B^{μ} and the isospin fields W_i^{μ}. Yes, the former (true vacuum) fields are linear combinations of the latter (false vacuum) fields. But in both cases, we have two different kinds of fields. In other words, there is no true unification.

In some sense, one might argue there is some unification, as we now know the two fields are related, i.e., intertwined, so to speak. The A^{μ} field, for instance, is a combination of what, at high energy, is the weak and hypercharge fields. Ditto for the Z^{μ}. So, each type of low energy field is a result of combining both types of high energy fields. But, it is still two types of fields transforming to two types of fields.

Grand unified theories (GUTs), on the other hand, actually unify the three force fields (weak, strong, e/m) into a single type of force at high energy. The difference is that the GSW theory has been verified by experiment, and appears unique, whereas the extant GUT theories fall short in these regards. Similarly, superstring (or M) theories appear to unite all four forces (above plus gravity) into a single type force.

Nevertheless, the GSW theory is a giant step for mankind, as it correctly describes the current properties, and evolution, of the electromagnetic and weak interactions.

8.4 Symmetry Breaking is Apparent, Not Real.

The Lagrangian actually never loses its symmetry. It only appears to. It is always symmetric with respect to the high energy, false vacuum. If we express $\mathcal{L}$ in terms of the ϕ_i, instead of the η_i and σ, we get the original high energy $\mathcal{L}$. It is never really lost, and it is symmetric about the false vacuum.

At the true vacuum, we merely express $\mathcal{L}$ in terms of the fields η_i and σ, which have zero expectation values at the true (present day, low energy) vacuum. There, $\mathcal{L}$ is not symmetric (in terms of the ϕ_i and also in terms of the η_i and σ). But it is only because our point of view is different that we seem to have lost symmetry.

By analogy, consider you are at a point in the middle of a huge bowl. You rotate yourself about the vertical axis through that point and the bowl still looks the same to you. It is symmetric. Now move up the side of the bowl to another point and again rotate yourself about the vertical axis through this point. The bowl obviously looks to you like it is oriented differently. It does not maintain its appearance during the rotation. It appears unsymmetric. But, it really is still a symmetric bowl. It just depends on where you rotate yourself about.

Similar logic holds for $\mathcal{L}$ about two different points in our Fock space of fields. The symmetry is not really broken (although it is commonly stated that way), it is hidden.

8.5 Yang-Mills Fields

You will no doubt see the W_i^μ fields referred to as Yang-Mills fields. They are the massless fields associated with an SU(2) theory, whose generators (the Pauli matrices in this case) are non-Abelian (do not commute).

In early 1954, Chen Ning Yang and Robert Mills extended the concept of gauge theory for Abelian groups, e.g. quantum electrodynamics, to non-Abelian groups to provide an explanation for strong interactions. The quanta of a Yang–Mills field must be massless in order to maintain gauge invariance of the Lagrangian.

In general, any massless set of fields (in SU(2), the set comprises three fields W_i^μ) in a non-Abelian SU(n) theory that leaves the Lagrangian invariant under an SU(n) transformation is comprised of Yang-Mills fields. The associated theory is called a Yang-Mills theory.

8.6 Non-Abelian Fields and Non-linearity

From (79), (80), we saw that the non-Abelian group SU(2) gives rise to extra terms in the interaction Lagrangian (see RHS of (79)), with more than two of the same type field (the W_i^μ field) as factors in that term. That is, the terms are greater than bilinear in the fields. Abelian groups, like U(1), do not have such terms.

When we substitute the Lagrangian $\mathcal{L}$ into the Euler-Lagrange equation, we take derivatives of $\mathcal{L}$ with respect to the field and the derivatives of the field. For only bilinear terms in $\mathcal{L}$, this results in a field equation that is linear in the field. For trilinear, quadrilinear, etc terms, it results in a field equation that is non-linear in the field. In a non-linear equation, the dependent variable (field for us) affects itself, i.e., it is self-interacting.

Quantum chromodynamics (QCD) turns out to be an SU(3) theory, with a number of (non-commuting) matrices in a 3D space as generators, parallel to the Pauli matrices in electroweak SU(2). Thus, QCD has a number of different gauge fields comparable to the W_i^μ fields of SU(2) electroweak theory. And thus, QCD has similar extra terms arising in the QCD $\mathcal{L}$ that are not bilinear in the QCD gauge fields. So, QCD is a non-Abelian, and non-linear, field theory.

The bottom line: Non-Abelian field theories are non-linear, or self-interacting, theories. Abelian field theories are linear, and not self-interacting. Yang-Mills theories are non-linear theories. See Wholeness Chart 7 for a summary overview.

U(1)		SU(n)	
↙	↘	↙	↘
Abelian	1 field, e.g., B^μ in high energy e/w theory	Non-Abelian	>1 field, e.g., 3 W_i^μ in high energy e/w theory
transformations commute in U(1) space	↓	transformations do not commute in SU(n) space	↓
e.g., $e^{i\alpha}$ in e/w U(1) theory	no extra term in $\mathcal{L}$	e.g. Pauli matrices τ_i in e/w SU(2) theory	extra terms in $\mathcal{L}$
	↓		↓
	linear field equation e.g., in B^μ		non-linear field equation e.g., in W_i^μ

Wholeness Chart 7. Non-Abelian Theories Have Non-Linear Field Equations

8.7 Being Clear on Commutation and Non-Commutation

Take care with the context in which the term "non-commutation" is used. With regard to any bosonic field in QFT, we have non-commutation between the field (such as ϕ), and its conjugate momentum (such as π_S), as in,

$$\left[\phi_r(\mathbf{x},t),\pi_s(\mathbf{y},t)\right]=i\delta_{rs}\delta(\mathbf{x}-\mathbf{y}), \tag{116}$$

and this is true regardless of whether the field is of a U(1), SU(2), SU(3) or other special unitary group theory. That is, (116) is a basic postulate of all of QFT for any boson field. For fermions, we have anti-commutation in place of (116).

However, with regard to the transformations of fields in Fock space, there are some transformation types (like those in a U(1) theory such as $e^{i\alpha}$ and $e^{i\beta}$) that commute. There are others (like those in an SU(2) theory such as τ_1 and τ_2) that do not. U(1) theories have non-commutation of fields and their conjugate momenta, but commutation of their transformations in U(1) space.

8.8 Charges: Weak Hypercharge, Weak Isospin, Electric

8.8.1 The Simpler View

Recall from QED, the Lagrangian interaction term (we only had one then … how simple!) had the form

$$\mathcal{L}_I=e\bar{\psi}\gamma^\mu\psi A_\mu \qquad \rightarrow \qquad \mathcal{H}_I=-e\bar{\psi}\gamma^\mu\psi A_\mu\ , \tag{117}$$

where the charge on the electron is – e. In other words, we could read the charge on the fermion off as the negative of the factor in front of the Lagrangian term describing the interaction vertex between two electrons and a photon.

In high energy electroweak theory, we can do a similar thing. Consider an e/w interaction term in (75) XXX change this ref to equation with L written out when that is incorporated XXX comparable to (117) such as (118) (see (66))

$$-\frac{g}{2}\bar{\Psi}_e^L\gamma^\mu\tau_3\Psi_e^L W_{\mu3}=-\frac{g}{2}\begin{pmatrix}\psi_{\nu_e}^L\\ \psi_e^L\end{pmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\left(\bar{\psi}_{\nu_e}^L,\bar{\psi}_e^L\right)W_{\mu3}=-\frac{g}{2}\bar{\psi}_{\nu_e}^L\gamma^\mu\psi_{\nu_e}^L W_{\mu3}+\frac{g}{2}\bar{\psi}_e^L\gamma^\mu\psi_e^L W_{\mu3} \tag{118}$$

Comparing (118) to (117) we could surmise the weak isospin charge on the electron is – $g/2$, and on the electron neutrino + $g/2$. And we would be correct. In similar fashion, we can deduce the weak isospin charges and weak hypercharges on all particles, and thus construct Wholeness Chart 8 below.

As one more example, consider the right handed fields ψ_e^R and $\psi_{\nu_e}^R$. They have no terms of form like (118), so their weak isospin charges are zero. So, where we generalize from the electron e and its neutrino to all 3 families with $l = e,\mu,\tau$,

$$\text{for physical weak isospin charge } gI_3^W, \qquad I_3^W=+\tfrac{1}{2}\text{ for }\psi_{\nu_l}^L \qquad -\tfrac{1}{2}\text{ for }\psi_l^L \qquad 0\text{ for }\psi_l^R \qquad 0\text{ for }\psi_{\nu_l}^R\,. \tag{119}$$

Note that it is common, as a sort of short hand notation, to call $\underline{I_3^W}$ the <u>weak isospin charge</u>, even though the actual charge one would measure with physical instruments would be gI_3^W. Note some authors use T_3 or another symbol for this, though Mandl and Shaw[1] use I_3^W, as we do. The symbol $\boldsymbol{I_3^W}$ is chosen as the "I " represents isospin; the "W", weak; and the "3", the third Paul matrix τ_3, which is used to determine the value in front of the respective terms in (118).

Similarly, <u>Y</u> is called the <u>weak hypercharge</u>, even though the actual physically measurable weak hypercharge is $g'Y$. And N_Q is commonly deemed the <u>electric charge</u>, even though the actual physically measurable electric charge is $Q = e\,N_Q$. (Some authors use Q for what we consider N_Q to be.)

With foresight, and for our choice of factors in the minimal substitution of (73), it turns out we find a consistent theory, where local symmetries hold and we get the correct interactions, if

$$Y=N_Q-I_3^W\,. \tag{120}$$

Be aware that authors using a different convention for the factors in minimal substitution define $Y = 2(N_Q - I_3^W)$.

Thus, by using (119) and our knowledge of $Q = e\,N_Q$ for the elementary particles, we can construct the lepton parts for the high energy part of Wholeness Chart 8. The quark values are beyond the scope of this chapter, but as a little hint, I note that the up and down quarks form an SU(2) doublet like the electron neutrino and electron do. Similar parallels exist for the other families. So, the quark weak isospin charges parallel those of the leptons.

As an exercise, from the Higgs part of the high energy Lagrangian (81) and (54) along with the unitary gauge (106), deduce the weak isospin charge and the weak hypercharge for the Higgs particle. Note that because the Higgs is real, it has zero electric charge. The solution to this exercise is shown in Appendix A, pg. 43.

Note that the right-handed neutrinos have no charge of any kind and so do not interact at high energy via the weak isospin or weak hypercharge interactions. Being leptons, they also do not interact via the QCD color force and have zero color charge. Hence, we do not even know if such particles exist, as there is no way to detect them.

We can find the relations for the weak isospin and weak hypercharge boson mediators by considering interaction terms in the Lagrangian and assuming charge conservation. For example, consider the terms in the high energy interaction Lagrangian (75) of form (70) for W_1^μ and also for W_2^μ.

$$
\begin{aligned}
i\frac{g}{2}\bar{\Psi}^L_e\tau_1\not{W}_1\Psi^L_e &= i\frac{g}{2}\left(\bar{\psi}^L_{\nu_e},\bar{\psi}^L_e\right)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\not{W}_1\begin{pmatrix}\psi^L_{\nu_e}\\ \psi^L_e\end{pmatrix} = i\frac{g}{2}\left(\bar{\psi}^L_{\nu_e}\not{W}_1\psi^L_e + \bar{\psi}^L_e\not{W}_1\psi^L_{\nu_e}\right)\\
i\frac{g}{2}\bar{\Psi}^L_e\tau_2\not{W}_2\Psi^L_e &= i\frac{g}{2}\left(\bar{\psi}^L_{\nu_e},\bar{\psi}^L_e\right)\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\not{W}_2\begin{pmatrix}\psi^L_{\nu_e}\\ \psi^L_e\end{pmatrix} = i\frac{g}{2}\left(-i\bar{\psi}^L_{\nu_e}\not{W}_2\psi^L_e + i\bar{\psi}^L_e\not{W}_2\psi^L_{\nu_e}\right)
\end{aligned}
\tag{121}
$$

If we add these (which they are in the Lagrangian), we get

$$
= i\frac{g}{\sqrt{2}}\left(\bar{\psi}^L_{\nu_e}\gamma^\mu\frac{\left(W_{1\mu}-iW_{2\mu}\right)}{\sqrt{2}}\psi^L_e + \bar{\psi}^L_e\gamma^\mu\frac{\left(W_{1\mu}+iW_{2\mu}\right)}{\sqrt{2}}\psi^L_{\nu_e}\right) = i\frac{g}{\sqrt{2}}\bar{\psi}^L_{\nu_e}\gamma^\mu W_\mu\psi^L_e + i\frac{g}{\sqrt{2}}\bar{\psi}^L_e\gamma^\mu W^\dagger_\mu\psi^L_{\nu_e} \;.\tag{122}
$$

In the spirit of Section 1.3, pg. 2, the field W_μ destroys a <u>W particle</u> (which we label as <u>W^+</u>) and creates a <u>W antiparticle</u> (which we label as <u>W^-</u>). The complex conjugate field $W^\dagger_\mu$ does the reverse. It destroys a W^- and creates a W^+.

A Feynman diagram for the last term in (122) looks like Figure 6, where an incoming left hand neutrino and an incoming W^- are destroyed, with a left handed electron created.

