Original Prob 14 of 1st edition below.

Problem 14. Use Noether's theorem for scalars and the transformation $x^i \rightarrow x^i + \alpha^i$ to show that three-momentum k_i is conserved. Then, show the same result via commutation of the three-momentum operator of Chap. 3 (which can be found in Wholeness Chart 5-4 at the end of Chap. 5) with the Hamiltonian.

Prob 14, Correction version of 2nd edition.

Problem 14. Show that the total (not density) 3-momentum k^i for free scalars is conserved. Use our knowledge that the conjugate momentum for x^i is k_i, the total (not density) 3-momentum (expressed in covariant components), and it is conserved if L is symmetric (invariant) under the coordinate translation transformation $x^i \rightarrow x'^i = x^i + \alpha^i$, where α^i is a constant 3D vector. Then, show the same result via commutation of the three-momentum operator of Chap. 3 (see Wholeness Chart 5-4, pg. 158) with the Hamiltonian. (Solution is posted on book website. See pg.xvi, opposite pg. 1.)

Ans. (first part).

The Lagrangian density is $\mathcal{L}_0^0 = \phi^\dagger{}_{,\mu}\phi^{,\mu} - \mu^2\phi^\dagger\phi$. We must integrate this over all volume to get the total Lagrangian L. $L = \int \mathcal{L}_0^0 dV$. If k_i is conserved, then of course, so is k^i. So, we need to show L is invariant under $x^i \rightarrow x'^i = x^i + \alpha^i$.

The 1st term in $\mathcal{L}_0^0$, $\phi^\dagger{}_{,\mu}\phi^{,\mu}$

$$\phi = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right) \qquad \phi^\dagger = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$

$$\phi_{,\mu} = \sum_{\mathbf{k}} \frac{i k_\mu}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$

$$\phi^{,\mu} = \sum_{\mathbf{k}} \frac{i k^\mu}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-a(\mathbf{k})e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right) \qquad \phi^\dagger{}_{,\mu} = \sum_{\mathbf{k}} \frac{i k_\mu}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-b(\mathbf{k})e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k})e^{ik_\mu x^\mu}\right)$$

$$\phi^\dagger{}_{,\mu}\phi^{,\mu} = \sum_{\mathbf{k}}\sum_{\mathbf{k}''} \frac{-1}{2V}\frac{k_\mu k''^\mu}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}''}}}\Big(b(\mathbf{k})a(\mathbf{k}'')e^{-ik_\mu x^\mu}e^{-ik''_\mu x^\mu} - b(\mathbf{k})b^\dagger(\mathbf{k}'')e^{-ik_\mu x^\mu}e^{ik''_\mu x^\mu}$$
$$- a^\dagger(\mathbf{k})a(\mathbf{k}'')e^{ik_\mu x^\mu}e^{-ik''_\mu x^\mu} + a^\dagger(\mathbf{k})b^\dagger(\mathbf{k}'')e^{ik_\mu x^\mu}e^{ik''_\mu x^\mu}\Big)$$

We have to integrate each term in $\mathcal{L}$ over all volume to find L. When we do this to the first term $\phi^\dagger{}_{,\mu}\phi^{,\mu}$ above, the first sub-term on the RHS inside the parentheses immediately above will only survive if $k_i = -k''_i$. The same is true of the last sub-term. The 2nd and 3rd sub-terms will only survive if $k_i = k''_i$. So, therefore (where we note that for $k_i = -k''_i$, $k_\mu k''^\mu = \omega_{\mathbf{k}}^2 + k_i k''^i = \omega_{\mathbf{k}}^2 + k_i(-k^i) = \omega_{\mathbf{k}}^2 + k_i k_i = k_\mu k_\mu$),

$$\underbrace{\int \phi^\dagger{}_{,\mu}\phi^{,\mu}dV}_{\text{original term in } L} = \sum_{\mathbf{k}} \frac{-1}{2\omega_{\mathbf{k}}}\begin{pmatrix} k_\mu k_\mu e^{-i2\omega_{\mathbf{k}}t}b(\mathbf{k})a(-\mathbf{k}) - k_\mu k^\mu b(\mathbf{k})b^\dagger(\mathbf{k}) \\ - k_\mu k^\mu a^\dagger(\mathbf{k})a(\mathbf{k}) + k_\mu k_\mu e^{i2\omega_{\mathbf{k}}t}a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k}) \end{pmatrix} \qquad \text{(A)}$$

