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Derivation of the Feynman Propagator

From Chapter 3 of Student Guide to Quantum Field Theory*, by Robert D. Klauber* ©

3.0 The Scalar Feynman Propagator

The Feynman propagator, the mathematical formulation representing a virtual particle, such as the one represented by the wavy line in Fig. 1-1 of Chap. 1, is the toughest thing, in my opinion, to learn and feel comfortable with in QFT. If you don't feel comfortable with it right away, don't worry about it. That is how virtually everyone feels. Over time, it will become more familiar, and if you are lucky and work hard, maybe even easy.

Feynman propagator not simple to understand

Use wholeness chart as you study the derivation

I have tried to take the derivation of the propagator one step at a time, and emphasize what each step entails. Wholeness Chart 5-0X (also at Free Fields Wholeness Chart link at www.quantumfieldtheory.info) breaks these steps out clearly, and should be used as an aid when studying the propagator derivation.

Propagators: NRQM vs QFT and Real vs Virtual Particles

Note that the propagator for real particles, which you may have studied in NRQM, is *not* the same as the Feynman propagator, which is explicitly for virtual particles in QFT. It may be confusing, but the Feynman propagator is often simply called, "the propagator". You will have to get used to discerning the difference from context.

Feynman propagator for QFT virtual particles is different from propagator for real particles of NRQM & RQM

In QFT, as we will see when we study interactions, a propagator for real particles is not generally needed, and we will not derive one here.

3.0.1 The Approach

The first part of QFT is a free particle theory (no interactions, as in this chapter and the next three). After this, interactions are introduced. In the course of deriving the interaction theory, a mathematical relationship arises that is called the Feynman propagator. Physically, it can be visualized as representing a virtual particle that exists fleetingly and carries energy, momentum, and in some cases, charge from one real particle to another. Thus, it is the carrier, or mediator, of force (interaction.) See the virtual photon of Fig. 1-1 in Chap. 1.

We'll use the Feynman propagator when we get to interaction theory

It will help us pedagogically to derive the Feynman propagator now, rather than when we get to interactions. The derivation of interaction theory is fairly complicated and it will be easier, as we develop it, if we already know the mathematical relation for the Feynman propagator, rather than diverting our attention for several pages to derive it then.

But it's easier in the long run if we derive it here

Heuristically, it may help to consider the virtual particle as created at a particular spacetime point and destroyed at a later spacetime point, and this is how Feynman diagrams portray it. From this (heuristic) perspective the operator field $\phi^\dagger(y)$ can be considered to create a virtual scalar particle at event y (we used the symbol x_2 in Fig. 1-1), and the field operator $\phi(x)$ destroys that virtual particle at event x (x_1 in Fig. 1-1.) The scalar propagator incorporates these two field operators in a sort of "short-hand" way.

Note that the above "creation/destruction at a point" perspective can help initially in understanding the derivation of the propagator, but we caution that it will have to be modified and refined. We will save that to the end when, after digesting the derivation to follow, this modification will be easier to understand.

We will now derive a relationship for the propagator using the field operators acting on the vacuum, and will later see (Chap. 7? XXX) that this derived relationship arises naturally in the full mathematical development of the interaction theory.

3.0.2 Milestones in the Derivation

We start with the coefficient commutation relations and proceed in five distinct steps. The entire derivation is for continuous (not discrete) eigenstate solutions of the field equation (Klein-Gordon here), since the propagator represents a virtual particle in the vacuum and the vacuum is not confined to a volume V. We represent the scalar Feynman propagator with the symbol $\Delta_F(x-y)$.

Start with the coefficient commutators and follow 5 distinct steps

As noted, our starting point is the coefficient commutators for continuous solutions,

Use continuous solutions to field equation

$$\left[a(\mathbf{k}),a^\dagger(\mathbf{k}')\right]=\left[b(\mathbf{k}),b^\dagger(\mathbf{k}')\right]=\delta(\mathbf{k}-\mathbf{k}') \quad \text{(continuous)}, \tag{3-1}$$

and from there we follow five steps. The first two are purely mathematical, and will serve as background that will help us with the remaining steps. The third step comprises a physical interpretation of the propagator, and the last two steps are mathematical manipulations to get that interpretation into convenient form (the form that is used in QFT interaction analysis.)

