Continuous Solutions Creation and Destruction Operators Derivation

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1 Delta Function Units

Note first that the Dirac delta function has inverse units of its argument. For example, in

$$\int \delta(x-x')\,dx = 1 \ (\text{unitless}) \qquad \int f(x)\delta(x-x')\,dx = f(x') \ (\text{units of function } f) \ , \tag{1}$$

$\delta(x - x')$ has units of $1/x$ (cm^{-1} in cgs units, M^{1} in natural units). Similarly, $\delta(k - k')$ has units of $1/k$ (cm in cgs, since units of wave number k are the inverse of length units; M^{-1} in natural units).

In 3D, $\delta^{(3)}(\mathbf{x} - \mathbf{x}')$ [often just written as $\delta(\mathbf{x} - \mathbf{x}')$] has units of 1/volume (cm^{-3} in cgs, M^{3} in natural units); $\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ [often just written as $\delta(\mathbf{k} - \mathbf{k}')$] has units of volume (cm^{3} in cgs, M^{-3} in natural units).

2 Creation and Destruction Operator Units

Thus, in the 3D relation for the coefficients of the continuous solutions,

$$\left[a(\mathbf{k}), a^{\dagger}(\mathbf{k}')\right] = \delta(\mathbf{k} - \mathbf{k}') \ , \tag{2}$$

the delta function has units of $1/\mathbf{k}^3$ (M^{-3} in natural units), so the $a(\mathbf{k})$ and $a^{\dagger}(\mathbf{k})$ operators each have units of $1/\sqrt{\mathbf{k}^3}$ ($M^{-3/2}$ in natural units).

So, in

$$\begin{aligned}
\phi(x) &= \underbrace{\int \frac{d^3\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}\, a(\mathbf{k}) e^{-ikx}} + \underbrace{\int \frac{d^3\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}\, b^{\dagger}(\mathbf{k}) e^{ikx}} && \text{(a)} \\
&= \qquad \phi^{+} \qquad + \qquad \phi^{-} \\
\phi^{\dagger}(x) &= \underbrace{\int \frac{d^3\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}\, b(\mathbf{k}) e^{-ikx}} + \underbrace{\int \frac{d^3\mathbf{k}}{\sqrt{2(2\pi)^3\omega_{\mathbf{k}}}}\, a^{\dagger}(\mathbf{k}) e^{ikx}} && \text{(b)} \\
&= \qquad \phi^{\dagger +} \qquad + \qquad \phi^{\dagger -} \ ,
\end{aligned} \qquad \text{(3-37) in text} \tag{3}$$

given that $\omega_{\mathbf{k}}$ and $\mathbf{k}$ have the same units, and $a(\mathbf{k})$ has units as shown just above (3), then ϕ has units of $\mathbf{k}$ (M^{1} in natural units). $b(\mathbf{k})$

Note that ϕ having units of $\mathbf{k}$ (M^{1}) in the discrete case

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} a(\mathbf{k}) e^{-ikx} + \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} b^{\dagger}(\mathbf{k}) e^{ikx} \ , \qquad \text{(3-37a) in text,} \tag{4}$$

means that $a(\mathbf{k})$ in that case is unitless. This follows from the commutation relation

$$\left[a(\mathbf{k}),a^{\dagger}(\mathbf{k}')\right]=\delta_{\mathbf{k}\mathbf{k}'} \qquad \text{discrete case,} \tag{5}$$

since $\delta_{\mathbf{k}\mathbf{k}'}$ is unitless.

3 Hamiltonian for Continuous Solutions

From www.quantumfieldtheory.info/H_operator_and_noneigen_states.pdf, equation (37) therein, for continuous solutions, the Hamiltonian operator is

$$\begin{aligned} H &= \int \omega_{\mathbf{k}}\left(\mathcal{N}_a(\mathbf{k})+\tfrac{1}{2}\delta(0)+\mathcal{N}_b(\mathbf{k})+\tfrac{1}{2}\delta(0)\right)d^3k \\ \mathcal{N}_a(\mathbf{k}) &= a^{\dagger}(\mathbf{k})a(\mathbf{k}) \qquad \mathcal{N}_b(\mathbf{k})=b^{\dagger}(\mathbf{k})b(\mathbf{k}) \quad (\text{continuous}) \end{aligned} \qquad \text{(10-8) in } \textit{SFQFT} \text{ text.} \tag{6}$$