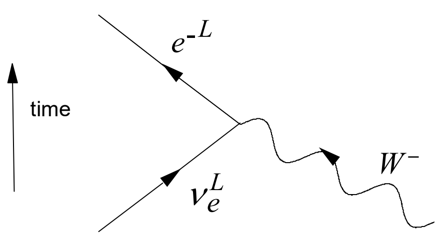


Figure 6. Weak Interaction Feynman Diagram for LH Neutrino and LH Electron

The total incoming charges (on the neutrino) must equal the total outgoing charges on the other two particles. The superscript W^- refers to the W^- particle, of course.

$$
\begin{array}{llcll}
\text{Electric charge} & \rightarrow & 0 + N_{Q\,W^-} = -1 & \rightarrow & N_{Q\,W^-} = -1\\
\text{Weak isospin charge} & \rightarrow & \frac{1}{2} + I_3^{W^-} = -\frac{1}{2} & \rightarrow & I_3^{W^-} = -1\\
\text{Weak hypercharge} & \rightarrow & -\frac{1}{2} + Y^{W^-} = -\frac{1}{2} & \rightarrow & Y^{W^-} = 0\,.
\end{array}
\tag{123}
$$

In similar fashion, we can find the electric and weak isospin charges for the W^+, W_3, and B particles shown in the wholeness chart.

For the Higgs as the ϕ_3 field in the unitary gauge, as an exercise, use the lepton-Higgs coupling term of (75)

$$
-g_l\bar{\Psi}^L_l\psi^R_l\Phi \tag{124}
$$

to show is weak isospin charge is – ½, as shown in Wholeness Chart 8. (Answer in Appendix A, pg. 43.)

For low energy particles:

Though the way the leptons interact changes in degree and manner as energy falls, they do not change their identity, so the charges they carry do not change. However, the gauge bosons we work with at low energy are different particles than those we considered at high energy. The W_μ field is a combination of the W_1 and W_2 fields as in (110) and the photon and Z fields are combinations of the W_3 and B fields as in the LH part of (109). The deductions above related to the W_μ field and its associated W particle hold at high or low energy. In passing we note that since the $W_{1\mu}$ and $W_{2\mu}$ fields are each a linear combination of the W_μ and $W_\mu^\dagger$ fields, whose associated particles have different charges, the particles of the $W_{1\mu}$ and $W_{2\mu}$ fields are not in charge eigenstates.

The A^μ and the Z^μ fields are linear combinations of B^μ and W_3^μ, so if the particles associated with the latter have zero electric and weak isospin charges, then so will those of the former.

Fermion Family	**Left-chiral Fermions**	**Electric Charge** NQ	**Weak Isospin** I_3^W	**Weak Hyper-charge** Y	**Right-chiral Fermions**	**Electric Charge** NQ	**Weak Isospin** I_3^W	**Weak Hyper-charge** Y
Leptons	$\nu_{eL}, \nu_{\mu L}, \nu_{\tau L}$	0	+ 1/2	– 1/2	$\nu_{eR}, \nu_{\mu R}, \nu_{\tau R}$	0	0	0
	e_L^-, μ_L^-, τ_L^-	– 1	– 1/2	– 1/2	e_R^-, μ_R^-, τ_R^-	– 1	0	– 1
Quarks	u_L, c_L, t_L	+ 2/3	+ 1/2	+ 1/6	u, c, t	+ 2/3	0	+ 2/3
	d_L, s_L, b_L	– 1/3	– 1/2	+ 1/3	d, s, b	– 1/3	0	– 1/3

Original $\mathcal{L}$ Gauge Fields	**Boson**	**Electric Charge** NQ	**Weak Isospin** I_3^W	**Weak Hyper-charge** Y
Weak	W_1	Not in electric or weak isospin charge eigenstates		0
	W_2			0
	W_3	0	0	0
Hypercharge	B	0	0	0
Higgs	ϕ_3	0	– 1/2	+ 1/2

Contemporary $\mathcal{L}$ Gauge Fields	**Boson**	**Electric Charge** NQ	**Weak Isospin** I_3^W	**Weak Hyper-charge** Y
Weak	W^+	+ 1	+ 1	0
	W^-	– 1	– 1	0
	Z	0	0	0
E/m	γ	0	0	0
Higgs	σ	0	– 1/2	+ 1/2

Wholeness Chart 8. Various Charges for Elementary Particles ($Y = N_Q - I_3^W$)

We can now see the reason for the letter designations of the weak fields. W_i is obviously for "weak". W^+ is the positively charged (both electric and weak isospin) weak force carrier; W^-, its negatively charged antiparticle. Z stands for "zero" charge.

8.8.2 The More Sophisticated, but More Complicated, View

Actually, there is a more elegant (but perhaps less easy to grok) method for determining charge on all particles similar to what we did in QED*. That is, we find the conserved, weak 4 current operators (often just called "4-currents") for the relevant fields. Then we integrate the $\mu = 0$ component over all space to get the charge operator. We then operate on a given ket (representing a given particle) with the charge operator and get the eigenvalue, which is the charge of that particle. This

* See Klauber[2], pgs. 112-113, 173-176.

is a relatively involved process that we will not do here, though someday I would like to include it. Hopefully, the simple analogy presented here between (118) and (117) will suffice to give us some confidence in how we could obtain Wholeness Chart 8. The more sophisticated method with conserved 4 currents also proves conservation of charge for weak interactions.

8.8.3 Plotting Charges

From Figure 7 one can note that in QED only the antiparticles had opposite electric charge from particles, but for the weak interaction, we have particles (neutrinos, specifically) that have opposite weak charge of other particles (electrons).

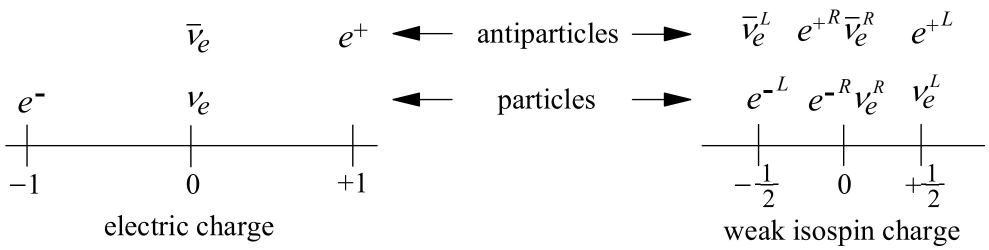


Figure 7. Electric and Weak Isospin Charges

8.9 Reason for Nomenclature "Weak Isospin"

Note that we can write the left chiral electron doublet as

$$\begin{pmatrix}\psi^L_{\nu_e}\\ \psi^L_e\end{pmatrix}=\begin{pmatrix}\psi^L_{\nu_e}\\ 0\end{pmatrix}+\begin{pmatrix}0\\ \psi^L_e\end{pmatrix}, \qquad (125)$$

where the first term after the equal sign in (125) represents the electron neutrino field and the second term the electron field.

Note further that the operation of the third Pauli matrix, divided by two, on a doublet having only one component gives us the weak isospin charge I_3^W for that component.

$$\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}\psi^L_{\nu_e}\\ 0\end{pmatrix}=\underbrace{\frac{1}{2}}_{I_3^W}\begin{pmatrix}\psi^L_{\nu_e}\\ 0\end{pmatrix} \qquad \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}0\\ \psi^L_e\end{pmatrix}=\underbrace{-\frac{1}{2}}_{I_3^W}\begin{pmatrix}0\\ \psi^L_e\end{pmatrix} \qquad (126)$$

The third Pauli matrix acting on a one component doublet gives us the eigenvalue I_3^W .

Recall from non-relativistic quantum mechanics (NRQM) that our spin operator was ½ times the 3rd Pauli matrix. That is

$$\frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\underbrace{\frac{1}{2}}_{\text{spin}}\begin{pmatrix}1\\ 0\end{pmatrix} \qquad \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}=\underbrace{-\frac{1}{2}}_{\text{spin}}\begin{pmatrix}0\\ 1\end{pmatrix} \qquad \text{NRQM} . \qquad (127)$$

The weak charge associated with the lepton field doublet components in weak interaction theory behaves like the spin associated with spinor components of the wave function in NRQM. So, the word "isospin", where the prefix "iso" means equal, is used to describe a system that behaves mathematically much like spin in NRQM. The adjective "weak" is added to "isospin" in the present case to pin down the particular system that parallels NRQM spin. Another example of isospin is nucleon isospin (often called isotopic spin), which is used in a particular theory with a doublet comprising components of a proton and a neutron.

So, the 3rd Pauli matix multiplied by ½ is the weak isospin operator. The top component of a doublets will have eigenvalues ½; the bottom component, – ½ . Given that and (54) in the unitary gauge, deduce the eigenvalue for the Higgs particle ϕ_3. (Answer is in Appendix A, pg. 43.)

In mathematical terminology, any spin or isospin system is a manifestation in nature of SU(2) group theory.

8.10 Phenomenology and Terminology: "The Vacuum Eats Charge"

Although weak charge is conserved in interactions like those of Figure 6, which is one Feynman diagram associated with (122), an interesting effect arises due to the mass term for chiral fields, in which weak charge is not actually conserved.

The low energy Lagrangian (112) has the term, due to symmetry breaking of the Higgs field, (see (143) for the relation after the equal sign)

$$m_e\bar{\psi}_e\psi_e = m_e\bar{\psi}_e^L\psi_e^R + m_e\bar{\psi}_e^R\psi_e^L \qquad (\text{no sum on } e). \qquad (128)$$

Consider the second term after the equal sign. It can be represented by a Feynman diagram like Figure 8, where a left hand electron is destroyed and a right hand electron is created. Since a left hand electron has weak isospin charge of – ½ and a right hand electron has zero, a half unit of negative charge has been lost.

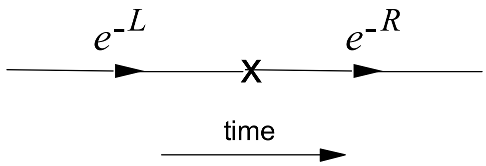


Figure 8. Weak Isospin Charge is "Eaten" (Not Conserved)

This is sometimes referred to as the vacuum eating (weak) charge. It is due to the mass term and only occurs there. Note that from the 1st term after the equal sign in (128), we get the opposite effect. An additional – ½ unit of weak isospin charge is added. The net effect of many such interactions sums to zero overall charge created or destroyed.

In reality, if we were to consider things from the point of view of the false vacuum, instead of the X in Figure 8 (which represents the mass that came from the constant v in ϕ_3 when symmetry broke), we would have the Higgs particle from the field ϕ_3 being created. And that carries off the – ½ unit of weak isospin charge. In other words, it is only an illusion from the point of view of the true vacuum that charge is not conserved. From the point of view of the false vacuum, it is conserved.

There are, of course, similar terms to (128), and similar behavior, for muons and taus.

8.11 Symmetry and "Rotations" in SU(2) Space

Our symmetry transformation set of Section 5.4 included SU(2) transformations on doublets like those in the first two rows of (83) [finite transformation] and the first two rows of (91) [infinitesimal form of the same transformation]. Looking more closely at the finite case (because we can make our point more easily with that), we can express it in more detail as

$$\begin{aligned}\begin{pmatrix}\psi_{\nu_e}^{L\,\prime}\\ \psi_e^{L\prime}\end{pmatrix} &= \Psi_l^{L\prime} = SU(2)\Psi_l^L = e^{i\omega_i(x)\tau_i/2}\Psi_l^L = \left(1+\tfrac{i}{2}\omega_i(x)\tau_i+....\right)\Psi_l^L = \left(I+\tfrac{i}{2}\omega_1\tau_1+\tfrac{i}{2}\omega_2\tau_2+\tfrac{i}{2}\omega_3\tau_3+...\right)\Psi_l^L\\ &= \left(\begin{bmatrix}1&0\\0&1\end{bmatrix}+\tfrac{i}{2}\omega_1\begin{bmatrix}0&1\\1&0\end{bmatrix}+\tfrac{i}{2}\omega_2\begin{bmatrix}0&-i\\i&0\end{bmatrix}+\tfrac{i}{2}\omega_3\begin{bmatrix}1&0\\0&-1\end{bmatrix}+\;.....\right)\begin{pmatrix}\psi_{\nu_e}^L\\ \psi_e^L\end{pmatrix}\\ &= \begin{bmatrix}A&B\\C&D\end{bmatrix}\begin{pmatrix}\psi_{\nu_e}^L\\ \psi_e^L\end{pmatrix},\end{aligned} \qquad (129)$$

where A, B, C, and D are complex numbers equal to the sum of the contributions of their respective matrix position values from all of the matrices in the second row of (129). So, the column matrix in the last row is transformed by the 2X2 matrix in front of it to the primed quantities column matrix at the very beginning of (129). The degree of transformation depends on the three $\omega_i(x)$. The Pauli matrices τ_i plus the identity matrix form a basis for the 2X2 transformation matrix. Note that we have earlier referred to the Pauli matrices as generators of SU(2) group theory. So, the generators plus the identity matrix comprise a basis for the transformations in the space.

This transformation parallels that of a rotation in 2D Euclidean space,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \cos\theta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \sin\theta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \sin\theta \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \qquad (130)$$

where the degree of transformation (rotation) depends on θ. In the second row, we show the four basis matrices for the transformation.

So, we can think heuristically of the SU(2) transformations as a sort of rotation in the SU(2) space of the elementary particle fields. The 2D position vector in Euclidean space corresponds to the fermion doublet in SU(2) space.

A scalar in 2D Euclidean space is not changed under a rotation. Similarly, a singlet such as either of the right hand fields in the 3rd and 4th rows of (83) does not change under an SU(2) transformation. That is,

$$\psi_l^{R'} = \psi_l^R \qquad \bar{\psi}_l^{R'} = \bar{\psi}_l^R \qquad \text{singlets under SU(2) tranformation} \; . \qquad (131)$$

Note that just as the 2D Euclidean rotations rotate a given vector, but do not change its length (maintains its inner product), so too do 2D SU(2) space "rotations" maintain the "length" (inner product) of SU(2) doublets. In Euclidean space we call such transformations orthogonal; in SU(2) space, we call them unitary transformations. (See Klauber[2], pg. 27.) See Wholeness Chart 9.