Overview of the 5 steps

Step 1: Use (3-1) to find commutation relations for positive and negative frequency solutions (see (3-37) in Chap. 3), i.e., determine

$$\left[\phi^{\pm}(x),\phi^{\dagger\mp}(y)\right]=i\Delta^{\pm}(x-y), \tag{3-2}$$

where $i\Delta^{\pm}(x-y)$ is the symbol representing the two commutators on the LHS of (3-2). $i\Delta^+$ represents the upper + and – signs on the LHS; $i\Delta^-$ represents the lower – and + signs. $\Delta^{\pm}$ is slightly different from, and will be used to find, the Feynman propagator Δ_F.

Step 2: Express $i\Delta^{\pm}(x-y)$ in terms of a contour integral in the complex plane.

Step 3: Express the Feynman propagator Δ_F as a mathematical representation of a particle or antiparticle created at one point in space and time in the vacuum and destroyed at another place and time.

Step 4: Use $i\Delta^{\pm}(x-y)$ of Step 2 to express Δ_F as a contour integral in the complex plane.

Step 5: Re-express Δ_F as an integral in real (rather than complex) space. This is the form most suitable for analysis.

3.0.3 The Derivation

Step 1: Commutation Relations for Positive/Negative Frequency Solutions

Define the symbol $i\Delta^+$ as the commutator of the field type a solutions, i.e.,

$$i\Delta^+(x-y)=\left[\phi^+(x),\phi^{\dagger-}(y)\right], \tag{3-3}$$

Step 1, first part, find $i\Delta^+$ = commutation relation for type a fields

where the solutions used on the RHS are the integral (continuous) form for the Klein-Gordon solutions (see (3-37) in Chap. 3). It is common usage to use a + sign to designate (3-3), rather than the letter a, which would be easier to remember. Just think "a type field" when you see +. Equation (3-3) is thus

$$\begin{aligned} i\Delta^+(x-y)&=\frac{1}{2(2\pi)^3}\iint\left[a(\mathbf{k}),a^\dagger(\mathbf{k}')\right]\frac{e^{-ikx}e^{ik'y}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}d^3\mathbf{k}\,d^3\mathbf{k}' \\ &=\frac{1}{2(2\pi)^3}\int\left(\int\frac{e^{ik'y}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\delta(\mathbf{k}-\mathbf{k}')\,d^3\mathbf{k}'\right)e^{-ikx}d^3\mathbf{k}, \end{aligned} \tag{3-4}$$

and hence,

$$i\Delta^{+}\left(x-y\right)=\frac{1}{2(2\pi)^{3}}\int\frac{e^{-ik(x-y)}}{\omega_{\mathbf{k}}}d^{3}\mathbf{k}\,. \tag{3-5}$$

Similarly, where a minus sign stands for b type fields (since they are associated with antiparticles, the minus makes some sense),

Step 1, second part, find $i\Delta^{-}$ = commutation relation for type b fields

$$\begin{aligned}i\Delta^{-}\left(x-y\right)&=\left[\phi^{-}\left(x\right),\phi^{\dagger+}\left(y\right)\right]=\frac{1}{2(2\pi)^{3}}\iint\left[b^{\dagger}\left(\mathbf{k}\right),b\left(\mathbf{k}'\right)\right]\frac{e^{ikx}e^{-ik'y}}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}d^{3}\mathbf{k}\,d^{3}\mathbf{k}'\\&=\frac{-1}{2(2\pi)^{3}}\int\frac{e^{ik(x-y)}}{\omega_{\mathbf{k}}}d^{3}\mathbf{k}\,,\end{aligned} \tag{3-6}$$

which leads to

$$i\Delta^{\pm}\left(x-y\right)=\left[\phi^{\pm}\left(x\right),\phi^{\dagger\mp}\left(y\right)\right]=\frac{\pm 1}{2\left(2\pi\right)^{3}}\int\frac{e^{\mp ik(x-y)}}{\omega_{\mathbf{k}}}d^{3}\mathbf{k} \tag{3-7}$$