For the vacuum terms in (6), the $\delta(0)$ factor of infinity has units of 1/(**k** space vol), or equivalently, since $|\mathbf{k}| = 2\pi/\lambda$ (λ is wavelength), units of **x** space volume. The $\delta(0)$ represents the infinite volume of space over which our Hamiltonian H represents the total energy. Thus, the **x** (physical) space Hamiltonian volume density for the vacuum energy is

$$\mathcal{H}_{vac}=\frac{H_{vac}}{V_{\infty}}=\int \omega_{\mathbf{k}}\left(\tfrac{1}{2}+\tfrac{1}{2}\right)d^3k=\int \omega_{\mathbf{k}}d^3k\,. \tag{7}$$

Note, from (6), that the vacuum terms factor of ½ is dimensionless (no units). That is,

$$\begin{aligned} \tfrac{1}{2} &= \left(\left(\text{number of vacuum particle states}\right)/\left(\text{unit vol in } \mathbf{k} \text{ space}\right)\right)/\left(\text{unit vol in } \mathbf{x}\,\text{space}\right), \\ &\qquad \text{dimensions } M^3/M^3=\text{dimensionless}\,. \end{aligned} \tag{8}$$

4 Number Operator Units

For the number operator relations

$$N_a(\mathbf{k})=a^{\dagger}(\mathbf{k})a(\mathbf{k}) \quad (\text{discrete}) \qquad \mathcal{N}_a(\mathbf{k})=a^{\dagger}(\mathbf{k})a(\mathbf{k}) \quad (\text{continuous})\,, \tag{9}$$

we know that $N_a(\mathbf{k})$ is unitless (a number operator without dimensions). From the Hamiltonian operator for continuous solution scalars (6), we see that $\mathcal{N}_a(\mathbf{k})$ is a number density operator (density in **k** space). That is,

$$\mathcal{N}_a(\mathbf{k}),\mathcal{N}_b(\mathbf{k})=\left(\text{number of real particle states}\right)/\text{unit vol in } \mathbf{k} \text{ space}, \text{dimensions } 1/M^3 \tag{10}$$

These operators act on all states spread through all of **x** space, as our integrations in **x** space (as well as **k** space) in the continuous solution approach are from – ∞ to + ∞ in all directions.

5 Creation and Destruction Operators Derived: Continuous Case

In parallel with the discrete solution derivation of Sect. 3.6.1, pg. 58, of *SFQFT*, consider the **k**th wave component of a wave packet ket as $|n_{\mathbf{k}}\rangle$. Then

$$\mathcal{N}_a(\mathbf{k})|n_{\mathbf{k}}\rangle=n_{\mathbf{k}}|n_{\mathbf{k}}\rangle\,, \tag{11}$$

where $n_{\mathbf{k}}$ has the same units as $\mathcal{N}_a(\mathbf{k})$, i.e., (number of real particle states)/(**k** space vol). Note carefully, that $n_{\mathbf{k}}$ is for all space (infinite spatial volume).

We then ask, what is

$$a(\mathbf{k})|n_{\mathbf{k}}\rangle = |m_{\mathbf{k}}\rangle \, ? \qquad \text{parallel to (3-71) for discrete case in text} \qquad (12)$$

$$\mathcal{N}_a(\mathbf{k})|m_{\mathbf{k}}\rangle = \mathcal{N}_a(\mathbf{k})a(\mathbf{k})|n_{\mathbf{k}}\rangle = \underbrace{a^{\dagger}(\mathbf{k})a(\mathbf{k})}_{\text{use commutator}}a(\mathbf{k})|n_{\mathbf{k}}\rangle \qquad \text{parallel to (3-72)} \qquad (13)$$