One might wonder why the Pauli matrices are chosen as bases matrices for the SU(2) "rotations" rather than some simpler form like the matrices shown in the last row of (130). The shortest answer is that one could indeed have done that, but the theory comes out in simplest form (simplest representation of the gauge bosons, etc) using the Pauli matrices.

	2D Cartesian Space (Real)	**SU(2) Space** (Complex)
Magnitude conserving transformation	Orthogonal (rotation matrix) $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$	Unitary (any Pauli matrix) $\tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Inverse	$\mathrm{A^T A = I} \rightarrow \mathrm{A^{-1} = A^T}$	$\tau_2^\dagger \tau_2 = \mathrm{I} \rightarrow \tau_2^{-1} = \tau_2^\dagger$
Inner product invariant under transformation	$\begin{pmatrix} x' & y' \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} A^T A \begin{pmatrix} x \\ y \end{pmatrix}$ $= \begin{pmatrix} x & y \end{pmatrix} A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$	$\begin{pmatrix} \bar{\psi}_{\nu_e}^{L\,\prime} & \bar{\psi}_e^{L\prime} \end{pmatrix} \begin{pmatrix} \psi_{\nu_e}^{L\,\prime} \\ \psi_e^{L\prime} \end{pmatrix} = \begin{pmatrix} \bar{\psi}_{\nu_e}^L & \bar{\psi}_e^L \end{pmatrix} \tau_2^\dagger \tau_2 \begin{pmatrix} \psi_{\nu_e}^L \\ \psi_e^L \end{pmatrix}$ $= \begin{pmatrix} \bar{\psi}_{\nu_e}^L & \bar{\psi}_e^L \end{pmatrix} \tau_2^{-1} \tau_2 \begin{pmatrix} \psi_{\nu_e}^L \\ \psi_e^L \end{pmatrix} = \begin{pmatrix} \bar{\psi}_{\nu_e}^L & \bar{\psi}_e^L \end{pmatrix} \begin{pmatrix} \psi_{\nu_e}^L \\ \psi_e^L \end{pmatrix}$
Effect of transformation	rotates 2D vector in real space	"rotates" doublet in SU(2) space

Wholeness Chart 9. Unitary vs Orthogonal Transformations

Note from matrix theory, when the LHS of (132) is true for the column matrix vector V, the square matrix M, and the transformed vector V', then the absolute values of those quantities are related by the RHS, where Det means determinant.

$$V' = MV \qquad \rightarrow \qquad |V'| = |Det\, M|\,|V| \qquad (132)$$

Thus, the magnitude (or the inner product of the column matrix with itself) is unchanged by any matrix transformation whose determinant has an absolute value of 1. This is true for the matrices A and τ_i of Wholeness Chart 9. In other words, they rotate the vector in the given space without changing its length.

8.12 SU(2) and U(1) Symmetries Hold Independently

Note that because each of the $f(x)$, $\omega_i(x)$ in transformations (99) to (102) can be varied independently, each of the SU(2) and U(1) transformation sets acting alone leaves the Lagrangian invariant. For example, consider $f = 0$. The Lagrangian is still symmetric under the SU(2) transformation provided by non-zero values for the ω_i. That is, it is invariant under just the SU(2) transformation part of (99) to (102), ignoring the U(1) part. The same logic works in reverse. $\mathcal{L}$ is symmetric for those transformations of the fields where $f \neq 0$ and $\omega_i = 0$. So, $\mathcal{L}$ is invariant under just the U(1) transformation without the SU(2) transformation.

8.13 Gauge Boson Fields Don't Actually Change with Energy Level

We commonly talk of the B^μ and W_i^μ as the high energy (or false vacuum) fields, and the A^μ, W^μ, and Z^μ fields as the low energy (or true vacuum) fields. However, this is not really true, as via (109), each set of fields is really a set of linear combinations of the fields in the other set, and this is true at any energy level. Our theory is a gauge theory, and the fields are gauge fields, which are unmeasurable directly and can be different, yet predict the same measured results. So, we could actually work with either set at high energy, or either set at low energy.

However, analysis is much simpler if we use the B^μ and W_i^μ set at high energy (to deduce the form of $\mathcal{L}$, and show its invariance, for examples), and the A^μ, W^μ, and Z^μ set at low energy (as they reflect the properties of particles we envision intuitively as interacting at low energies.) As we've noted before, don't confuse the form of the theory with the phenomenology it predicts. Different forms of the former can yield the same for the latter.

8.14 Right Chiral Particles Can Have Weak Interactions at Low Energy

As we have seen, the photon and Z particles seen at contemporary energy levels are really linear combinations of the B and W_3 particles at high energy. That is, from the inverse of (109),

$$\begin{bmatrix} A_\mu \\ Z_\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{bmatrix} \begin{bmatrix} B_\mu \\ W_{3\mu} \end{bmatrix}. \tag{133}$$

We know the W_3 serves as a gauge boson mediating part of the weak force between left chiral fermions (but not right chiral ones). But since, from (133),

$$Z_\mu = -\sin\theta_W B_\mu + \cos\theta_W W_{3\mu}\ , \tag{134}$$

then the Z field is partly composed of the W_3 field, and some of the W_3 interaction properties must be carried by the Z. And they are. In fact, in the low energy interaction Lagrangian, we have a term

$$-e\tan\theta_W \bar{\psi}_e^R \gamma^\mu Z_\mu \psi_e^R \qquad (\text{term in low energy Lagrangian}), \tag{135}$$

that couples the weak force carrying Z to the right chiral electron. In a Feynman diagram, this looks like Figure 9.

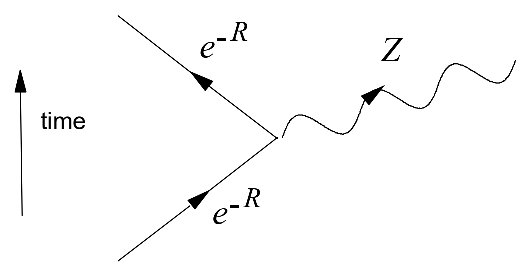


Figure 9. Right Chiral Electron Feels the Weak Z Force

<u>Bottom line:</u> Even though one hears everywhere that the weak force only couples to left chiral fermions, and not to right chiral ones, that is only true at high energy for the weak force mediated by the W_i gauge bosons. It is not true at low energy for the weak force mediated by the Z boson.

This was the reason why we earlier cautioned that certain statements made were only true for the high energy fields.

8.15 Any Candidate Higgs-like Particle Can Only Be a Real Scalar

For reasons we won't get into deeply here, because they are fairly complicated, only a scalar field of type (104) in a potential of form (103) leads to a realistic, renormalizable symmetry breaking theory. Scalar field terms to other powers, as well as spinor and vector fields, and will not work.

As one simpler example, if we were to have a term of form $\phi^\dagger\phi\phi$ added to (103), under a U(1) symmetry transformation (i.e., such as U(1) = $e^{i\alpha}$), we would get a complex term [$|\phi|^2(i\sin\,\alpha)$] in the Lagrangian (even if ϕ were real). But the Lagrangian term we started from was real and of different form. So, it would not be symmetric under this transformation. Hence, we cannot have any terms in it like $\phi^\dagger\phi\phi$.

Other possible terms in $\mathcal{V}$ can be shown, via more involved arguments, to give rise to an invalid theory. For more on this see http://edu.itp.phys.ethz.ch/hs12/qft1/Chapter07.pdf .

8.16 Right Chiral Neutrinos May Not Exist

Note from Wholeness Chart 8, pg. 35, that chiral RH neutrinos (not helicity RH neutrinos) have no electric, weak, or hyper charge. Since they are leptons, we know they do not interact via the strong force, so have zero strong force charge. Hence, they simply do not interact with other particles via any of the three standard model forces. Neutrino masses are so small, and gravitational coupling is so weak, that it is virtually impossible to detect whether they interact gravitationally. Since we have no way to detect them, we really don't know if they even exist. We show them throughout our development of the theory, in order to be complete, but it is certainly possible there are no such animals.

8.17 Flavor and Generation at a Vertex

8.17.1 No Generation Changing at a Vertex

Interaction terms such as (122) for electrons and electron neutrinos arise for each of the three generations (families) of leptons. Generalizing the last term in (122), as an example, we get

$$i\frac{g}{\sqrt{2}}\bar{\psi}_l^L\gamma^\mu\psi_{\nu_l}^L W_\mu^\dagger = i\frac{g}{\sqrt{2}}\bar{\psi}_e^L\gamma^\mu\psi_{\nu_e}^L W_\mu^\dagger + i\frac{g}{\sqrt{2}}\bar{\psi}_\mu^L\gamma^\mu\psi_{\nu_\mu}^L W_\mu^\dagger + i\frac{g}{\sqrt{2}}\bar{\psi}_\tau^L\gamma^\mu\psi_{\nu_\tau}^L W_\mu^\dagger \,. \tag{136}$$

The first term after the equal sign in (136), the term for the electron generation, has one of its associated Feynman diagram shown in Figure 7, pg. 36. For the second term, each e symbol in that figure would be a μ. That is, an incoming muon neutrino would absorb a W^-, and turn into an outgoing muon. Similar behavior occurs for the tau, via the third term in (136).

The important thing to note is than no mixing of generations occurs. We never get an incoming electron absorbing a W^- and then turning into an outgoing muon. One might think it possible, since the charges would still balance (total charge would still be conserved). But the theory tells us that it does not occur.

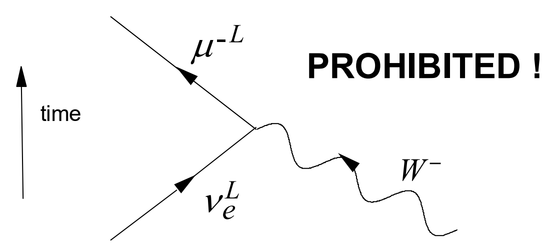


Figure 10. No Generation (Family) Changing at a Vertex

As an exercise, generalize (135) and Figure 9, pg. 39, for the Z boson interaction with electrons to all three lepton families. Then justify to yourself, at least for right hand electrons, muons, and taus, that no generation mixing can occur at a vertex having a (neutral) Z boson.

With more than one vertex, we can, however, have mixing of generations, as evidenced by the very first figure in Klauber[2] (pg. 2 therein). An incoming electron and positron annihilate at a vertex with a gauge boson (photon in the cited figure, but it could be a Z), and that boson subsequently creates an outgoing muon and anti-muon at a second vertex.

8.17.2 Flavor Changing at a Vertex Only with Charged IVBs

From Figure 7, pg. 36, we see that we can have flavor changing at a vertex within the same family. That is, an electron neutrino changes into an electron by absorbing a W^-. The flavor changes, but not the generation/family.

Z boson interactions cannot even change flavor within a generation. For example, see the Z interaction term (135) and an associated Feynman diagram of Figure 9, pg. 39. The Z is neutral for all types of charges, so it cannot change, for example, an electron into a neutrino at a vertex. In $\mathcal{L}$, we only have terms coupling the Z^μ field with a single type of lepton, similar to the term (135), so, as we said, Z boson interactions do not change flavor. We need a charged intermediate vector boson, like W^+ or W^- to do that, and then it is only within the same family.

8.17.3 Summary of Flavor and Generation Changes at a Vertex

We just covered first two lines of the summary below. As an exercise, use the photon interaction terms in $\mathcal{L}$ to justify the last.

Vertices with a Z $\rightarrow$ No flavor changing of any kind
Vertices with a $W \rightarrow$ Flavor changing inside a generation, but not from one generation to another.
Vertices with a γ $\rightarrow$ No flavor changing of any kind

8.17.4 Four-Currents and Flavor Changing

Recall from QED (see Klauber[2], pg. 112), that the electron four-current (to be strictly precise, the four-current *operator*) had form $j_e^\mu = \bar{\psi}_e \gamma^\mu \psi_e$ and the interaction term in the Lagrangian could be expressed in terms of it, as in

$$-e\bar{\psi}_e \gamma^\mu \psi_e A_\mu = -e\, j_e^\mu A_\mu \,. \tag{137}$$

In similar fashion, we can consider terms with the Z^μ field to have

$$\text{For } j_e^{\mu R} = \bar{\psi}_e^R \gamma^\mu \psi_e^R \qquad \rightarrow \qquad -e\tan\theta_W \bar{\psi}_e^R \gamma^\mu \psi_e^R Z_\mu = -e\tan\theta_W\, j_e^{\mu R} Z_\mu \,. \tag{138}$$

You will often see the Lagrangian terms written with four-currents as factors, as in the RHS of (137) and (138). In those cases, because no changes in charge are involved, the four currents are called neutral currents. They couple with neutral gauge bosons and have a neutral effect on charges.

In a similar spirit, terms like the first after the equal sign of (136) can be written as

$$i\frac{g}{\sqrt{2}}\bar{\psi}_e^L \gamma^\mu \psi_{\nu_e}^L W_\mu^\dagger = i\frac{g}{\sqrt{2}}\, j_{e,\nu_e}^{\mu L} W_\mu^\dagger \quad \text{where } j_{e,\nu_e}^{\mu L} = \bar{\psi}_e^L \gamma^\mu \psi_{\nu_e}^L \,, \tag{139}$$

and $j_{e,\nu_e}^{\mu L}$ considered a four current, as well. However, from the form of that four current, plus Figure 6 of pg. 34, we can see that it does not keep charges the same. It changes a neutrino (electron type) with zero electric charge and + ½ weak charge into an electron with – 1 electric charge and – ½ weak charge. (It can only do this with the help of an incoming W^- or an outgoing W^+, of course.) So $j_{e,\nu_e}^{\mu L}$ is not designated a neutral current, but a charged current.

This nomenclature is often used to restate our conclusions from Section 8.17.3 above as follows.