Step 1, final part, combine above parts into one symbol $i\Delta^{\pm}$

Step 2: Expressing the Two $i\Delta^{\pm}$ as Contour Integrals

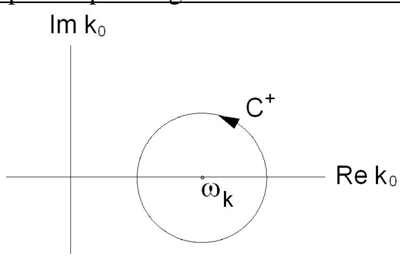

Figure 3-3. Contour Integral for Real, Positive Frequency

Consider the complex plane for a function f of the complex variable k_0, i.e., $f(k_0)$. Here, the symbol k_0 is not a pole (poles are usually designated with null subscript), but represents a complex number generalization of the zeroth component (the energy) of 4-momentum k. We concern ourselves with the particular case where k_0 takes on the real value $\omega_{\mathbf{k}}$.

Review of integral in the complex plane

From complex variable theory,

$$f\left(\omega_{\mathbf{k}}\right)=\frac{1}{i2\pi}\int_{C^{+}}\frac{f\left(k_{0}\right)}{k_{0}-\omega_{\mathbf{k}}}dk_{0}\,. \tag{3-8}$$

Now, re-express (3-5) as

$$i\Delta^{+}\left(x-y\right)=\frac{1}{\left(2\pi\right)^{3}}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\underbrace{\left\{\frac{e^{-i\omega_{\mathbf{k}}\left(t_{x}-t_{y}\right)}}{2\omega_{\mathbf{k}}}\right\}}_{f\left(\omega_{\mathbf{k}}\right)}d^{3}\mathbf{k}\,, \tag{3-9}$$

Step 2, first part, express $i\Delta^{+}$ as a contour integral

where we take the bracketed quantity as equal to $f(\omega_{\mathbf{k}})$, and where

$$f\left(k_{0}\right)=\frac{e^{-ik_{0}\left(t_{x}-t_{y}\right)}}{k_{0}+\omega_{\mathbf{k}}}\,. \tag{3-10}$$

We can then use (3-10) in (3-8) to re-express $f(\omega_{\mathbf{k}})$ in terms of a contour integral. Using this for the bracket in (3-9), we find (3-9) becomes

$$
\begin{aligned}
i\Delta^{+}(x-y) &= \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left\{\frac{1}{i2\pi}\int_{C^+}\frac{f(k_0)}{k_0-\omega_{\mathbf{k}}}dk_0\right\}d^3\mathbf{k} \\
&= \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\left\{\frac{1}{i2\pi}\int_{C^+}\frac{e^{-ik_0(t_x-t_y)}}{(k_0-\omega_{\mathbf{k}})(k_0+\omega_{\mathbf{k}})}dk_0\right\}d^3\mathbf{k} \qquad (3\text{-}11)\\
&= \frac{-i}{(2\pi)^4}\int_{C^+}\frac{e^{-ik(x-y)}}{(k_0)^2-(\omega_{\mathbf{k}})^2}d^4k\,.
\end{aligned}
$$

where the integral notation now implies integration over four dimensions of the 4-momentum, with the 3-momentum part from $-\infty$ to $+\infty$ in real space and the energy part a contour integral in complex space. Note that the integral does not "blow up" because $k_0 \neq \omega_{\mathbf{k}}$ over the contour integral. We are using a mathematical trick that works, though it jars our usual understanding that, for real particles, the zeroth component of 4-momentum equals energy. k_0 has at this point become, for us, a variable that generally does not equal energy $\omega_{\mathbf{k}}$.