The time dependent terms may seem strange until we remember that L here is an operator and its expectation value is what we would be related to our real-world measurement. For any state $|\phi_1\phi_2\,..\rangle$, the contribution to the expectation value from the first and last terms in (A) is zero since, for example, $\langle\phi_1...|a^\dagger(\mathbf{k})b^\dagger(\mathbf{k})|\phi_1...\rangle = \langle\phi_1...||\phi_{\mathbf{k}}\bar{\phi}_{\mathbf{k}}\phi_1...\rangle = 0$.

Now, let's see what we get when we transform the spatial coordinates via $x^i \rightarrow x'^i = x^i + \alpha^i$.

$$\phi^\dagger{}_{,\mu}\phi^{,\mu} \xrightarrow{x^i \rightarrow x^i = x'^i - \alpha^i} = \phi'^\dagger{}_{,\mu}\phi'^{,\mu} = \sum_{\mathbf{k}}\sum_{\mathbf{k}''}\frac{-1}{2V}\frac{k_\mu k''^\mu}{\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}''}}}\Big(b(\mathbf{k})a(\mathbf{k}'')e^{-ik_\mu x'^\mu}e^{ik_i\alpha^i}e^{-ik''_\mu x'^\mu}e^{ik''_i\alpha^i}$$
$$- b(\mathbf{k})b^\dagger(\mathbf{k}'')e^{-ik_\mu x'^\mu}e^{ik_i\alpha^i}e^{ik''_\mu x'^\mu}e^{-ik''_i\alpha^i} - a^\dagger(\mathbf{k})a(\mathbf{k}'')e^{ik_\mu x'^\mu}e^{-ik_i\alpha^i}e^{-ik''_\mu x'^\mu}e^{ik''_i\alpha^i}$$
$$+ a^\dagger(\mathbf{k})b^\dagger(\mathbf{k}'')e^{ik_\mu x'^\mu}e^{-ik_i\alpha^i}e^{ik''_\mu x'^\mu}e^{-ik_i\alpha^i}\Big)$$

Once again, the first and last sub-terms above, when integrated over all space, can only be non-zero if $k_i = -k''_i$, and in those cases $e^{ik_i\alpha^i}e^{ik''_i\alpha^i} = 1$. The 2nd and 3rd sub-terms will only survive if $k_i = k''_i$. In that case, $e^{ik_i\alpha^i}e^{-ik''_i\alpha^i} = 1$. When we do this, we get

$$\underbrace{\int \phi'^{\dagger}{}_{,\mu}\phi'^{,\mu}dV}_{\text{transformed term in } L} = \sum_{\mathbf{k}} \frac{-1}{2\omega_{\mathbf{k}}}\begin{pmatrix} k_\mu k_\mu e^{-i2\omega_{\mathbf{k}}t}b(\mathbf{k})a(-\mathbf{k}) - k_\mu k^\mu b(\mathbf{k})b^\dagger(\mathbf{k}) \\ -k_\mu k^\mu a^\dagger(\mathbf{k})a(\mathbf{k}) + k_\mu k_\mu e^{i2\omega_{\mathbf{k}}t}a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})\end{pmatrix}. \qquad \text{(B)}$$

Since (A) and (B) are the same, the first term in L is symmetric under the transformation.