There are no flavor changing neutral currents.
There are no generation changing currents, charged or neutral.

8.17.5 Note on Derivation of Weak Four-Currents

Four-currents for weak interactions can be derived from the symmetry of $\mathcal{L}$, and Noether's theorem. (See Klauber[2], Chapters 6 and 11, where it was done for QED.) We will not delve further into this at this time. Hopefully, at some point in the future I can present the derivations for weak currents and discuss relevant conservation laws.

8.18 Running Coupling Constant

The interaction terms in $\mathcal{L}$ of the Glashow/Salam/Weinberg model (112) give rise to radiative type corrections to the effective Higgs field coupling, via adding up relevant Feynman diagram sub-amplitudes, which vary as the energy level changes. This is similar to QED (see Klauber[2], pgs 304-317) where the QED coupling constant varies with energy level, due to the contributions of Feynman diagrams beyond tree level. Thus, we find our SU(2) and U(1) coupling constants g and g' become functions of energy level (represented by p)

$$g = g(p) \qquad g' = g'(p) . \tag{140}$$

We will not delve more deeply into this issue here. It is quite complex.

8.19 Renormalization

Renormalization of electroweak theory was long an intractable problem, but was brilliantly solved in the 1970s by Gerardus ‘t Hooft and Martinus Veltman, and in 1999, they received a Nobel Prize for their work. We do not consider what they did in depth, but simply note, very briefly, the approach they took.

‘t Hooft and Veltman realized that renormalization is easier to prove by choosing a different gauge, other than the unitary gauge, in which η_1, η_2, and $\eta_3 \neq 0$. This results in unphysical (ghost) fields we don’t observe. Ghost fields are linear combinations of the 12 degrees of freedom of the other fields of the unitary gauge [1 massive Higgs σ (1 DOF), 3 massive W and Z (9 DOF), and 1 massless photon (2 DOF)]. Conceptually, and practically for calculating cross sections, that approach is more complicated. However, it provides advantages that facilitate proof of the renormalizabilty of the theory.

For more on this subject, see, for examples, Peskin & Schroeder[5] and Itzykson and Zuber[6].

8.20 Gauge Hierarchy

Radiative corrections also affect the Higgs couplings and thus, particle masses. It turns out that when they are calculated, one gets a Higgs mass that is enormously larger (on the order of the grand unified scale of 10^{16} GeV or higher) than what it must be via experiment (~125 GeV as determined at CERN in 2012).

The two leading candidate theories to solve this problem are 1) supersymmetry and 2) extra dimensions. SUSY, however, is not finding much experimental support these days at CERN. Without a natural solution like these, one ends up with a need for some very unnatural, very fine-tuned cancellation of sub-amplitudes (plus and minus amplitude values on the order 10^{16} GeV cancelling to leave a term on order 10^2 GeV.)

The immense difference in hierarchical levels of energy in a gauge theory of particles leads to the designation “gauge hierarchy problem” for this issue.

At some point in the future, I may expand on this section and show Feynman diagrams and amplitudes responsible for the expected huge mass of the Higgs.

8.21 Mass of the Higgs

Hopefully, I will elaborate on the following some day. The first two paragraphs are lifted, word-for-word from https://www.theorie.physik.uni-muenchen.de/lsfrey/teaching/archiv/sose_09/rng/higgs_mechanism.pdf (2009). Note that greater Higgs coupling for a particle means greater mass at low energy. So, the heaviest elementary particles (top quark is heaviest) contribute the most to the radiative correction that modify the effective Higgs mass

“If we were to compute radiative corrections (e.g. to the Weinberg angle), we would discover that heavy quarks give rise to large corrections. This result is a direct consequence of a gauge theory with SSB. In a renormalizable gauge theory without SBB, heavy quarks would decouple at energy scales much smaller than their masses. However, in the GSW (Glashow/Salam/Weinberg) model, the longitudinal components of the W and Z bosons are generated by the Higgs mechanism, and their coupling increases with the masses. Thus, heavy quarks do not decouple in the SM.

The top quark, due to its large mass, plays an essential role both in and beyond the SM. Precision measurements of m_t set contraints on the masses of particles to which the top quark makes radiative corrections, including the unobserved Higgs boson and new particles that might contribute additional radiative corrections. The precision electroweak measurements indicate that $m_h \leq 163$ GeV (one-sided 95% confidence level), and according to the LEP Higgs Working Group m_h must be heavier than 114.4 GeV (95% confidence level).”

Of course, we know now, from the CERN discovery, that the Higgs mass is about 125 GeV.

8.22 Why is the Higgs Particle Called the "God Particle"?

You may have heard the Higgs particle referred to as the "God particle". It is called this, at least in part, for the following reason.

Prior to symmetry breaking, no particles had mass. So, all traveled at the speed of light. But particles traveling at the speed of light are not going to coalesce into nuclei and atoms, as they are simply moving too fast. So, nothing we know of as our world could ever form. Once the particles gain mass, via the coupling to the Higgs field, they no longer travel at the speed of light, so they effectively slow down enough to join together to form the building blocks of our universe. Without the Higgs, we would have no universe as we know it. It is the foundation of all, the source of all. Hence, the allusion to God, the source of everything according to the world's major religions.

Additionally, when a (massless) particle travels at the speed of light, time does not progress on that particle. In effect, time is frozen and nothing happens. From this perspective, the Higgs gives rise to time itself. Without any mass, no particles would experience the passage of time. The "God particle", by bestowing mass, bestows time on the participants in the evolution of the universe, something for which many would credit God.

9 Appendix A. Answers to Some Exercises

Derivation of (62)

Using (60) and (61), we have

$$\begin{aligned}
&\psi_l^L = \tfrac{1}{2}\left(1-\gamma^5\right)\psi_l \qquad \psi_l^R = \tfrac{1}{2}\left(1+\gamma^5\right)\psi_l \qquad \gamma^{5\dagger} = \gamma^5 \qquad \gamma^5\gamma^0 = -\gamma^0\gamma^5 \qquad \gamma^5\gamma^5 = 1 \\
&\bar{\psi}_l^L = \psi_l^{L\dagger}\gamma^0 = \psi_l^\dagger \tfrac{1}{2}\left(1-\gamma^{5\dagger}\right)\gamma^0 = \psi_l^\dagger \tfrac{1}{2}\left(1-\gamma^5\right)\gamma^0 = \psi_l^\dagger\gamma^0 \tfrac{1}{2}\left(1+\gamma^5\right) = \bar{\psi}_l \tfrac{1}{2}\left(1+\gamma^5\right) \\
&\bar{\psi}_l^R = \psi_l^{R\dagger}\gamma^0 = \psi_l^\dagger \tfrac{1}{2}\left(1+\gamma^{5\dagger}\right)\gamma^0 = \psi_l^\dagger \tfrac{1}{2}\left(1+\gamma^5\right)\gamma^0 = \psi_l^\dagger\gamma^0 \tfrac{1}{2}\left(1-\gamma^5\right) = \bar{\psi}_l \tfrac{1}{2}\left(1-\gamma^5\right).
\end{aligned} \tag{141}$$

Derivation of (92)

Starting with (84)

$$\begin{aligned}
W_i^{\mu\prime}\tau_i &= \left(W_i^\mu + \delta W_i^\mu\right)\tau_i = e^{i\omega_i(x)\tau_i/2}\left(W_j^\mu\tau_j\right)e^{-i\omega_k(x)\tau_k/2} \\
&= \left(1 + i\frac{\omega_i}{2}\tau_i + \dots\right)W_j^\mu\tau_j\left(1 - i\frac{\omega_k}{2}\tau_k - \dots\right) \approx W_j^\mu\tau_j + i\frac{\omega_i}{2}\tau_i W_j^\mu\tau_j - i\frac{\omega_k}{2}\tau_j W_j^\mu\tau_k \\
&= W_i^\mu\tau_i + \tfrac{i}{2}W_j^\mu\left(\omega_i\tau_i\tau_j - \omega_k\tau_j\tau_k\right).
\end{aligned} \tag{84}$$

Then, using the first relation in the second row of (69),

$$\tau_i\tau_j = \mathrm{I}\delta_{ij} + i\varepsilon_{ijk}\tau_k \qquad \text{part of (69)}$$

$$\begin{aligned}
W_i^{\mu\prime}\tau_i &= W_i^\mu\tau_i + \tfrac{i}{2}W_j^\mu\left(\omega_i\left(\delta_{ij} + i\varepsilon_{ijk}\tau_k\right) - \omega_k\left(\delta_{jk} + i\varepsilon_{jkm}\tau_m\right)\right) \\
&= W_i^\mu\tau_i + \tfrac{i}{2}W_j^\mu\left(\omega_j + i\omega_i\varepsilon_{ijk}\tau_k - \omega_j - i\omega_k\varepsilon_{jkm}\tau_m\right) = W_i^\mu\tau_i + \tfrac{i}{2}W_j^\mu\left(i\omega_i\varepsilon_{ijk}\tau_k - i\omega_i\varepsilon_{jim}\tau_m\right) \\
&= W_i^\mu\tau_i - \tfrac{1}{2}W_j^\mu\left(\omega_i\varepsilon_{ijk}\tau_k + \omega_i\varepsilon_{ijm}\tau_m\right) = W_i^\mu\tau_i - \tfrac{1}{2}W_j^\mu\left(\omega_i\varepsilon_{ijk}\tau_k + \omega_i\varepsilon_{ijk}\tau_k\right) \\
&= W_i^\mu\tau_i - \tfrac{1}{2}W_j^\mu 2\omega_i\varepsilon_{ijk}\tau_k = W_i^\mu\tau_i - W_j^\mu\omega_i\varepsilon_{ijk}\tau_k = W_i^\mu\tau_i - W_k^\mu\omega_j\varepsilon_{ijk}\tau_i \\
&= \left(W_i^\mu - \varepsilon_{ijk}\omega_j W_k^\mu\right)\tau_i \quad \rightarrow \quad \delta W_i^\mu = -\varepsilon_{ijk}\omega_j W_k^\mu .
\end{aligned} \tag{142}$$

Derivation of (115)

From (60), (61), and (62) (with no sum on l),

$$\bar{\psi}_l^L\psi_l^R = \tfrac{1}{2}\bar{\psi}_l\left(1+\gamma^5\right)\tfrac{1}{2}\left(1+\gamma^5\right)\psi_l = \tfrac{1}{4}\bar{\psi}_l\psi_l + \tfrac{2}{4}\bar{\psi}_l\gamma^5\bar{\psi}_l + \tfrac{1}{4}\bar{\psi}_l\underbrace{\gamma^5\gamma^5}_{1}\psi_l$$

$$\bar{\psi}_l^R\psi_l^L = \tfrac{1}{2}\bar{\psi}_l\left(1-\gamma^5\right)\tfrac{1}{2}\left(1-\gamma^5\right)\psi_l = \tfrac{1}{4}\bar{\psi}_l\psi_l - \tfrac{2}{4}\bar{\psi}_l\gamma^5\bar{\psi}_l + \tfrac{1}{4}\bar{\psi}_l\underbrace{\gamma^5\gamma^5}_{1}\psi_l \qquad (143)$$

$$\rightarrow \quad \bar{\psi}_l^L\psi_l^R + \bar{\psi}_l^R\psi_l^L = \bar{\psi}_l\psi_l \qquad (\text{no sum on } l).$$

Derivation of Higgs Isospin of Wholeness Chart 8, pg. 35.

To deduce the Higgs isospin charge, start with the first lepton-Higgs coupling term of (75) for electrons/positrons,

$$-g_e\bar{\Psi}_e^L\psi_e^R\Phi = -g_e\begin{pmatrix}\bar{\psi}_{\nu_e}^L & \bar{\psi}_e^L\end{pmatrix}\psi_e^R\begin{pmatrix}0\\ \phi_3\end{pmatrix} = -g_e\bar{\psi}_e^L\psi_e^R\phi_3. \qquad (144)$$

In the spirit of the Feynman diagram of Figure 6, pg. 34, (144) represents an incoming right hand electron and an incoming Higgs both destroyed at a vertex with a left hand electron created. The weak isospin charges incoming are zero for the RH electron and the charge on the Higgs. The outgoing charge is – ½, so the Higgs charge must be – ½.

Derivation of Higgs Isospin a Different Way

From (54) in the unitary gauge, use the weak isospin operator $\tau_3/2$ to find the weak isospin charge on the Higgs field ϕ_3.

$$\Phi(x) = \begin{bmatrix}\phi_a\\ \phi_b\end{bmatrix} = \begin{bmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{bmatrix} \xrightarrow[\text{gauge}]{\text{unitary}} \Phi(x) = \begin{bmatrix}0\\ \phi_3\end{bmatrix} \qquad (145)$$

$$\tfrac{1}{2}\tau_3\Phi = \frac{1}{2}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0\\ \phi_3\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}0\\ \phi_3\end{bmatrix} \qquad \text{eigenvalue} = \text{weak isospin charge} = -\frac{1}{2}, \qquad (146)$$

Showing the first and last terms of the last row in (162) cancel

Substituting the values for the variations found in (99) into the first and last terms of the last row of (161), we find

$$g'\not{B}\bar{\psi}_l^R\left(\delta\psi_l^R\right) + g'\not{B}\left(\delta\bar{\psi}_l^R\right)\psi_l^R = g'\not{B}\left(\bar{\psi}_l^R\left(-if\psi_l^R\right) + \left(if\bar{\psi}_l^R\right)\psi_l^R\right) = g'\not{B}\left(-if\bar{\psi}_l^R\psi_l^R + if\bar{\psi}_l^R\psi_l^R\right) = 0, \quad (147)$$

i.e., they cancel.