We modify (3-11) a little by noting what is always true mathematically for any four vector, and thus true for 4-momentum components,

Modifying terms in our result a little

$$k^2 = (k_0)^2 - (\mathbf{k})^2 \quad \rightarrow \quad (k_0)^2 = k^2 + (\mathbf{k})^2 \qquad (3\text{-}12)$$

and what is physically true for rest mass, energy, and 3-momentum,

$$\omega_{\mathbf{k}}^2 - (\mathbf{k})^2 = \mu^2 \quad \rightarrow \quad \omega_{\mathbf{k}}^2 = \mu^2 + (\mathbf{k})^2\,. \qquad (3\text{-}13)$$

Substitute the RH expressions of (3-12) and (3-13) into the last line of (3-11) to get

$$i\Delta^{+}(x-y) = \frac{-i}{(2\pi)^4}\int_{C^+}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k\,. \qquad (3\text{-}14)$$

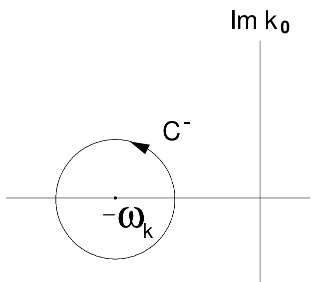

Figure 3-4. Contour Integral for Real, Negative Frequency

For $i\Delta^{-}(x-y)$, we carry out similar steps except that the contour integral (still c.c.w.) is now about $-\omega_{\mathbf{k}}$. When all is said and done, we find the only difference from (3-14) to be the contour, which is now about the negative frequency value and designed by C^{-}.

Step 2, second part, express $i\Delta^{-}$ as a contour integral

$$i\Delta^{-}(x-y) = \frac{-i}{(2\pi)^4}\int_{C^-}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k\,. \qquad (3\text{-}15)$$

Step 3: The Feynman Propagator as the VEV of a Time Order Operator

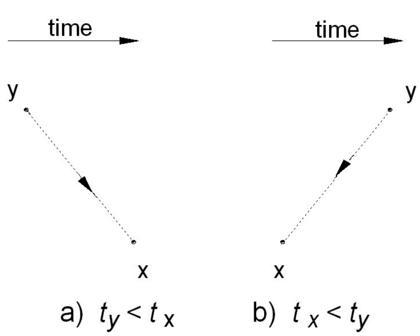

Figure 3-5. Creation & Destruction of Virtual Particle/Antiparticle

Fig 3-5a represents creation of a particle, which will be virtual, at y and destruction of it at x. Figure 3-5b represents creation of an antiparticle at x and destruction of it at y. Virtual particles are never detected when real particles interact, so the same effect on the real particles could be realized by either of the processes in Figure 3. For example, a virtual particle carrying charge from y to x would represent the same charge exchanges as an antiparticle carrying opposite

Step 3, first part, defining the time ordered operator T and seeing how it represents creation of either a virtual particle or antiparticle followed by its destruction charge from x to y. Thus, we need a relationship for the propagator that includes both scenarios as possibilities.

That is, we need an operator that will create a particle first if $t_y < t_x$, but create an antiparticle first if $t_x < t_y$. Our Klein-Gordon solutions of Chap. 3, (3-37), (3-84), and (3-85), provide the means for the desired creation and destruction operations. But these have to be arranged to provide us with the time ordering dependence of Fig. 3-5. To this end, consider the time ordered operator T, defined as follows.

If $t_y < t_x$, $\phi^\dagger(y)$ operates first (creates a particle) and is placed on the right, with $\phi(x)$ operating second (destroys the particle) and placed on the left.

$$\text{for} \quad t_y < t_x \qquad T\left\{\phi(x)\phi^\dagger(y)\right\} = \phi(x)\phi^\dagger(y)\,. \tag{3-16}$$

If $t_x < t_y$, $\phi(x)$ operates first (creates an antiparticle) and is placed on the right with $\phi^\dagger(y)$ operating second (destroying the antiparticle) and placed on the left.

$$\text{for} \quad t_x < t_y \qquad T\left\{\phi(x)\phi^\dagger(y)\right\} = \phi^\dagger(y)\phi(x)\,. \tag{3-17}$$

Step 3, second part, defining the transition amplitude density as equal to the VEV of the time ordered operator T

We now define what is called the transition amplitude density, which equals the vacuum expectation value (VEV) of the above time ordered operator. It is an amplitude, similar to the amplitude of a wave function in NRQM, because, as we will shortly see, the square of its magnitude equals the probability density of it being observed. (As the square of the magnitude of the amplitude for a component of the wave function equals the probability of it being observed.)

