Page 17 Derivation of (2-28), the Gell-Mann *SU*(3) matrix. by Luc Longtin

We recall that this matrix (2-28) is valid only in the limit of *small independent parameters*.

We start from the totally general 3X3 complex matrix in the form of (2-26), to which we impose the conditions (2-27) for a *special unitary group*. With a view of writing the matrix in a form that will reduce to the identity matrix in the limit where the independent parameters vanish, we write:

$$N=\begin{bmatrix}1+\delta_1 & \delta_4 & \delta_5\\ \delta_7 & 1+\delta_2 & \delta_6\\ \delta_8 & \delta_9 & 1+\delta_3\end{bmatrix} \qquad \ldots\text{with conditions } N^\dagger N = I \text{ and Det } N = 1$$

From the *special* condition, Det $N = 1$, we must have:

$$(1+\delta_1)[(1+\delta_2)(1+\delta_3)-\delta_6\delta_9]-\delta_4[\delta_7(1+\delta_3)-\delta_6\delta_8]+\delta_5[\delta_7\delta_9-\delta_8(1+\delta_2)]=1$$

Now, this equation must be valid for *all* values of the variable parameters. In particular, it must *also* be valid in the limit of *small* parameters. We can use this fact to find relations that must be satisfied among the parameters *in that limit*. Thus, keeping *only first-order* terms, we get:

$$(1+\delta_1)(1+\delta_2)(1+\delta_3)\approx(1+\delta_1)(1+\delta_2+\delta_3)\approx 1+\delta_1+\delta_2+\delta_3=1$$

So: $\delta_1+\delta_2+\delta_3\approx 0$

From the *unitary* condition, $N^\dagger N = I$, we must thus have:

$$N^\dagger N=\begin{bmatrix}1+\delta_1^* & \delta_7^* & \delta_8^*\\ \delta_4^* & 1+\delta_2^* & \delta_9^*\\ \delta_5^* & \delta_6^* & 1+\delta_3^*\end{bmatrix}\begin{bmatrix}1+\delta_1 & \delta_4 & \delta_5\\ \delta_7 & 1+\delta_2 & \delta_6\\ \delta_8 & \delta_9 & 1+\delta_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Again, this equation must be valid for *all* values of the variable parameters. In particular, it must *also* be valid in the limit of *small* parameters. Thus, keeping *only first-order* terms, we get:

Element	Expression		
1-1	$(1+\delta_1^*)(1+\delta_1)\approx 1+\delta_1+\delta_1^*\approx 1$	so	$\delta_1+\delta_1^*\approx 0$
2-2	$(1+\delta_2^*)(1+\delta_2)\approx 1+\delta_2+\delta_2^*\approx 1$	so	$\delta_2+\delta_2^*\approx 0$
3-3	$(1+\delta_3^*)(1+\delta_3)\approx 1+\delta_3+\delta_3^*\approx 1$	so	$\delta_3+\delta_3^*\approx 0$

1-2	$\delta_4+\delta_7^*\approx 0$	1-3	$\delta_5+\delta_8^*\approx 0$	2-3	$\delta_6+\delta_9^*\approx 0$
2-1	$\delta_4^*+\delta_7\approx 0$	3-1	$\delta_5^*+\delta_8\approx 0$	3-2	$\delta_6^*+\delta_9\approx 0$

The first three results tell us that $\delta_1, \delta_2, \delta_3$ are *pure imaginary* complex numbers. Making use of the last six results (three of which are independent), and the special condition, we can write:

$$N=\begin{bmatrix}1+\delta_1 & \delta_4 & \delta_5\\ -\delta_4^* & 1+\delta_2 & \delta_6\\ -\delta_5^* & -\delta_6^* & 1-(\delta_1+\delta_2)\end{bmatrix} \qquad \ldots\text{a } \textit{friendlier form} \text{ than (2-28)!}$$

NOTE: This matrix has *eight* independent (real) parameters, namely the real and imaginary parts of $\delta_4, \delta_5, \delta_6$ (which count for *six*) and the pure imaginary δ_1, δ_2 (which count for *two*).

We can reproduce Gell-Mann's form, (2-28), via the following change of variables:

$$\delta_1 = i\alpha_3 + i\frac{\alpha_8}{\sqrt{3}} \qquad \delta_2 = -i\alpha_3 + i\frac{\alpha_8}{\sqrt{3}} \qquad \text{so} \qquad \delta_1 + \delta_2 = i\frac{2\alpha_8}{\sqrt{3}}$$

NOTE: α_3, α_8 are both *real* parameters; this ensures that δ_1, δ_2 are both pure imaginary

$$\delta_4 = \alpha_2 + i\alpha_1 \qquad \delta_5 = \alpha_5 + i\alpha_4 \qquad \delta_6 = \alpha_7 + i\alpha_6$$

NOTE: In these relations, *all* the α variables are *real* parameters. Therefore, we can write:

$$-\delta_4^* = -\alpha_2 + i\alpha_1 \qquad -\delta_5^* = -\alpha_5 + i\alpha_4 \qquad -\delta_6^* = -\alpha_7 + i\alpha_6$$

So: $$N = \begin{bmatrix} 1 + i\alpha_3 + i\frac{\alpha_8}{\sqrt{3}} & \alpha_2 + i\alpha_1 & \alpha_5 + i\alpha_4 \\ -\alpha_2 + i\alpha_1 & 1 - i\alpha_3 + i\frac{\alpha_8}{\sqrt{3}} & \alpha_7 + i\alpha_6 \\ -\alpha_5 + i\alpha_4 & -\alpha_7 + i\alpha_6 & 1 - i\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix}$$ …factoring out a factor of i…

Or: $$N = i\begin{bmatrix} -i + \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -i - \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -i - \frac{2\alpha_8}{\sqrt{3}} \end{bmatrix}$$

Or: $$N = N(0) + i\begin{bmatrix} \alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_1 - i\alpha_2 & \alpha_4 - i\alpha_5 \\ \alpha_1 + i\alpha_2 & -\alpha_3 + \frac{\alpha_8}{\sqrt{3}} & \alpha_6 - i\alpha_7 \\ \alpha_4 + i\alpha_5 & \alpha_6 + i\alpha_7 & -\frac{2\alpha_8}{\sqrt{3}} \end{bmatrix}$$ …where $N(0) = I$, the identity

The advantage of this form is that *all* the α parameters are *real*.

NOTE: We recall that this derivation of the matrix representation for the *SU*(3) group is valid only in the limit of *small* independent parameters. That is, for $|\alpha_i| \ll 1$.