Derivation of the result of (182) starting with a 3D vector identity

From

$$\mathbf{a}\bullet\left(\mathbf{b}\times\mathbf{a}\right) = 0 \quad \rightarrow \quad a_i\left(\varepsilon_{ijk}b_ja_k\right) = \varepsilon_{ijk}b_ja_ka_i \qquad (148)$$

note the first term in the second line of (182)

$$\tfrac{1}{2}\varepsilon_{ijk}\omega_jG_k^{\mu\nu}G_{i\mu\nu} = \underbrace{\tfrac{1}{2}G_{i\mu\nu}\varepsilon_{ijk}\omega_jG_k^{\mu\nu} \quad \text{for} \quad \mathbf{a}\rightarrow\mathbf{G} \quad \mathbf{b}\rightarrow\boldsymbol{\omega}}_{\frac{1}{2}\mathbf{G}\bullet\left(\boldsymbol{\omega}\times\mathbf{G}\right)=0} \qquad (149)$$

Showing the part of (197) we ignored is also zero

$$\delta\left(\bar{\Psi}_{\nu_l}^L\psi_{\nu_l}^R\tilde{\Phi} + \tilde{\Phi}^\dagger\bar{\psi}_{\nu_l}^R\Psi_{\nu_l}^L\right) = \begin{pmatrix}\left(\delta\bar{\Psi}_{\nu_l}^L\right)\psi_{\nu_l}^R\tilde{\Phi} + \bar{\Psi}_{\nu_l}^L\underbrace{\left(\delta\psi_{\nu_l}^R\right)}_{=0}\tilde{\Phi} + \bar{\Psi}_{\nu_l}^L\psi_{\nu_l}^R\left(\delta\tilde{\Phi}\right)\\ +\left(\delta\tilde{\Phi}^\dagger\right)\bar{\psi}_{\nu_l}^R\Psi_{\nu_l}^L + \tilde{\Phi}^\dagger\underbrace{\left(\delta\bar{\psi}_{\nu_l}^R\right)}_{=0}\Psi_{\nu_l}^L + \tilde{\Phi}^\dagger\bar{\psi}_{\nu_l}^R\left(\delta\Psi_{\nu_l}^L\right)\end{pmatrix} \qquad (150)$$

$$=\begin{pmatrix}\left(-\frac{i}{2}\omega_i\bar{\Psi}^L_{\nu_l}\tau_i+\frac{i}{2}f\bar{\Psi}^L_{\nu_l}\right)\psi^R_{\nu_l}\tilde{\Phi}+\bar{\Psi}^L_{\nu_l}\psi^R_{\nu_l}\left(\frac{i}{2}\omega_i\tau_i\tilde{\Phi}-\frac{i}{2}f\tilde{\Phi}\right)\\ +\left(-\frac{i}{2}\omega_i\tilde{\Phi}^\dagger\tau_i+\frac{i}{2}f\tilde{\Phi}^\dagger\right)\bar{\psi}^R_{\nu_l}\Psi^L_{\nu_l}+\tilde{\Phi}^\dagger\bar{\psi}^R_{\nu_l}\left(\frac{i}{2}\omega_i\tau_i\Psi^L_{\nu_l}-\frac{i}{2}f\Psi^L_{\nu_l}\right)\end{pmatrix} \tag{151}$$

$$\begin{aligned}&=-\frac{i}{2}\omega_i\bar{\Psi}^L_{\nu_l}\tau_i\psi^R_{\nu_l}\tilde{\Phi}+\frac{i}{2}f\bar{\Psi}^L_{\nu_l}\psi^R_{\nu_l}\tilde{\Phi}+\underbrace{\frac{i}{2}\omega_i\bar{\Psi}^L_{\nu_l}\psi^R_{\nu_l}\tau_i\tilde{\Phi}}_{\frac{i}{2}\omega_i\bar{\Psi}^L_{\nu_l}\tau_i\psi^R_{\nu_l}\tilde{\Phi}}-\frac{i}{2}f\bar{\Psi}^L_{\nu_l}\psi^R_{\nu_l}\tilde{\Phi}\\ &-\frac{i}{2}\omega_i\tilde{\Phi}^\dagger\tau_i\bar{\psi}^R_{\nu_l}\Psi^L_{\nu_l}+\frac{i}{2}f\tilde{\Phi}^\dagger\bar{\psi}^R_{\nu_l}\Psi^L_{\nu_l}+\underbrace{\frac{i}{2}\omega_i\tilde{\Phi}^\dagger\bar{\psi}^R_{\nu_l}\tau_i\Psi^L_{\nu_l}}_{\frac{i}{2}\omega_i\tilde{\Phi}^\dagger\tau_i\bar{\psi}^R_{\nu_l}\Psi^L_{\nu_l}}-\frac{i}{2}f\tilde{\Phi}^\dagger\bar{\psi}^R_{\nu_l}\Psi^L_{\nu_l}\ .\end{aligned} \tag{152}$$

In the first row of (152) the first and third terms cancel. So, do the second and last terms. The same thing is true for the second row. So,

$$\delta\left(\bar{\Psi}^L_{\nu_l}\psi^R_{\nu_l}\tilde{\Phi}+\tilde{\Phi}^\dagger\bar{\psi}^R_{\nu_l}\Psi^L_{\nu_l}\right)=0\delta\left(\bar{\Psi}^L_{\nu_l}\psi^R_{\nu_l}\tilde{\Phi}+\tilde{\Phi}^\dagger\bar{\psi}^R_{\nu_l}\Psi^L_{\nu_l}\right)=0\,. \tag{153}$$

10 Appendix B. Showing the Symmetry of the High Energy Lagrangian

10.1 Expanded Form of High Energy Lagrangian

To show the invariance of the high energy Lagrangian under the transformations of Section 5.4, we need to first show the expanded version of the Lagrangian of (75). We do this by substituting (76) through (82) into (75).

10.1.1 For $\mathcal{L}^L$ (from (75), (77), and (78)

$$\left.\begin{aligned}&i\bar{\Psi}^L_l\not{D}\Psi^L_l=i\bar{\Psi}^L_l\left(\not{\partial}+\frac{i}{2}g\tau_j\not{W}_j-\frac{i}{2}g'\not{B}\right)\Psi^L_l=i\bar{\Psi}^L_l\not{\partial}\Psi^L_l-\frac{g}{2}\bar{\Psi}^L_l\tau_j\not{W}_j\Psi^L_l+\frac{g'}{2}\bar{\Psi}^L_l\not{B}\Psi^L_l\\ &i\bar{\psi}^R_l\not{D}\psi^R_l=i\bar{\psi}^R_l\left(\not{\partial}-ig'\not{B}\right)\psi^R_l=i\bar{\psi}^R_l\not{\partial}\psi^R_l+g'\bar{\psi}^R_l\not{B}\psi^R_l\\ &i\bar{\psi}^R_{\nu_l}\not{D}\psi^R_{\nu_l}=i\bar{\psi}^R_{\nu_l}\not{\partial}\psi^R_{\nu_l}\end{aligned}\right\}\mathcal{L}^L \tag{154}$$

$$\mathcal{L}^L=i\bar{\Psi}^L_l\not{\partial}\Psi^L_l+i\bar{\psi}^R_l\not{\partial}\psi^R_l+i\bar{\psi}^R_{\nu_l}\not{\partial}\psi^R_{\nu_l}-\frac{g}{2}\bar{\Psi}^L_l\tau_j\not{W}_j\Psi^L_l+\frac{g'}{2}\bar{\Psi}^L_l\not{B}\Psi^L_l+g'\bar{\psi}^R_l\not{B}\psi^R_l$$

10.1.2 For $\mathcal{L}^B$ (from (75))

$$\begin{gathered}\mathcal{L}^B=\overbrace{-\frac{1}{4}B^{\mu\nu}B_{\mu\nu}}^{\boxed{9}}\overbrace{-\frac{1}{4}G_i^{\mu\nu}G_{i\,\mu\nu}}^{\boxed{10}}\\ B^{\mu\nu}=\partial^\nu B^\mu-\partial^\mu B^\nu\qquad G_i^{\mu\nu}=\partial^\nu W_i^\mu-\partial^\mu W_i^\nu-ge_{ijk}W_j^\nu W_k^\mu\end{gathered} \tag{155}$$

10.1.3 For $\mathcal{L}^H$ (from (75) and (81)

$$\begin{gathered}\mathcal{L}^H=\left(D^\mu\Phi\right)^\dagger\left(D_\mu\Phi\right)-\mu^2\Phi^\dagger\Phi-\lambda\left(\Phi^\dagger\Phi\right)^2\\ D_\mu\Phi=\left(\partial_\mu+\frac{i}{2}g\tau_kW_{k\mu}+i\frac{g'}{2}B_\mu\right)\Phi\end{gathered} \tag{156}$$

10.1.4 For $\mathcal{L}^{LH}$ (from (75))

$$\mathcal{L}^{LH}=-g_l\left(\bar{\Psi}^L_l\psi^R_l\Phi+\Phi^\dagger\bar{\psi}^R_l\Psi^L_l\right)-g_{\nu_l}\left(\bar{\Psi}^L_{\nu_l}\psi^R_{\nu_l}\tilde{\Phi}+\tilde{\Phi}^\dagger\bar{\psi}^R_{\nu_l}\Psi^L_{\nu_l}\right) \tag{157}$$

Using the above (154) to (157),

$$\mathcal{L}=\mathcal{L}^L+\mathcal{L}^B+\mathcal{L}^H+\mathcal{L}^{LH} \tag{158}$$

10.2 Showing the Symmetry of the High Energy Lagrangian

It is easiest showing invariance under an infinitesimal form of a transformation set rather than the finite form, so we will use (99) to (102) in (154) to (158) above. Note that under any such transformation of the underlying fields, we have

$$\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}\,, \tag{159}$$

where $\delta\mathcal{L}$ is the variation in in the Lagrangian due to the variation in the underlying fields. If $\delta\mathcal{L} = 0$, then $\mathcal{L}' = \mathcal{L}$, and the transformation is symmetric (leaves $\mathcal{L}$ invariant). So, all we need to do is show $\delta\mathcal{L} = 0$ using (158). We will separately examine each of the terms in (158) to demonstrate that this does, indeed, happen.

10.2.1 For $\mathcal{L}^L$

From the last row of (154),

$$\begin{aligned}
\mathcal{L}'^{\,L} &= \mathcal{L}^L + \delta\mathcal{L}^L \\
&= i\left(\bar{\Psi}_l^L + \delta\bar{\Psi}_l^L\right)\not\partial\left(\Psi_l^L + \delta\Psi_l^L\right) + i\left(\bar{\psi}_l^R + \delta\bar{\psi}_l^R\right)\not\partial\left(\psi_l^R + \delta\psi_l^R\right) + i\left(\bar{\psi}_{\nu_l}^R + \delta\bar{\psi}_{\nu_l}^R\right)\not\partial\left(\psi_{\nu_l}^R + \delta\psi_{\nu_l}^R\right) \\
&\qquad -\tfrac{g}{2}g\left(\bar{\Psi}_l^L + \delta\bar{\Psi}_l^L\right)\tau_j\left(\not{W}_j + \delta\not{W}_j\right)\left(\Psi_l^L + \delta\Psi_l^L\right) + \tfrac{g'}{2}\left(\bar{\Psi}_l^L + \delta\bar{\Psi}_l^L\right)\left(\not{B} + \delta\not{B}\right)\left(\Psi_l^L + \delta\Psi_l^L\right) \\
&\qquad\qquad + g'\left(\bar{\psi}_l^R + \delta\bar{\psi}_l^R\right)\left(\not{B} + \delta\not{B}\right)\left(\psi_l^R + \delta\psi_l^R\right),
\end{aligned} \tag{160}$$

where for part of our chosen transformation set, $\delta\psi_l^R = \delta\bar{\psi}_l^R = 0$. Because the variation δ is infinitesimal, we can ignore all orders in δ higher than the first. Thus,

$$\begin{aligned}
\delta\mathcal{L}^L \approx\; & i\bar{\Psi}_l^L\not\partial\left(\delta\Psi_l^L\right) + i\left(\delta\bar{\Psi}_l^L\right)\not\partial\Psi_l^L + i\bar{\psi}_l^R\not\partial\left(\delta\psi_l^R\right) + i\left(\delta\bar{\psi}_l^R\right)\not\partial\psi_l^R \\
& -\tfrac{g}{2}\bar{\Psi}_l^L\tau_j\not{W}_j\left(\delta\Psi_l^L\right) - \tfrac{g}{2}\bar{\Psi}_l^L\tau_j\left(\delta\not{W}_j\right)\Psi_l^L - \tfrac{g}{2}\left(\delta\bar{\Psi}_l^L\right)\tau_j\not{W}_j\Psi_l^L \\
& + \tfrac{g'}{2}\bar{\Psi}_l^L\not{B}\left(\delta\Psi_l^L\right) + \tfrac{g'}{2}\bar{\Psi}_l^L\left(\delta\not{B}\right)\Psi_l^L + \tfrac{g'}{2}\left(\delta\bar{\Psi}_l^L\right)\not{B}\Psi_l^L \\
& + g'\bar{\psi}_l^R\not{B}\left(\delta\psi_l^R\right) + g'\bar{\psi}_l^R\left(\delta\not{B}\right)\psi_l^R + g'\left(\delta\bar{\psi}_l^R\right)\not{B}\psi_l^R\,.
\end{aligned} \tag{161}$$

Let's start with one of the easier parts of (161), the first and last terms of the next to last row of (161). It turns out, they cancel. That is, using (99),

$$\begin{aligned}
\tfrac{g'}{2}\bar{\Psi}_l^L\not{B}\left(\delta\Psi_l^L\right) + \tfrac{g'}{2}\left(\delta\bar{\Psi}_l^L\right)\not{B}\Psi_l^L &= \tfrac{g'}{2}\bar{\Psi}_l^L\not{B}\left(\tfrac{i}{2}\omega_i\tau_i\Psi_l^L + \tfrac{i}{2}f\Psi_l^L\right) + \tfrac{g'}{2}\left(-\tfrac{i}{2}\omega_i\bar{\Psi}_l^L\tau_i - \tfrac{i}{2}f\bar{\Psi}_l^L\right)\not{B}\Psi_l^L \\
&= \tfrac{g'}{2}\bar{\Psi}_l^L\left(\tfrac{i}{2}\omega_i\tau_i - \tfrac{i}{2}f\right)\not{B}\Psi_l^L + \tfrac{g'}{2}\bar{\Psi}_l^L\left(-\tfrac{i}{2}\omega_i\tau_i + \tfrac{i}{2}f\right)\not{B}\Psi_l^L = 0\,.
\end{aligned} \tag{162}$$

Showing the first and last terms of the last row in (162) cancel is even easier, so we leave it to you as an exercise. The answer can be found in Appendix A, pg. 43.