This transition amplitude density is

$$\left\langle 0\right| T\left\{\phi(x)\phi^\dagger(y)\right\}\left|0\right\rangle, \tag{3-18}$$

We use the VEV because we will be interested in the expectation of finding a virtual particle traveling in the vacuum

and this represents both possible scenarios of Fig. 3-5. In wave mechanics, the bracket above is an integration over all space. This is still true, but note carefully that the integration variable is over the space variable of the bra and ket (think $\mathbf{x}'$), but not the time ordered variables $\mathbf{x}$ and $\mathbf{y}$. In QFT notation, we tend to merely think of a bracket as equaling zero unless the bra and ket represent the same state.

Gaining insight into the time ordered operator T acting on the vacuum

To gain insight into (3-18), consider the transition amplitude density operating on the vacuum when a virtual particle is propagated. Then, where an overbar in a state represents an antiparticle,

$$\begin{aligned}
T\left\{\phi(x)\phi^\dagger(y)\right\}\left|0\right\rangle &= \phi(x)\phi^\dagger(y)\left|0\right\rangle \\
&= \Bigg(\underbrace{\phi^+(x)}_{\text{destroys particle}} + \underbrace{\phi^-(x)}_{\text{creates antiparticle}}\Bigg)\Bigg(\underbrace{\cancel{\phi^{\dagger+}(y)\left|0\right\rangle}}_{\text{destroys antiparticle, annihilates vacuum}} + \underbrace{\phi^{\dagger-}(y)}_{\text{creates particle}}\left|0\right\rangle\Bigg) \\
&= \left(\phi^+(x)+\phi^-(x)\right)F(y)\left|\phi\right\rangle \\
&= G(x)F(y)\left|0\right\rangle + H(x)F(y)\left|\bar{\phi}\phi\right\rangle.
\end{aligned} \tag{3-19}$$

G, F, and H are numeric factors that result from the creation and destruction operations (such as the normalization coefficients that are part of the field operators), which we will not express explicitly here. Thus, we have a general ket left, which in this case is part vacuum state, with the amplitude of the vacuum state part being GF, and the amplitude of the multiparticle state (scalar plus anti-scalar) part being HF. For appropriate normalization, the square of the magnitude of GF is the probability of observing the vacuum (no particles left after the transition.) To find the amplitude GF, we need only form an inner product of the last line of (3-19) with $\left\langle 0\right|$, i.e.,

$$\langle 0|T\{\phi(x)\phi^\dagger(y)\}|0\rangle = \langle 0|G(x)F(y)|0\rangle + \langle 0|H(x)F(y)|\bar{\phi}\phi\rangle$$
$$= G(x)F(y)\underbrace{\langle 0|0\rangle}_{=1} + H(x)F(y)\underbrace{\langle 0|\bar{\phi}\phi\rangle}_{=0} = G(x)F(y)\cdot \quad (3\text{-}20)$$

Taking the inner product of the above $T|0\rangle$ with $\langle 0|$ to get the transition amplitude density

Thus, the VEV of the time ordered operator is an amplitude, the square of whose magnitude is the probability of the transition from vacuum initially to vacuum finally. Actually, $|G(x)F(y)|^2$ is a probability density (to be precise, a double density), because it is a function of **x** and **y**. That is, the location **y** where the virtual particle is created could be anywhere, and so could the location **x** where it is destroyed. We would need to integrate the probability density over all possible **x** and all possible **y** to get the actual probability, and this is what one does in interaction theory to calculate probabilities and cross sections

The square of the absolute value of the transition amplitude density is a probability density (for the transition to occur)

In a similar way, the same time ordered operator can be used for antiparticle propagation (with time for x and y reversed) as in Fig 3-5b and (3-17). You can prove this to yourself by doing Prob. 17 of Chap. 3.

