Finding D = 26: A Summary of and Little Different Wrinkle from Zwiebach

Ref: Zwiebach, Sect. 12.4 pgs 250-253 and Sect. 12.5, pgs 259-262. Robert D. Klauber Feb 1, 2023

The present writer has big problems with (12.110) (he doesn't believe it and will never be convinced), so we ignore that result herein. It is not needed to form a meaningful theory.

The Hamiltonian of (12.101) is (where p has nothing to do with momentum and is simply an index label for the p^{th} mode, and $\mathbb{Z}$ comprises all real integers, positive, negative, and zero)

$$H = L_0^{\perp} = \tfrac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{-p}^I \alpha_p^I = \tfrac{1}{2}\alpha_0^I\alpha_0^I + \tfrac{1}{2}\sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \tfrac{1}{2}\sum_{p=1}^{\infty}\alpha_p^I\alpha_{-p}^I\,, \tag{12.101}$$

where the last term is not normal ordered. Zwiebach converts that term to a normal ordered terms plus an infinite constant, which converts (12.101) to (12.103), where D is the dimension of spacetime, i.e.,

$$H = L_0^{\perp} = \tfrac{1}{2}\sum_{p\in\mathbb{Z}} \alpha_{-p}^I \alpha_p^I = \tfrac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I + \tfrac{1}{2}(D-2)\sum_{p=1}^{\infty} p\,. \tag{12.103}$$

The last term is reminiscent of the infinite zero point energy (ZPE) of quantum field theory (QFT). That term is ignored in QFT, though its meaning is an unsettled issue with field theorists. So, we choose to follow tradition and simply ignore it here, without trying to force it into something it is not.

Jumping ahead to pg. 260, where the Lorentz generators $M^{\mu\nu}$ are discussed, we see that their commutation relations in the light-cone gauge must be parallel to the covariant formulation of the same relations, in order for Lorentz covariance to hold. In particular,

$$\left[M^{-I}, M^{-J}\right] = 0\,, \tag{12.148}$$

is true in the covariant formulation, so it must be true in the light-cone gauge formulation.

It turns out (it is not actually said in the text) that, for string theory as developed herein in the light-cone gauge, the LHS of (12.148) does not equal zero. It cannot be made to equal zero without redefining something in that theory.

The redefinition that makes (12.148) hold is the following, where a is, at this point, an undetermined constant.

$$\text{Old definition } p^- = \frac{1}{2\alpha' p^+}L_0^{\perp} \quad \Rightarrow \text{ new defintion } \; p^- = \frac{1}{2\alpha' p^+}\left(L_0^{\perp} + a\right) \tag{1}$$

Using this in (12.150) give us the M^{-I} of (12.151). When we plug that into (12.148), we get (12.152) on the RHS.

The only way that can equal zero is if $D = 26$, and $a = -1$.

Using the new definition in (1) with this value of a affects the mass squared, as shown in (12.108).

$$\text{Old relation } M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}L_0^{\perp} - p^Ip^I = \frac{1}{\alpha'}\sum_{n=1} n\alpha_n^{I\dagger}\alpha_n^I \tag{2}$$

$$\text{New relation } M^2 = -p^2 = 2p^+p^- - p^Ip^I = \frac{1}{\alpha'}\left(L_0^{\perp} + a\right) - p^Ip^I = \frac{1}{\alpha'}\left(a + \sum_{n=1}\alpha_n^{I\dagger}\alpha_n^I\right) = \frac{1}{\alpha'}\left(a + \sum_{n=1} n a_n^{I\dagger}a_n^I\right) \tag{12.108}$$

$$\text{Number operator } N^{\perp} = \sum_{n=1}\alpha_n^{I\dagger}\alpha_n^I = \sum_{n=1} n\,\underbrace{a_n^{I\dagger}a_n^I}_{\substack{\text{num oper}\\ \text{of QFT}}}\,. \tag{12.164}$$

Note that from $a_n^{I\dagger}a_n^I$, the familiar number operator of QFT, we get one unit of mass squared in (12.108) for each identical string in a multi-string ket, which are all in the same n=1 vibration mode. If there are 4 such identical states in the ket, for a given transverse oscillation I, $a_n^{I\dagger}a_n^I$ has eigenvalue 4. For the n=2 mode, if there were 4 such states in the ket, then the eigenvalue of $na_n^{I\dagger}a_n^I$ would be 2X4 = 8. For the n=3 mode, 3X4 =12. Then we have to sum over all of the I oscillations. Each such oscillation contributes in the same manner, but the $na_n^{I\dagger}a_n^I$ eigenvalue for each depends on the number of identical string states in the ket in the nth mode and the value of n (i.e., which mode it is in).