The first and last terms of the second row in (161) become, where we use (69) in the last line of (163),

$$\begin{aligned}
& -\tfrac{g}{2}\bar{\Psi}_l^L\tau_j\not{W}_j\left(\delta\Psi_l^L\right) - \tfrac{g}{2}\left(\delta\bar{\Psi}_l^L\right)\tau_j\not{W}_j\Psi_l^L \\
&\quad = -\tfrac{g}{2}\bar{\Psi}_l^L\tau_j\not{W}_j\left(\tfrac{i}{2}\omega_i\tau_i\Psi_l^L - \tfrac{i}{2}f\Psi_l^L\right) - \tfrac{g}{2}\left(-\tfrac{i}{2}\omega_i\bar{\Psi}_l^L\tau_i + \tfrac{i}{2}\bar{\Psi}_l^L\right)\tau_j\not{W}_j\Psi_l^L \\
&\quad = -i\tfrac{g}{4}\omega_i\bar{\Psi}_l^L\tau_j\tau_i\not{W}_j\Psi_l^L \underbrace{+ i\tfrac{g}{4}f\bar{\Psi}_l^L\tau_j\not{W}_j\Psi_l^L}_{\text{cancels}} + i\tfrac{g}{4}\omega_i\bar{\Psi}_l^L\tau_i\tau_j\not{W}_j\Psi_l^L \underbrace{- i\tfrac{g}{4}\bar{\Psi}_l^L\tau_j\not{W}_j\Psi_l^L}_{\text{cancels}} \\
&\quad = -i\tfrac{g}{4}\omega_i\bar{\Psi}_l^L\left(\tau_j\tau_i - \tau_i\tau_j\right)\not{W}_j\Psi_l^L = -\tfrac{g}{2}\bar{\Psi}_l^L\left(e_{ijk}\tau_k\right)\omega_i\not{W}_j\Psi_l^L\,.
\end{aligned} \tag{163}$$

The middle term of the second row in (161), using (100), is

$$\begin{aligned}
-\tfrac{g}{2}\bar{\Psi}_l^L\tau_j\left(\delta\not{W}_j\right)\Psi_l^L &= -\tfrac{g}{2}\bar{\Psi}_l^L\tau_j\left(-\tfrac{1}{g}\not\partial\omega_j - \varepsilon_{jkl}\omega_k\not{W}_l\right)\Psi_l^L = \tfrac{1}{2}\bar{\Psi}_l^L\tau_j\left(\not\partial\omega_j\right)\Psi_l^L + \tfrac{g}{2}\bar{\Psi}_l^L\varepsilon_{klj}\tau_j\omega_k\not{W}_l\,\Psi_l^L \\
&= \tfrac{1}{2}\bar{\Psi}_l^L\tau_j\left(\not\partial\omega_j\right)\Psi_l^L + \tfrac{g}{2}\bar{\Psi}_l^L\varepsilon_{ijk}\tau_k\omega_i\not{W}_j\,\Psi_l^L\,,
\end{aligned} \tag{164}$$

the last term of which cancels with the final result in (163), leaving us with the first term in the second row of (164) equal to the entire second row of (161)

Assimilating all of the results from (162) to (164) into (161), we have

$$\begin{aligned}\delta\mathcal{L}^L = {} & \overbrace{i\bar{\Psi}_l^L \not{\partial}\left(\delta\Psi_l^L\right)}^{\boxed{1}} + \overbrace{i\left(\delta\bar{\Psi}_l^L\right)\not{\partial}\Psi_l^L}^{\boxed{2}} + \overbrace{i\bar{\psi}_l^R \not{\partial}\left(\delta\psi_l^R\right)}^{\boxed{3}} + \overbrace{i\left(\delta\bar{\psi}_l^R\right)\not{\partial}\psi_l^R}^{\boxed{4}} \\ & + \overbrace{\tfrac{1}{2}\bar{\Psi}_l^L\tau_j\left(\not{\partial}\omega_j\right)\Psi_l^L}^{\boxed{5}} + \overbrace{\tfrac{g'}{2}\bar{\Psi}_l^L\left(\delta\not{B}\right)\Psi_l^L}^{\boxed{6}} + \overbrace{g'\bar{\psi}_l^R\left(\delta\not{B}\right)\psi_l^R}^{\boxed{7}} .\end{aligned} \tag{165}$$

Let's look at terms $\boxed{1}$ and $\boxed{2}$.

$$\begin{aligned}&\overbrace{i\bar{\Psi}_l^L \not{\partial}\left(\delta\Psi_l^L\right)}^{\boxed{1}} + \overbrace{i\left(\delta\bar{\Psi}_l^L\right)\not{\partial}\Psi_l^L}^{\boxed{2}} = i\bar{\Psi}_l^L\not{\partial}\tfrac{i}{2}\omega_i\tau_i\Psi_l^L + i\bar{\Psi}_l^L\not{\partial}i\left(-\tfrac{1}{2}\right)f\Psi_l^L - i\bar{\Psi}_l^L\tfrac{i}{2}\omega_i\tau_i\not{\partial}\Psi_l^L - i\bar{\Psi}_l^L i\left(-\tfrac{1}{2}\right)f\not{\partial}\Psi_l^L \\ &= -\tfrac{1}{2}\bar{\Psi}_l^L\left(\not{\partial}\omega_i\right)\tau_i\Psi_l^L \underbrace{-\tfrac{1}{2}\bar{\Psi}_l^L\omega_i\tau_i\not{\partial}\Psi_l^L}_{\text{cancels with 5th term}} + \tfrac{1}{2}\bar{\Psi}_l^L\left(\not{\partial}f\right)\Psi_l^L \underbrace{+\tfrac{1}{2}\bar{\Psi}_l^L f\not{\partial}\Psi_l^L}_{\text{cancels with last term}} + \underbrace{\tfrac{1}{2}\bar{\Psi}_l^L\omega_i\tau_i\not{\partial}\Psi_l^L}_{\text{cancels with 2nd term}} \underbrace{-\tfrac{1}{2}\bar{\Psi}_l^L f\not{\partial}\Psi_l^L}_{\text{cancels with 4th term}} \\ &= -\tfrac{1}{2}\bar{\Psi}_l^L\left(\not{\partial}\omega_i\right)\tau_i\Psi_l^L + \tfrac{1}{2}\bar{\Psi}_l^L\left(\not{\partial}f\right)\Psi_l^L .\end{aligned} \tag{166}$$

Now the $\boxed{5}$ and $\boxed{6}$ terms.

$$\overbrace{\tfrac{1}{2}\bar{\Psi}_l^L\tau_j\left(\not{\partial}\omega_j\right)\Psi_l^L}^{\boxed{5}} + \overbrace{\tfrac{g'}{2}\bar{\Psi}_l^L\left(\delta\not{B}\right)\Psi_l^L}^{\boxed{6}} = \tfrac{1}{2}\bar{\Psi}_l^L\tau_j\left(\not{\partial}\omega_j\right)\Psi_l^L + \tfrac{g'}{2}\bar{\Psi}_l^L\left(-\tfrac{1}{g'}\not{\partial}f\right)\Psi_l^L . \tag{167}$$

(167) cancels (166), and leaves us with

$$\delta\mathcal{L}^L = +\overbrace{i\bar{\psi}_l^R \not{\partial}\left(\delta\psi_l^R\right)}^{\boxed{3}} + \overbrace{i\left(\delta\bar{\psi}_l^R\right)\not{\partial}\psi_l^R}^{\boxed{4}} + \overbrace{g'\bar{\psi}_l^R\left(\delta\not{B}\right)\psi_l^R}^{\boxed{7}} , \tag{168}$$

which becomes

$$\begin{aligned}\delta\mathcal{L}^L &= i\bar{\psi}_l^R\not{\partial}\left(-if\psi_l^R\right) + i\left(if\bar{\psi}_l^R\right)\not{\partial}\psi_l^R + g'\bar{\psi}_l^R\left(-\tfrac{1}{g'}\not{\partial}f\right)\psi_l^R \\ &= \underbrace{\bar{\psi}_l^R\left(\not{\partial}f\right)\psi_l^R}_{\text{cancels with last term}} \underbrace{+\bar{\psi}_l^R f\not{\partial}\psi_l^R - \bar{\psi}_l^R f\not{\partial}\psi_l^R}_{=0} \underbrace{-\bar{\psi}_l^R\left(\not{\partial}f\right)\psi_l^R}_{\text{cancels with 1st term}} = 0 .\end{aligned} \tag{169}$$

<u>Result of this section:</u> We have just shown that the $\mathcal{L}^L$ part of $\mathcal{L}$ is symmetric under the transformation of Section 5.4.3.

10.2.2 <u>For $\mathcal{L}^B$</u>

Re-stating (155) for convenience,

$$\begin{aligned}&\mathcal{L}^B = \overbrace{-\tfrac{1}{4}B^{\mu\nu}B_{\mu\nu}}^{\boxed{9}}\overbrace{-\tfrac{1}{4}G_i^{\mu\nu}G_{i\,\mu\nu}}^{\boxed{10}} \\ &B^{\mu\nu} = \partial^\nu B^\mu - \partial^\mu B^\nu \qquad G_i^{\mu\nu} = \partial^\nu W_i^\mu - \partial^\mu W_i^\nu - g e_{ijk} W_j^\nu W_k^\mu\end{aligned} \tag{155}$$

For the $\boxed{9}$ term in (155), we find from (101) that B^μ only transforms under the U(1) part of the transformation, so the transformation parallels the symmetric transformation of the QED Lagrangian term for the free part of the photon field A^μ, as shown in Klauber[2] (11-36), pg. 294. So, from that, we recognize the $\boxed{9}$ term contributes zero to $\delta\mathcal{L}^B$ and only need to examine term $\boxed{10}$.

$$\delta\mathcal{L}^B = \text{ term } \boxed{10} \text{ variation } = -\tfrac{1}{4}\left(\left(\delta G_i^{\mu\nu}\right)G_{i\,\mu\nu} + G_i^{\mu\nu}\left(\delta G_{i\,\mu\nu}\right)\right) \quad \text{(sum on all indices)} \tag{170}$$

We need to find $\delta G_i^{\mu\nu}$ to evaluate (170). We know $G_i^{\mu\nu}$ from (155) and from that

$$\delta G_i^{\mu\nu} = \partial^\nu\left(\delta W_i^\mu\right) - \partial^\mu\left(\delta W_i^\nu\right) - g e_{ijk}\left(\delta W_j^\nu\right)W_k^\mu - g e_{ijk} W_j^\nu\left(\delta W_k^\mu\right) \tag{171}$$

With (101), (171) becomes

$$\begin{aligned}
\delta G_i^{\mu\nu} &= \partial^\nu\left(-\tfrac{1}{g}\partial^\mu\omega_i - \varepsilon_{ijk}\omega_j W_k^\mu\right) - \partial^\mu\left(-\tfrac{1}{g}\partial^\nu\omega_i - \varepsilon_{ijk}\omega_j W_k^\nu\right) \\
&\qquad - g\varepsilon_{ijk}\left(-\tfrac{1}{g}\partial^\nu\omega_j - \varepsilon_{jmn}\omega_m W_n^\nu\right)W_k^\mu - g\varepsilon_{ijk}W_j^\nu\left(-\tfrac{1}{g}\partial^\mu\omega_k - \varepsilon_{kmn}\omega_m W_n^\mu\right) \\
&= \underbrace{-\tfrac{1}{g}\partial^\nu\partial^\mu\omega_i}_{\text{cancels}} - \varepsilon_{ijk}\partial^\nu\left(\omega_j W_k^\mu\right) \underbrace{+\tfrac{1}{g}\partial^\mu\partial^\nu\omega_i}_{\text{cancels}} + \varepsilon_{ijk}\partial^\mu\left(\omega_j W_k^\nu\right) \\
&\qquad + \varepsilon_{ijk}\left(\partial^\nu\omega_j\right)W_k^\mu + g\varepsilon_{ijk}\varepsilon_{jmn}\omega_m W_n^\nu W_k^\mu + \varepsilon_{ijk}\left(\partial^\mu\omega_k\right)W_j^\nu + g\varepsilon_{ijk}\varepsilon_{kmn}\omega_m W_n^\mu W_j^\nu
\end{aligned} \tag{172}$$

$$\begin{aligned}
&= \underbrace{-\varepsilon_{ijk}\left(\partial^\nu\omega_j\right)W_k^\mu}_{\text{cancels}} \quad - \varepsilon_{ijk}\omega_j\left(\partial^\nu W_k^\mu\right) \quad \underbrace{+\varepsilon_{ijk}\left(\partial^\mu\omega_j\right)W_k^\nu}_{\text{cancels}} \quad + \varepsilon_{ijk}\omega_j\left(\partial^\mu W_k^\nu\right) \\
&\quad \overbrace{+\varepsilon_{ijk}\left(\partial^\nu\omega_j\right)W_k^\mu}^{\text{with}} \quad + g\varepsilon_{ijk}\varepsilon_{jmn}\omega_m W_n^\nu W_k^\mu \quad \underbrace{\overbrace{+\varepsilon_{ijk}\left(\partial^\mu\omega_k\right)W_j^\nu}^{\text{with}}}_{-\varepsilon_{ijk}\left(\partial^\mu\omega_j\right)W_k^\nu} \quad + g\varepsilon_{ijk}\varepsilon_{kmn}\omega_m W_n^\mu W_j^\nu \\
&= \underbrace{-\varepsilon_{ijk}\omega_j\left(\partial^\nu W_k^\mu - \partial^\mu W_k^\nu\right)}_{\boxed{10a}} \; \underbrace{+g\varepsilon_{ijk}\varepsilon_{jmn}\omega_m W_n^\nu W_k^\mu}_{\boxed{10b}} \; \underbrace{+g\varepsilon_{ijk}\varepsilon_{kmn}\omega_m W_n^\mu W_j^\nu}_{\boxed{10c}}.
\end{aligned} \tag{173}$$

So, slightly re-arranged,

$$\delta G_i^{\mu\nu} = \underbrace{-\varepsilon_{ijk}\omega_j\left(\partial^\nu W_k^\mu - \partial^\mu W_k^\nu\right)}_{\boxed{10a}} \; \underbrace{+g\varepsilon_{ijk}\left(\varepsilon_{jmn}\omega_m W_n^\nu\right)W_k^\mu}_{\boxed{10b}} \; \underbrace{+g\varepsilon_{ijk}W_j^\nu\left(\varepsilon_{kmn}\omega_m W_n^\mu\right)}_{\boxed{10c}}. \tag{174}$$

We will re-express the terms labeled $\boxed{10b}$ and $\boxed{10c}$ in (174), but to do this, we first need to deduce a particular mathematical relationship we will employ to do so.