Given all of this, we can define our mathematical relationship for the processes shown in Fig. 3-5 as the VEV of the time ordered operator T. This is called, in honor of its discoverer, the Feynman propagator Δ_F (which, by convention, actually includes an extra factor of i),

Redefine the transition amplitude density as the Feynman propagator

$$\boxed{i\Delta_F(x-y) = \langle 0|T\{\phi(x)\phi^\dagger(y)\}|0\rangle}. \quad (3\text{-}21)$$

Step 4: Expressing $i\Delta_F$ as the Contour Integrals $i\Delta^{\pm}(x\text{-}y)$

Note what the Feynman propagator equals for $t_y < t_x$, the case for a virtual particle (not antiparticle).

Step 4, expressing Feynman propagator as two contour integrals

$$i\Delta_F(x-y) = \langle 0|\phi(x)\phi^\dagger(y)|0\rangle$$
$$= \langle 0|\phi^+(x)\underbrace{\phi^{\dagger+}(y)|0\rangle}_{=0} + \langle 0|\phi^+(x)\underbrace{\phi^{\dagger-}(y)|0\rangle}_{=|\phi\rangle} + \langle 0|\phi^-(x)\underbrace{\phi^{\dagger+}(y)|0\rangle}_{=0} + \langle 0|\phi^-(x)\underbrace{\phi^{\dagger-}(y)|0\rangle}_{=|\phi\rangle} \quad (3\text{-}22)$$
$$= \langle 0|\phi^+(x)\phi^{\dagger-}(y)|0\rangle + \underbrace{\langle 0|\phi^-(x)|\phi\rangle}_{=\langle 0||\bar{\phi}\phi\rangle = 0} = \langle 0|\phi^+(x)\phi^{\dagger-}(y)|0\rangle\ .$$

Step 4, first part, expressing Feynman propagator for virtual particle (not antiparticle)

To the last part of (3-22), we can add zero in the form of

$$0 = \langle 0|-\phi^{\dagger-}(y)\underbrace{\phi^+(x)|0\rangle}_{=0}\ . \quad (3\text{-}23)$$

By adding a term equal to zero, we can use a commutator we derived in Step 1

Doing that, and using (3-3) of Step 1 to get the second line below, (3-22) becomes

$$i\Delta_F(x-y) = \langle 0|\phi^+(x)\phi^{\dagger-}(y) - \phi^{\dagger-}(y)\phi^+(x)|0\rangle = \langle 0|\left[\phi^+(x),\phi^{\dagger-}(y)\right]|0\rangle$$
$$= \langle 0|\underbrace{i\Delta^+(x-y)}_{\text{numeric}}|0\rangle = i\Delta^+(x-y)\langle 0||0\rangle \quad (3\text{-}24)$$
$$= i\Delta^+(x-y)\ .$$

In similar fashion, for $t_x < t_y$, the case for a virtual antiparticle, one finds, by doing Prob. 18 of Chap. 3, that

Step 4, second part, expressing Feynman propagator for virtual antiparticle

$$i\Delta_F(x-y) = \langle 0|\left[\phi^{\dagger+}(y),\phi^-(x)\right]|0\rangle$$
$$= -i\Delta^-(x-y). \quad (3\text{-}25)$$

In summary,

Step 4, summary

$$\Delta_F(x-y) = \Delta^+(x-y) \quad \text{if } t_y < t_x$$
$$= -\Delta^-(x-y) \quad \text{if } t_x < t_y \quad (3\text{-}26)$$

Feynman propagator in terms of the contour integrals of Step 2 where in Step 2, we expressed $\Delta^{\pm}(x-y)$ as contour integrals, and so here we express the Feynman propagator in terms of contour integrals. Note that the Feynman propagator, although encompassing operators in its initial definition, turns out to be simply a numeric quantity without operators.