<u>The 2nd term</u> in $\mathcal{L}_0^0$, $-\mu^2\phi^\dagger\phi$

The second term in L follows in almost identical fashion (and is simpler, since no derivatives exist in it) to the first.

$$-\mu^2\phi^\dagger\phi = -\sum_{\mathbf{k}}\sum_{\mathbf{k}''}\frac{\mu^2}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}''}}}\Big(b(\mathbf{k})a(\mathbf{k}'')e^{-ik_\mu x^\mu}e^{-ik''_\mu x^\mu} + b(\mathbf{k})b^\dagger(\mathbf{k}'')e^{-ik_\mu x^\mu}e^{ik''_\mu x^\mu}$$
$$+a^\dagger(\mathbf{k})a(\mathbf{k}'')e^{ik_\mu x^\mu}e^{-ik''_\mu x^\mu} + a^\dagger(\mathbf{k})b^\dagger(\mathbf{k}'')e^{ik_\mu x^\mu}e^{ik''_\mu x^\mu}\Big)$$

$$\underbrace{-\int \mu^2\phi^\dagger\phi\, dV}_{\text{original term}} = -\sum_{\mathbf{k}}\frac{\mu^2}{2\omega_{\mathbf{k}}}\left(e^{-i2\omega_{\mathbf{k}}t}b(\mathbf{k})a(-\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + e^{i2\omega_{\mathbf{k}}t}a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})\right). \qquad \text{(C)}$$

When we transform the spatial coordinates via $x^i \rightarrow x'^i = x^i + \alpha^i$, we get

$$-\mu^2\phi^\dagger\phi \xrightarrow{x^i \rightarrow x^i = x'^i - \alpha^i} = -\mu^2\phi'^\dagger\phi' = -\sum_{\mathbf{k}}\sum_{\mathbf{k}''}\frac{\mu^2}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}''}}}\Big(b(\mathbf{k})a(\mathbf{k}'')e^{-ik_\mu x'^\mu}e^{ik_i\alpha^i}e^{-ik''_\mu x'^\mu}e^{ik''_i\alpha^i}$$
$$+b(\mathbf{k})b^\dagger(\mathbf{k}'')e^{-ik_\mu x'^\mu}e^{ik_i\alpha^i}e^{ik''_\mu x'^\mu}e^{-ik''_i\alpha^i} + a^\dagger(\mathbf{k})a(\mathbf{k}'')e^{ik_\mu x'^\mu}e^{-ik_i\alpha^i}e^{-ik''_\mu x'^\mu}e^{ik''_i\alpha^i}$$
$$+a^\dagger(\mathbf{k})b^\dagger(\mathbf{k}'')e^{ik_\mu x'^\mu}e^{-ik_i\alpha^i}e^{ik''_\mu x'^\mu}e^{-ik_i\alpha^i}\Big).$$

When we integrate the above over space, the same sub-terms will drop out in the same way as did to get (B). Thus, we end up with

$$\underbrace{-\int \mu^2\phi^\dagger\phi\, dV}_{\text{transformed term}} = -\sum_{\mathbf{k}}\frac{\mu^2}{2\omega_{\mathbf{k}}}\left(e^{-i2\omega_{\mathbf{k}}t}b(\mathbf{k})a(-\mathbf{k}) + b(\mathbf{k})b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k})a(\mathbf{k}) + e^{i2\omega_{\mathbf{k}}t}a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})\right). \qquad \text{(D)}$$

Since (C) and (D) are the same, the second term in L is also symmetric under the transformation, and thus L is symmetric under it.

From macro variational mechanics, we know that if L is symmetric in some coordinate, then the conjugate momentum of that coordinate is conserved. k_i, the particle(s) 3-momentum is the conjugate momentum of x^i. Thus, k_i, is conserved.

Note one subtlety. To get to macro mechanics we integrated over all field coordinates x^i, so there was no x^i coordinate left in L. Macroscopically, we would then need to consider our x^i coordinate as that of the position of the center of mass of our solid body (particle). A transformation on the field coordinates x^i would then be the same transformation on the center of mass x^i coordinate used in macro, solid body, variational mechanics analysis.

Ans. (second part).

$$H = \sum_{\mathbf{k}}\omega_{\mathbf{k}}\left(N_a(\mathbf{k}) + N_b(\mathbf{k})\right) \qquad \mathbf{P} = \sum_{\mathbf{k}}\mathbf{k}\left(N_a(\mathbf{k}) + N_b(\mathbf{k})\right) \rightarrow [H, \mathbf{P}] = 0 \quad \begin{pmatrix}\text{because all number}\\ \text{operators commute}\end{pmatrix}$$

Thus $\mathbf{P}$ is conserved for the free Hamiltonian.