$$\begin{aligned}&= \left(a(\mathbf{k})a^{\dagger}(\mathbf{k}) - \delta(0)\right)a(\mathbf{k})|n_{\mathbf{k}}\rangle = a(\mathbf{k})a^{\dagger}(\mathbf{k})a(\mathbf{k})|n_{\mathbf{k}}\rangle - \delta(0)a(\mathbf{k})|n_{\mathbf{k}}\rangle = a(\mathbf{k})N_a(\mathbf{k})|n_{\mathbf{k}}\rangle - \delta(0)a(\mathbf{k})|n_{\mathbf{k}}\rangle \\ &= a(\mathbf{k})n_{\mathbf{k}}|n_{\mathbf{k}}\rangle - \delta(0)a(\mathbf{k})|n_{\mathbf{k}}\rangle = n_{\mathbf{k}}a(\mathbf{k})|n_{\mathbf{k}}\rangle - \delta(0)a(\mathbf{k})|n_{\mathbf{k}}\rangle = \left(n_{\mathbf{k}} - \delta(0)\right)a(\mathbf{k})|n_{\mathbf{k}}\rangle = \left(n_{\mathbf{k}} - \delta(0)\right)|m_{\mathbf{k}}\rangle.\end{aligned} \qquad (14)$$

So,

$$\mathcal{N}_a(\mathbf{k})|m_{\mathbf{k}}\rangle = \left(n_{\mathbf{k}} - \delta(0)\right)|m_{\mathbf{k}}\rangle = m_{\mathbf{k}}|m_{\mathbf{k}}\rangle \qquad m_{\mathbf{k}} = n_{\mathbf{k}} - \underbrace{\delta(0)}_{=V_{\infty}} \qquad \text{parallel to (3-74).} \qquad (15)$$

In (15), $m_{\mathbf{k}}$ and $n_{\mathbf{k}}$ are for all space (infinite in size in our continuous solution math where all integrals are over – ∞ to + ∞). Thus, for any finite number (however small) of **k** states per unit volume of space, $m_{\mathbf{k}}$ and $n_{\mathbf{k}}$ must be infinite.

If we define new quantities ${}_u n_{\mathbf{k}}$ and ${}_u m_{\mathbf{k}}$ as spatial densities, i.e., as number per unit spatial volume, such that

$$n_{\mathbf{k}} = {}_u n_{\mathbf{k}} V_{\infty} \qquad m_{\mathbf{k}} = {}_u m_{\mathbf{k}} V_{\infty} \qquad {}_u n_{\mathbf{k}}, {}_u m_{\mathbf{k}} = \frac{\text{number of } \mathbf{k} \text{ fields per unit } \mathbf{k} \text{ volume}}{\text{unit volume of space}} \;, \qquad (16)$$

then the RHS of (15) becomes

$$m_{\mathbf{k}} = n_{\mathbf{k}} - V_{\infty} = {}_u m_{\mathbf{k}} V_{\infty} = {}_u n_{\mathbf{k}} V_{\infty} - V_{\infty}\,. \qquad (17)$$

Dividing (17) by V_{∞} we have

$${}_u m_{\mathbf{k}} = {}_u n_{\mathbf{k}} - 1\,, \qquad (18)$$

and thus, from (15),

$$\frac{\mathcal{N}_a(\mathbf{k})}{V_{\infty}}|m_{\mathbf{k}}\rangle = \left({}_u n_{\mathbf{k}} - 1\right)|m_{\mathbf{k}}\rangle = {}_u m_{\mathbf{k}}|m_{\mathbf{k}}\rangle \qquad (19)$$

(18) and (19) remind us of (3-74) in *SFQFT*. But, whereas (3-74) implied $a(\mathbf{k})$ in the discrete solutions case decreased the total number of **k** states by one, here in the continuous solutions case, $a(\mathbf{k})$ decreases the **k** states per unit **k** volume per unit **x** volume by one.

Similar analysis leads to $a^{\dagger}(\mathbf{k})$ in the continuous state increasing the **k** states per unit **k** volume per unit **x** volume by one. Thus, the continuous solutions coefficients $a(\mathbf{k})$ and $a^{\dagger}(\mathbf{k})$ are respectively, destruction and creation operators. By parallel logic, $b(\mathbf{k})$ and $b^{\dagger}(\mathbf{k})$ behave in the same way for b type states.