Step 5: Re-express Δ_F in Most Convenient Form

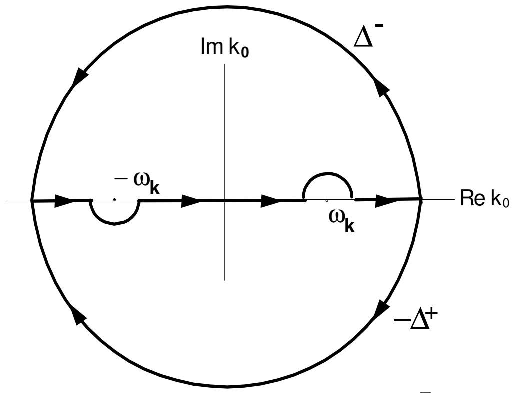

Figure 3-6. Contour Integrals for Δ^- and $-\Delta^+$

Step 5, expressing Feynman propagator as real, not complex (contour), integral

We would like two things more: 1) express the propagator as a single function so we don't have to keep track (while we are integrating over spacetime and doing other things) of whether the virtual field is a particle or antiparticle (i.e., whether to use the Δ^+ or Δ^- function), and 2) have all our integrations over real numbers rather than deal with contour integrals.

To do this, consider Figure 3-6, where we have shown two contour integrals. The top loop represents Δ^- and encloses $-\omega_{\mathbf{k}}$ with a ccw path. The lower loop encloses $+\omega_{\mathbf{k}}$, but since it has a cw integration path, represents $-\Delta^+$.

Thus, we can define the Feynman propagator Δ_F of (3-26) as proportional to the negative of the integral of Fig. 3-6 for either loop. We say "proportional" because we also have to include the concomitant integration over the 3D space of **k** not shown in Fig. 3-6, as well as the various constants involved.

So we can then re-write the Feynman propagator (3-26) with (3-14), (3-15), and Fig. 3-6, as

Two different contours for the Feynman propagator written with same integral, different meaning for path C_F

$$\Delta_F(x-y)=\frac{1}{(2\pi)^4}\int_{C_F}\frac{e^{-ik(x-y)}}{k^2-\mu^2}d^4k\ , \qquad (3\text{-}27)$$

where the C_F on the integral defines the route we take in the plane of Fig. 3-6.

Now, consider enlarging the outer hemispheric parts of the two loops in Fig. 3-6, so they extend essentially to infinity. The value of the contour integrals over them will remain unchanged. But the k^2 value in the denominator of (3-27) will become so large that any contribution to the integral over those parts of the path will become negligible. This means that we can effectively take the integral of (3-27) as extending only along the real axis from $-\infty$ to $+\infty$ as in Fig. 3- 7.

Extending all parts of contours to ∞ except along real axis

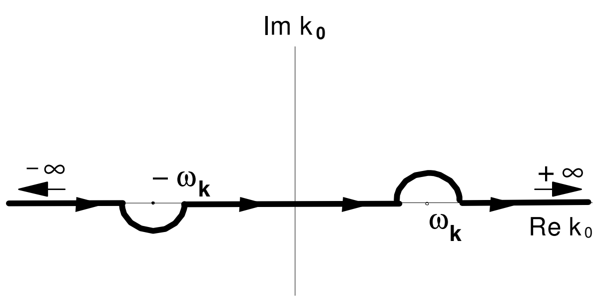

Fig. 3-7. Contour C_F for Δ_F

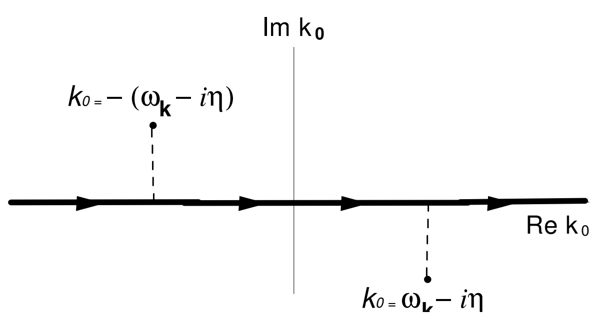

Fig. 3-8. Contour and Displaced Poles for Δ_F

Instead of integrating around poles on the axis, move poles slightly off the axis