<u>Digression to deduce a needed math relationship</u>

A well know relation between cross products of three 3D vectors **a**, **b**, and **c** is

$$\mathbf{a}\times\left(\mathbf{b}\times\mathbf{c}\right) + \mathbf{b}\times\left(\mathbf{c}\times\mathbf{a}\right) + \mathbf{c}\times\left(\mathbf{a}\times\mathbf{b}\right) = 0, \tag{175}$$

which can be re-arranged as

$$\left(\mathbf{a}\times\mathbf{b}\right)\times\mathbf{c} + \mathbf{b}\times\left(\mathbf{a}\times\mathbf{c}\right) = \mathbf{a}\times\left(\mathbf{b}\times\mathbf{c}\right). \tag{176}$$

Now re-write (176) in terms of vector indices as

$$\varepsilon_{ijk}\left(\varepsilon_{jmn}a_m b_n\right)c_k + \varepsilon_{ijk}b_j\left(\varepsilon_{kmn}a_m c_n\right) = \varepsilon_{ijk}a_j\left(\varepsilon_{kmn}b_m c_n\right). \tag{177}$$

Then, make the substitutions

$$a \to \omega \qquad b \to W^\nu \qquad c \to W^\mu \tag{178}$$

in (177), to get the relationship we seek to use in (174)

$$\underbrace{g\varepsilon_{ijk}\left(\varepsilon_{jmn}\omega_m W_n^\nu\right)W_k^\mu}_{\boxed{10b}} \; \underbrace{+g\varepsilon_{ijk}W_j^\nu\left(\varepsilon_{kmn}\omega_m W_n^\mu\right)}_{\boxed{10c}} = g\varepsilon_{ijk}\omega_j\left(\varepsilon_{kmn}W_m^\nu W_n^\mu\right). \tag{179}$$

<u>End of digression</u>

Thus, from (179), (174) becomes

$$\delta G_i^{\mu\nu} = \underbrace{-\varepsilon_{ijk}\omega_j\left(\partial^\nu W_k^\mu - \partial^\mu W_k^\nu\right)}_{\boxed{10a}} + g\varepsilon_{ijk}\omega_j\left(\varepsilon_{kmn}W_m^\nu W_n^\mu\right) = -\varepsilon_{ijk}\omega_j\left(\partial^\nu W_k^\mu - \partial^\mu W_k^\nu - g\left(\varepsilon_{kmn}W_m^\nu W_n^\mu\right)\right), \quad (180)$$

or from (155),

$$\delta G_i^{\mu\nu} = -\varepsilon_{ijk}\omega_j G_k^{\mu\nu}. \quad (181)$$

Thus, (170) becomes

$$\begin{aligned}\delta \mathcal{L}^B &= -\tfrac{1}{4}\left(\left(-\varepsilon_{ijk}\omega_j G_k^{\mu\nu}\right)G_{i\,\mu\nu} + G_i^{\mu\nu}\left(-\varepsilon_{ijk}\omega_j G_{k\,\mu\nu}\right)\right) = \tfrac{1}{4}\left(\left(\varepsilon_{ijk}\omega_j G_k^{\mu\nu}\right)G_{i\,\mu\nu} + G_{i\,\mu\nu}\left(\varepsilon_{ijk}\omega_j G_k^{\mu\nu}\right)\right)\\ &= \tfrac{1}{2}\varepsilon_{ijk}\omega_j G_k^{\mu\nu}G_{i\,\mu\nu} = \tfrac{1}{2}\omega_j\varepsilon_{jki}G_k^{\mu\nu}G_{i\,\mu\nu} = \tfrac{1}{2}\sum_{i,k=1}^{3}\sum_{j=1}^{3}\omega_j\left(G_k^{\mu\nu}G_{i\,\mu\nu} - G_i^{\mu\nu}G_{k\,\mu\nu}\right)\\ &= \tfrac{1}{2}\sum_{i,k=1}^{3}\sum_{j=1}^{3}\omega_j\left(G_k^{\mu\nu}G_{i\,\mu\nu} - G_{i\,\mu\nu}G_k^{\mu\nu}\right) = 0.\end{aligned} \quad (182)$$

<u>Exercise</u>

In a manner similar to what we did in the digression above, derive the result of (182) starting with the 3D vector identity $\mathbf{a}\cdot\left(\mathbf{b}\times\mathbf{a}\right) = 0$ expressed in index notation. Answer is in Appendix A, pg. 43.

<u>Result of this section:</u> We have shown $\mathcal{L}^B$ is invariant under the transformation of Section 5.4.3.

10.2.3 <u>For $\mathcal{L}^H$</u>

Re-stating (156) here for convenience,

$$\begin{aligned}\mathcal{L}^H &= \left(D^\mu\Phi\right)^\dagger\left(D_\mu\Phi\right) - \mu^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2\\ D_\mu\Phi &= \left(\partial_\mu + \tfrac{i}{2}g\tau_k W_{k\,\mu} + i\tfrac{g'}{2}B_\mu\right)\Phi\end{aligned} \quad (156)$$

We first show the invariance of $\Phi^\dagger\Phi$.

$$\left(\Phi^\dagger\Phi\right)' \approx \left(\Phi^\dagger + \delta\Phi^\dagger\right)\left(\Phi + \delta\Phi\right) \approx \Phi^\dagger\Phi + \left(\delta\Phi^\dagger\right)\Phi + \Phi^\dagger\left(\delta\Phi\right), \quad (183)$$

So,

$$\delta\left(\Phi^\dagger\Phi\right) \approx \left(\delta\Phi^\dagger\right)\Phi + \Phi^\dagger\left(\delta\Phi\right). \quad (184)$$

From (102),

$$\begin{aligned}\delta\left(\Phi^\dagger\Phi\right) &\approx \left(-\tfrac{i}{2}\omega_i\Phi^\dagger\tau_i^\dagger - \tfrac{i}{2}f\Phi^\dagger\right)\Phi + \Phi^\dagger\left(\tfrac{i}{2}\omega_i\tau_i\Phi + \tfrac{i}{2}f\Phi\right) \qquad \left(\text{where } \tau_i^\dagger = \tau_i\right)\\ &= -\tfrac{i}{2}\omega_i\Phi^\dagger\tau_i\,\Phi - \tfrac{i}{2}f\Phi^\dagger\Phi + \tfrac{i}{2}\omega_i\Phi^\dagger\tau_i\Phi + \tfrac{i}{2}f\Phi^\dagger\Phi = 0,\end{aligned} \quad (185)$$

and $\Phi^\dagger\Phi$ is unchanged under the transformation, so we can ignore the last two terms in (156). Thus,

$$\delta\mathcal{L}^H = \delta\left(\left(D^\mu\Phi\right)^\dagger\left(D_\mu\Phi\right)\right) = \left(\delta\left(D^\mu\Phi\right)^\dagger\right)\left(D_\mu\Phi\right) + \left(D^\mu\Phi\right)^\dagger\left(\delta\left(D_\mu\Phi\right)\right). \quad (186)$$

We need to show (186) is zero, and I attempt that below. However, I cannot get the math to work out in the time I have available, so I warn the reader ahead of time. I invite interested readers to try and find where I went wrong and notify me via the feedback link on my book website shown on pg. 1.

As I attempt to shown (186) is zero, I start by finding $\delta\left(D_\mu\Phi\right)$ via the following.

$$
\begin{aligned}
\left(D_\mu\Phi\right)' &= \left(\partial_\mu\Phi + \tfrac{i}{2}g\tau_k W_{k\mu}\Phi + \tfrac{i}{2}g'B_\mu\Phi\right)' \\
&= \partial_\mu\left(\Phi+\delta\Phi\right) + \tfrac{i}{2}g\tau_k\left(W_{k\mu}+\delta W_{k\mu}\right)\left(\Phi+\delta\Phi\right) + \tfrac{i}{2}g\tau_k\left(W_{k\mu}+\delta W_{k\mu}\right)\left(\Phi+\delta\Phi\right) + i\tfrac{g'}{2}\left(B_\mu+\delta B_\mu\right)\left(\Phi+\delta\Phi\right) \\
&= \underbrace{\partial_\mu\Phi}_{\boxed{11}} + \underbrace{\partial_\mu\left(\delta\Phi\right)}_{\boxed{12}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\Phi}_{\boxed{13}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\left(\delta\Phi\right)}_{\boxed{14}} + \underbrace{\tfrac{i}{2}g\tau_k\left(\delta W_{k\mu}\right)\Phi}_{\boxed{15}} + \underbrace{\tfrac{i}{2}g\tau_k\left(\delta W_{k\mu}\right)\left(\delta\Phi\right)}_{\approx 0} \\
&\quad + \underbrace{\tfrac{i}{2}g'B_\mu\Phi}_{\boxed{16}} + \underbrace{\tfrac{i}{2}g'B_\mu\left(\delta\Phi\right)}_{\boxed{17}} + \underbrace{\tfrac{i}{2}g'\left(\delta B_\mu\right)\Phi}_{\boxed{18}} + \underbrace{\tfrac{i}{2}g'\left(\delta B_\mu\right)\left(\delta\Phi\right)}_{\approx 0}.
\end{aligned}
\tag{187}
$$

Re-arranging, and using, (102), we have

$$
\begin{aligned}
\left(D_\mu\Phi\right)' &= \overbrace{\underbrace{\partial_\mu\Phi}_{\boxed{11}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\Phi}_{\boxed{13}} + \underbrace{\tfrac{i}{2}g'B_\mu\Phi}_{\boxed{16}}}^{D_\mu\Phi} + \underbrace{\partial_\mu\left(\delta\Phi\right)}_{\boxed{12}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\left(\delta\Phi\right)}_{\boxed{14}} + \underbrace{\tfrac{i}{2}g'B_\mu\left(\delta\Phi\right)}_{\boxed{17}} \\
&\quad + \underbrace{\tfrac{i}{2}g\tau_k\left(\delta W_{k\mu}\right)\Phi}_{\boxed{15}} + \underbrace{\tfrac{i}{2}g'\left(\delta B_\mu\right)\Phi}_{\boxed{18}}
\end{aligned}
\tag{188}
$$

$$
\begin{aligned}
&= D_\mu\Phi + \underbrace{\partial_\mu\left(\tfrac{i}{2}\omega_i\tau_i\Phi + \tfrac{i}{2}f\Phi\right)}_{\boxed{12}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\left(\tfrac{i}{2}\omega_i\tau_i\Phi + \tfrac{i}{2}f\Phi\right)}_{\boxed{14}} + \underbrace{\tfrac{i}{2}g'B_\mu\left(\tfrac{i}{2}\omega_i\tau_i\Phi + \tfrac{i}{2}f\Phi\right)}_{\boxed{17}} \\
&\quad + \underbrace{\tfrac{i}{2}g\tau_k\left(-\tfrac{1}{g}\partial^\mu\omega_k - \varepsilon_{kmn}\omega_m W_n^\mu\right)\Phi}_{\boxed{15}} + \underbrace{\tfrac{i}{2}g'\left(-\tfrac{1}{g'}\partial^\mu f\right)\Phi}_{\boxed{18}}
\end{aligned}
$$

$$
\begin{aligned}
&= D_\mu\Phi + \underbrace{\partial_\mu\tfrac{i}{2}\omega_i\tau_i\Phi}_{\boxed{12a}} + \underbrace{\partial_\mu\left(\tfrac{i}{2}f\Phi\right)}_{\boxed{12b}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\left(\tfrac{i}{2}\omega_i\tau_i\Phi\right)}_{\boxed{14a}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\left(\tfrac{i}{2}f\Phi\right)}_{\boxed{14b}} \\
&\quad + \underbrace{\tfrac{i}{2}g'B_\mu\left(\tfrac{i}{2}\omega_i\tau_i\Phi\right)}_{\boxed{17a}} + \underbrace{\tfrac{i}{2}g'B_\mu\left(\tfrac{i}{2}f\Phi\right)}_{\boxed{17b}} \\
&\quad + \underbrace{\tfrac{i}{2}g\tau_k\left(-\tfrac{1}{g}\partial^\mu\omega_k\right)\Phi}_{\boxed{15a}} + \underbrace{\tfrac{i}{2}g\tau_k\left(-\varepsilon_{kmn}\omega_m W_n^\mu\right)\Phi}_{\boxed{15b}} + \underbrace{\tfrac{i}{2}g'\left(-\tfrac{1}{g'}\partial^\mu f\right)\Phi}_{\boxed{18}}
\end{aligned}
\tag{189}
$$

