# *Chapter 3 Problem 12 Correction*

**Problem 3-12.** Substitute the free field solutions (3-36) to the Klein-Gordon equation into the probability density operator relation (3-89) and then insert that into (3-90) to find the effective probability density operator expressed in terms of number operators (3-91). It will help you in doing so to note that for any term where $\mathbf{k} \neq \mathbf{k}'$, the destruction and creation operators will cause the ket to be different from (orthogonal to) the bra, so the resulting term in the expectation value $\bar{\rho}$ will be zero. Hence, those terms can be ignored in determining an effective $\rho$.

Note that the result you get is restricted to situations where all particles (in the ket) are in **k** eigenstates, which is almost invariably the case in QFT problems and applications. With particles in general (non **k** eigen) states, $\rho$ becomes more complicated.

**Ans.** The probability density operator (3-89) is

$$\rho = j^0 = i\left(\frac{\partial \phi}{\partial t}\phi^\dagger - \frac{\partial \phi^\dagger}{\partial t}\phi\right). \tag{3-89}$$

Substituting (3-36) into (3-89), yields

$$\begin{aligned}\rho = i\Bigg\{&\left(\sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ikx} - b^\dagger(\mathbf{k})e^{ikx}\right)\right)\left(\sum_{\mathbf{k}'} \frac{1}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(b(\mathbf{k}')e^{-ik'x} + a^\dagger(\mathbf{k}')e^{ik'x}\right)\right)\\ &-\left(\sum_{\mathbf{k}'} \frac{-i\,\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(b(\mathbf{k}')e^{-ik'x} - a^\dagger(\mathbf{k}')e^{ik'x}\right)\right)\left(\sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ikx} + b^\dagger(\mathbf{k})e^{ikx}\right)\right)\Bigg\}.\end{aligned}$$

$$\begin{aligned}= \sum_{\mathbf{k}}\sum_{\mathbf{k}'} \frac{1}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\Big\{&\omega_{\mathbf{k}} a(\mathbf{k})b(\mathbf{k}')e^{-ikx}e^{-ik'x} + \omega_{\mathbf{k}} a(\mathbf{k})a^\dagger(\mathbf{k}')e^{-ikx}e^{ik'x}\\ &-\omega_{\mathbf{k}} b^\dagger(\mathbf{k})b(\mathbf{k}')e^{ikx}e^{-ik'x} - \omega_{\mathbf{k}} b^\dagger(\mathbf{k})a^\dagger(\mathbf{k}')e^{ikx}e^{ik'x}\\ &-\omega_{\mathbf{k}'} b(\mathbf{k}')a(\mathbf{k})e^{-ik'x}e^{-ikx} - \omega_{\mathbf{k}'} b(\mathbf{k}')b^\dagger(\mathbf{k})e^{-ik'x}e^{ikx}\\ &+\omega_{\mathbf{k}'} a^\dagger(\mathbf{k}')a(\mathbf{k})e^{ik'x}e^{-ikx} + \omega_{\mathbf{k}'} a^\dagger(\mathbf{k}')b^\dagger(\mathbf{k})e^{ik'x}e^{ikx}\Big\}.\end{aligned} \tag{A}$$

In (A) we have a whole lot of complicated terms. But if we use (A) to find the expectation value

$$\bar{\rho} = \langle \phi_1\phi_2 \ldots | \rho | \phi_1\phi_2 \ldots \rangle\,, \tag{B}$$

which reflects what we measure, then all terms in (A) for which $\mathbf{k} \neq \mathbf{k}'$ will drop out in (B), because the resulting ket will not match (will not be orthogonal to) the bra. Thus, we can consider an *effective* probability density ("effective" because it will provide us with what we measure although strictly speaking, it is not equal to (A)) of

effective $\rho =$

$$\begin{aligned}\sum_{\mathbf{k}} \frac{1}{2V\omega_{\mathbf{k}}}\Big\{&\omega_{\mathbf{k}} a(\mathbf{k})b(\mathbf{k})e^{-ikx}e^{-ikx} + \omega_{\mathbf{k}} \underbrace{a(\mathbf{k})a^\dagger(\mathbf{k})}_{a^\dagger(\mathbf{k})a(\mathbf{k})+1}e^{-ikx}e^{ikx} - \omega_{\mathbf{k}} b^\dagger(\mathbf{k})b(\mathbf{k})e^{ikx}e^{-ikx} - \omega_{\mathbf{k}} b^\dagger(\mathbf{k})a^\dagger(\mathbf{k})e^{ikx}e^{ikx}\\ &-\omega_{\mathbf{k}} b(\mathbf{k})a(\mathbf{k})e^{-ikx}e^{-ikx} - \omega_{\mathbf{k}} \underbrace{b(\mathbf{k})b^\dagger(\mathbf{k})}_{b^\dagger(\mathbf{k})b(\mathbf{k})+1}e^{-ikx}e^{ikx} + \omega_{\mathbf{k}} a^\dagger(\mathbf{k})a(\mathbf{k})e^{ikx}e^{-ikx} + \omega_{\mathbf{k}} a^\dagger(\mathbf{k})b^\dagger(\mathbf{k})e^{ikx}e^{ikx}\Big\}.\end{aligned} \tag{C}$$

The only terms in (C) that will survive in (B) are those having factors of $a$ and $a^\dagger$, or $b$ and $b^\dagger$, and for them, the exponential factors cancel. The result is an *effective* (streamlined) probability density operator of form

$$\rho = \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}}{V\omega_{\mathbf{k}}}\left\{\; a^\dagger(\mathbf{k})a(\mathbf{k})e^0 - b^\dagger(\mathbf{k})b(\mathbf{k})e^0 \;\right\} = \sum_{\mathbf{k}} \frac{1}{V}\left\{N_a(\mathbf{k}) - N_b(\mathbf{k})\right\}.$$