We can further simplify by moving the poles an infinitesimal distance η off the real axis as shown in Fig. 3-8 and deform the contour so that it is all along the real axis. In the limit as $\eta \to 0$, the integral will have the same value, though we must now include this slight pole shift in the propagator expression (3-27). We do this by recalling from (3-12) and (3-13) that we used

$$k^2 - \mu^2 = (k_0)^2 - (\omega_{\mathbf{k}})^2 \tag{3-28}$$

to obtain the denominator of (3-27), so we must temporarily restate (3-27) using the right hand side of (3-28), then shift the poles. Thus, (3-27) becomes

$$\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{-ik(x-y)}}{(k_0)^2 - (\omega_{\mathbf{k}} - i\eta)^2}d^4k\,. \tag{3-29}$$

If we then use (3-28) again, ignore second order terms in η, and take $\varepsilon = 2\eta\omega_{\mathbf{k}}$, we have our final result for the Feynman scalar propagator

Yields a single, real integral representing both virtual particle and antiparticle, the most convenient form for the Feynman propagator

$$\boxed{\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{-ik(x-y)}}{k^2 - \mu^2 + i\varepsilon}d^4k\,.}\,. \tag{3-30}$$

Note the advantages of this form. We now have a single mathematical relationship that automatically describes both a particle propagating from y to x *and* an antiparticle propagating from x to y. We also have done away with the cumbersome contour integrals in favor of a simple 4D integral over the entire 4-momentum space. In practice, we can evaluate this integral then take ε to zero after the integration is carried out.

3.0.4 Comments on the Propagator and Its Derivation

The Propagator and Interaction Theory

Our definition of Feynman propagator here will pop up in our formal derivation of interaction theory

The derivation above was formulated with an eye to interaction theory. In that theory, amplitudes are derived for various kinds of interactions between various particles. The square of the magnitude of each amplitude turns out to be the probability of that particular interaction (transition) occurring. These transition amplitudes each depend on the initial real particles, the final real particles, and the virtual particle(s) that mediate the transition. It turns out that the factor in the amplitude representing the virtual particle contribution is identical to the Feynman propagator Δ_F as we defined it in the VEV of the time ordered operator (3-21). Thus, it is also equal to (3-30), so we can simply plug the RHS of (3-30) into the overall transition amplitude as part of our analysis.

This is one reason we started with the relation $\phi\phi^\dagger$ to create and destroy a virtual scalar particle, rather than what one might initially expect, the simpler creation and destruction operator relation $a(\mathbf{k})a^\dagger(\mathbf{k})$. Our heuristic approach was tailored to match what we knew would be coming in the mathematical development of interaction theory.

Meaning of Spacetime Points y and x

Feynman diagrams, and our derivation, imply creation and destruction at a point, but more properly, waves are created and destroyed, and they are spread out over space.

In Fig. 3-5, we imply the virtual particle is created at y and destroyed at x. In Feynman diagrams virtual particles are depicted in this way, and at least one real incoming particle can be thought of as being destroyed at y, as in Fig. 1-1 of Chap. 1, with a virtual particle created simultaneously at y. At x the virtual particle is destroyed, with the simultaneous creation of at least one outgoing real particle at x..

We are really finding probability density as functions of **x** *and* **y**

To be precise, it is more correct to think of the incoming, outgoing, and virtual particles as moving waves spread out in space. What we calculate for a given y and x is the probability density for the interaction as a function of the coordinates y and x. If **y** and **x** are closer, one would find the probability density for the interaction to occur is greater; if farther away, the probability density is less. Integrating over all **x** and **y** gives the total probability for observing the interaction.

Momentum Space Form of the Propagator

From (3-30), we can readily write down the 4-momentum space form of the propagator, the Fourier transform of (3-30), which will be very useful,

Momentum space form of the propagator

$$\boxed{\Delta_F(k) = \frac{1}{k^2 - \mu^2 + i\varepsilon}}\,. \tag{3-31}$$

Earlier version was physical space form