$$
\begin{aligned}
&= D_\mu\Phi + \underbrace{\tfrac{i}{2}\tau_k\left(\partial^\mu\omega_k\right)\Phi}_{\substack{\boxed{12a-1} \\ \text{cancels } \boxed{15a}}} + \underbrace{\tfrac{i}{2}\omega_i\tau_i\left(\partial_\mu\Phi\right)}_{\boxed{12a-2}} + \underbrace{\tfrac{i}{2}\left(\partial_\mu f\right)\Phi}_{\substack{\boxed{12b-1} \\ \text{cancels } \boxed{18}}} + \underbrace{\tfrac{i}{2}f\left(\partial_\mu\Phi\right)}_{\boxed{12b-1}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\left(\tfrac{i}{2}\omega_i\tau_i\Phi\right)}_{\boxed{14a}} + \underbrace{\tfrac{i}{2}g\tau_k W_{k\mu}\left(\tfrac{i}{2}f\Phi\right)}_{\boxed{14b}} \\
&\quad + \underbrace{\tfrac{i}{2}g'B_\mu\left(\tfrac{i}{2}\omega_i\tau_i\Phi\right)}_{\boxed{17a}} + \underbrace{\tfrac{i}{2}g'B_\mu\left(\tfrac{i}{2}f\Phi\right)}_{\boxed{17b}} \\
&\quad - \underbrace{\tfrac{i}{2}\tau_k\left(\partial^\mu\omega_k\right)\Phi}_{\substack{\boxed{15a} \\ \text{cancels } \boxed{12a-1}}} + \underbrace{\tfrac{i}{2}g\tau_k\left(-\varepsilon_{kmn}\omega_m W_n^\mu\right)\Phi}_{\boxed{15b}} - \underbrace{\tfrac{i}{2}\left(\partial^\mu f\right)\Phi}_{\substack{\boxed{18} \\ \text{cancels } \boxed{12b-1}}}
\end{aligned}
\tag{190}
$$

$$
\begin{aligned}
&= D_\mu\Phi + \underbrace{\tfrac{i}{2}\omega_i\tau_i\left(\partial_\mu\Phi\right)}_{\boxed{12a-2}} + \underbrace{\tfrac{i}{2}\omega_i\tau_i\left(\tfrac{i}{2}g\tau_k W_{k\mu}\Phi\right)}_{\boxed{14a}} + \underbrace{\tfrac{i}{2}\omega_i\tau_i\left(\tfrac{i}{2}g'B_\mu\Phi\right)}_{\boxed{17a}} \\
&\quad + \underbrace{\tfrac{i}{2}f\left(\partial_\mu\Phi\right)}_{\boxed{12b-1}} + \underbrace{\tfrac{i}{2}f\left(\tfrac{i}{2}g\tau_k W_{k\mu}\Phi\right)}_{\boxed{14b}} + \underbrace{\tfrac{i}{2}f\left(\tfrac{i}{2}g'B_\mu\Phi\right)}_{\boxed{17b}} + \underbrace{\tfrac{i}{2}g\tau_k\left(-\varepsilon_{kmn}\omega_m W_n^\mu\right)\Phi}_{\boxed{15b}}
\end{aligned}
\tag{191}
$$

$$
= D_\mu\Phi + \tfrac{i}{2}\omega_i\tau_i\left(D_\mu\Phi\right) + \tfrac{i}{2}f\left(D_\mu\Phi\right) + \underbrace{\tfrac{i}{2}g\tau_k\left(-\varepsilon_{kmn}\omega_m W_n^\mu\right)\Phi}_{\boxed{15b}}
$$

$$\left(D_\mu\Phi\right)' = D_\mu\Phi + \overbrace{\tfrac{i}{2}\omega_i\tau_i\left(D_\mu\Phi\right) + \tfrac{i}{2}f\left(D_\mu\Phi\right) + \underbrace{\tfrac{i}{2}g\tau_k\left(-\varepsilon_{kmn}\omega_m W_n^\mu\right)\Phi}_{\boxed{15b}}}^{\delta\left(D_\mu\Phi\right)}. \tag{192}$$

Hence, where $\tau_i = \tau_i^\dagger$,

$$\delta\left(D^\mu\Phi\right)^\dagger = -\left(D^\mu\Phi\right)^\dagger \tfrac{i}{2}\omega_i\tau_i - \tfrac{i}{2}f\left(D^\mu\Phi\right)^\dagger + \tfrac{i}{2}g\Phi^\dagger\tau_k\left(\varepsilon_{kmn}\omega_m W_n^\mu\right) \tag{193}$$

So, (186) becomes

$$\begin{aligned}\delta\mathcal{L}^H &= \left(-\left(D^\mu\Phi\right)^\dagger \tfrac{i}{2}\omega_i\tau_i - \tfrac{i}{2}f\left(D^\mu\Phi\right)^\dagger + \tfrac{i}{2}g\Phi^\dagger\tau_k\left(\varepsilon_{kmn}\omega_m W_n^\mu\right)\right)\left(D_\mu\Phi\right)\\ &\quad + \left(D^\mu\Phi\right)^\dagger\left(\tfrac{i}{2}\omega_i\tau_i\left(D_\mu\Phi\right) + \tfrac{i}{2}f\left(D_\mu\Phi\right) - \tfrac{i}{2}g\tau_k\left(\varepsilon_{kmn}\omega_m W_{n\,\mu}\right)\Phi\right)\\ &= -\left(D^\mu\Phi\right)^\dagger \tfrac{i}{2}\omega_i\tau_i\left(D_\mu\Phi\right) - \tfrac{i}{2}f\left(D^\mu\Phi\right)^\dagger\left(D_\mu\Phi\right) + \tfrac{i}{2}g\Phi^\dagger\tau_k\left(\varepsilon_{kmn}\omega_m W_n^\mu\right)\left(D_\mu\Phi\right)\\ &\quad + \left(D^\mu\Phi\right)^\dagger \tfrac{i}{2}\omega_i\tau_i\left(D_\mu\Phi\right) + \tfrac{i}{2}f\left(D^\mu\Phi\right)^\dagger\left(D_\mu\Phi\right) - \tfrac{i}{2}g\left(D^\mu\Phi\right)^\dagger\tau_k\left(\varepsilon_{kmn}\omega_m W_{n\,\mu}\right)\Phi.\end{aligned} \tag{194}$$

The first term in the next to last row of (194) cancels with the first term in the second row. The second terms in each of those rows also cancel and leave us with

$$\delta\mathcal{L}^H = \tfrac{i}{2}g\left(\varepsilon_{kmn}\omega_m W_n^\mu\right)\left(\Phi^\dagger\tau_k\left(D_\mu\Phi\right) - \left(D_\mu\Phi\right)^\dagger\tau_k\Phi\right). \tag{195}$$

With (73), this becomes

$$\delta\mathcal{L}^H = \tfrac{i}{2}g\left(\varepsilon_{kmn}\omega_m W_n^\mu\right)\begin{pmatrix}\Phi^\dagger\tau_k\left(\partial_\mu\Phi + \tfrac{i}{2}g\tau_r W_{r\,\mu}\Phi + \tfrac{i}{2}g'B_\mu\Phi\right)\\ -\left(\partial_\mu\Phi^\dagger - \tfrac{i}{2}g\Phi^\dagger\tau_r W_{r\,\mu} - \tfrac{i}{2}g'B_\mu\Phi^\dagger\right)\tau_k\Phi\end{pmatrix} \tag{196}$$

I'm sorry, but I'm drawing a blank on how this could be equal to zero. Perhaps I made a sign mistake somewhere, but I don't have time right now to try to determine why this does not vanish. I've decided to post this anyway in hopes that one of you readers can show me how to do this correctly so $\delta\mathcal{L}^H = 0$, as it must.

Result we would like to get from this section: We hoped to have shown $\mathcal{L}^H$ is invariant under the transformation of Section 5.4.3.

10.2.4 For $\mathcal{L}^{LH}$

We repeat (157) below for convenience.

$$\mathcal{L}^{LH} = -g_l\left(\bar{\Psi}_l^L\psi_l^R\Phi + \Phi^\dagger\bar{\psi}_l^R\Psi_l^L\right) - g_{\nu_l}\left(\bar{\Psi}_{\nu_l}^L\psi_{\nu_l}^R\tilde{\Phi} + \tilde{\Phi}^\dagger\bar{\psi}_{\nu_l}^R\Psi_{\nu_l}^L\right) \tag{157}$$

$$\delta\mathcal{L}^{LH} = -g_l\,\delta\left(\bar{\Psi}_l^L\psi_l^R\Phi + \Phi^\dagger\bar{\psi}_l^R\Psi_l^L\right)\underbrace{-g_{\nu_l}\delta\left(\bar{\Psi}_{\nu_l}^L\psi_{\nu_l}^R\tilde{\Phi} + \tilde{\Phi}^\dagger\bar{\psi}_{\nu_l}^R\Psi_{\nu_l}^L\right)}_{\text{ignore for now}} \tag{197}$$

$$\begin{aligned}\delta\left(\bar{\Psi}_l^L\psi_l^R\Phi + \Phi^\dagger\bar{\psi}_l^R\Psi_l^L\right) &= \begin{pmatrix}\left(\delta\bar{\Psi}_l^L\right)\psi_l^R\Phi + \bar{\Psi}_l^L\left(\delta\psi_l^R\right)\Phi + \bar{\Psi}_l^L\psi_l^R\left(\delta\Phi\right)\\ +\left(\delta\Phi^\dagger\right)\bar{\psi}_l^R\Psi_l^L + \Phi^\dagger\left(\delta\bar{\psi}_l^R\right)\Psi_l^L + \Phi^\dagger\bar{\psi}_l^R\left(\delta\Psi_l^L\right)\end{pmatrix}\\ &= \begin{pmatrix}\left(-\tfrac{i}{2}\omega_i\bar{\Psi}_l^L\tau_i + \tfrac{i}{2}f\bar{\Psi}_l^L\right)\psi_l^R\Phi + \bar{\Psi}_l^L\left(-if\psi_l^R\right)\Phi + \bar{\Psi}_l^L\psi_l^R\left(\tfrac{i}{2}\omega_i\tau_i\Phi + \tfrac{i}{2}f\Phi\right)\\ +\left(-\tfrac{i}{2}\omega_i\Phi^\dagger\tau_i - \tfrac{i}{2}f\Phi^\dagger\right)\bar{\psi}_l^R\Psi_l^L + \Phi^\dagger\left(if\bar{\psi}_l^R\right)\Psi_l^L + \Phi^\dagger\bar{\psi}_l^R\left(\tfrac{i}{2}\omega_i\tau_i\Psi_l^L - \tfrac{i}{2}f\Psi_l^L\right)\end{pmatrix}\end{aligned} \tag{198}$$

Writing out (198), where the underbrackets in (199) reflect the fact that a singlet is not a two component object in SU(2) space like a doublet, so it commutes (like any scalar in Euclidean space) with the 2X2 Pauli matrices.

$$\begin{aligned}&= -\tfrac{i}{2}\omega_i \bar{\Psi}_l^L \tau_i \psi_l^R \Phi + \tfrac{i}{2} f \bar{\Psi}_l^L \psi_l^R \Phi - if \bar{\Psi}_l^L \psi_l^R \Phi + \underbrace{\tfrac{i}{2}\omega_i \bar{\Psi}_l^L \psi_l^R \tau_i \Phi}_{\frac{i}{2}\omega_i \bar{\Psi}_l^L \tau_i \psi_l^R \Phi} + \tfrac{i}{2} f \bar{\Psi}_l^L \psi_l^R \Phi \\ &\quad - \tfrac{i}{2}\omega_i \Phi^\dagger \tau_i \bar{\psi}_l^R \Psi_l^L - \tfrac{i}{2} f \Phi^\dagger \bar{\psi}_l^R \Psi_l^L + if \Phi^\dagger \bar{\psi}_l^R \Psi_l^L + \underbrace{\tfrac{i}{2}\omega_i \Phi^\dagger \bar{\psi}_l^R \tau_i \Psi_l^L}_{\frac{i}{2}\omega_i \Phi^\dagger \tau_i \bar{\psi}_l^R \Psi_l^L} - \tfrac{i}{2} f \Phi^\dagger \bar{\psi}_l^R \Psi_l^L \;.\end{aligned} \tag{199}$$

In the first row of (199), the first and fourth terms cancel. So do the 2nd, 3rd, and last terms. The same thing is true for the second row. So, in (197),

$$\delta\left(\bar{\Psi}_l^L \psi_l^R \Phi + \Phi^\dagger \bar{\psi}_l^R \Psi_l^L\right) = 0 \tag{200}$$

As an exercise, you can show the part of (197) we ignored above is also zero. (Answer in Appendix A, pg. 43.)

<u>Result of this section:</u> $\mathcal{L}^{LH}$ is invariant under the transformation of Section 5.4.3.

10.2.5 Final Conclusion

We have shown (except for what at this time is a missing part of the derivation for $\mathcal{L}^{H}$) that the high energy Lagrangian $\mathcal{L}$ is symmetric under the transformation of Section 5.4.3.

11 Appendix C. Transforming the Lagrangian to the True Vacuum

To come someday.

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