Bottom line: To keep Lorentz invariance (maintain (12.148) in the light-cone gauge, we needed to introduce a constant in the definition of p^{-}, as in (1). The Lorentz invariance constraint of (12.148) forced two things upon us. 1) a must equal – 1, and 2) the dimension of spacetime must equal 26. That, in turn, forced a shift (a 3rd thing) in the mass squared operator by $a/\alpha' = 1/\alpha'$.

Related Points

The Mass Squared Relation

In (12.108), the factor in front is (pg. 168), where T_0 is the string tension,

$$\frac{1}{\alpha'} = 2\pi T_0 \hbar c \quad \left(= 2\pi T_0 \text{ in natural units}\right). \tag{8.76}$$

In natural units, T_0, and thus $1/\alpha'$, has units of energy (dimension M^{1}) divided by length (dimension M^{-1}), and hence, dimension M^2. For a and the operator coefficients of (12.108) as dimensionless, this gives the mass squared of (12.108) the proper units.

Further, the string vibration frequency ω_n is related to string tension by (pg. 77)

$$\omega_n = \sqrt{\frac{T_0}{\mu_0}}\left(\frac{n\pi}{l}\right) \qquad n = 1,2,3,\ldots \quad l = \text{string length} \qquad \mu_0 = \text{ mass/unit length}. \tag{4.17}$$

(4.17) is for classical, non-relativistic strings, but general principle for proportionality to the square root of tension holds relativistically, as well. So, T_0 is proportional to the square of the natural frequency ω_n. Quantum mechanically, the energy is $\hbar\omega_n$, with $\hbar$=1 in natural units. Thus, we find

$$\frac{1}{\alpha'} = 2\pi T_0 \propto \omega_n^2 = E_n^2 \propto M_n^2 \quad \left(\text{natural units}\right), \tag{3}$$

telling us that (12.108) makes a lot of sense. For each integer value $n=n'$ term, for each transverse mode $I=I'$, in the number operator, we get a $n' M_{n'}^2$ contribution to the total mass-energy squared M^2.

Converting Back to Minkowski Coordinates

$$p^{-} = \tfrac{1}{\sqrt{2}}\left(p^0 - p^1\right) \qquad p^0 = \tfrac{1}{\sqrt{2}}\left(p^{+} + p^{-}\right) \tag{4}$$

with (1) (new definition part) and

$$p^{+} = \frac{X^{+}}{2\alpha'\tau} \tag{9.62}$$

$$\begin{aligned} p^0 &= \tfrac{1}{\sqrt{2}}\left(p^{+} + p^{-}\right) = \tfrac{1}{\sqrt{2}}\left(p^{+} + \frac{1}{2\alpha' p^{+}}\left(L_0^{\perp} + a\right)\right) = \tfrac{1}{\sqrt{2}}\left(\frac{X^{+}}{2\alpha'\tau} + \frac{1}{2\alpha' p^{+}}\left(L_0^{\perp} + a\right)\right) \\ &= \tfrac{1}{\sqrt{2}}\left(\frac{X^{+}}{2\alpha'\tau} + \frac{2\alpha'\tau}{2\alpha' X^{+}}\left(L_0^{\perp} + a\right)\right) = \tfrac{1}{\sqrt{2}}\left(\frac{X^{+}}{2\alpha'\tau} + \frac{\tau}{X^{+}}\left(L_0^{\perp} + a\right)\right) = \tfrac{1}{\sqrt{2}}\left(p^{+} + \frac{\tau}{X^{+}}\left(L_0^{\perp} + a\right)\right) \end{aligned} \tag{5}$$

Not sure what this means, but the a shows up in energy in Minkowski coordinates.