Comments on Equation (2-37) in SFQFT

First comment: partial vs total derivatives

In equation (2-37), pg. 27, one might think that, according to the product differentiation rule, the factor $\frac{\partial \mathcal{O}^S}{\partial t}$ should be a total derivative, as in $\frac{d\mathcal{O}^S}{dt}$, rather than a partial derivative.

But as long as our operators are functions of fields, using similar logic to that of of Box 2.1, we can take total time derivatives of operators on the RHS of (2-37) as partial time derivatives.

To see this, consider the following, where *f* and *g* would be fields (in classical field theory, for example).

$$\mathcal{O} = f(x,t)\,g(x,t) \qquad x \neq x(t) \tag{1}$$

$\mathcal{O}$ in (1) is analogous to $\mathcal{O}^H = U^\dagger \mathcal{O}^S U$ in (2-37). *f* and *g* are analogous to $U^\dagger, \mathcal{O}^S$ or *U*. (We only use two factors instead of three to keep it simple.)

$$\begin{aligned}\frac{d\mathcal{O}}{dt} &= \frac{\partial \mathcal{O}}{\partial f}\frac{df}{dt} + \frac{\partial \mathcal{O}}{\partial g}\frac{dg}{dt} = g\frac{df}{dt} + f\frac{dg}{dt} \\ &= g\left(\frac{\partial f}{\partial t}\frac{dt}{dt} + \frac{\partial f}{\partial x}\frac{dx}{dt}\right) + f\left(\frac{\partial g}{\partial t}\frac{dt}{dt} + \frac{\partial g}{\partial x}\frac{dx}{dt}\right) = g\frac{\partial f}{\partial t} + f\frac{\partial g}{\partial t}.\end{aligned} \tag{2}$$

So, as long as $\mathcal{O}$ is a function of fields (as it typically is in QFT), the partial time and total time derivatives on the RHS can be interchanged in these sorts of expressions. See end of first and second rows in (2). More generally, as long as *x* is not a function of *t*, then this holds true.

If *f* and *g* are operators (as in QFT), we have to more careful about the order above (always keeping the factor with *f* in it to the left of the factor with *g*.)

The above comments hold elsewhere in the text, for example, (2-26).

Second comment: changing symbolism for special definition in (2-37)

In (2-37), the text employed a special, unorthodox definition of the symbol $\frac{\partial \mathcal{O}^H}{\partial t}$, where $\mathcal{O}^H = e^{iHt}\mathcal{O}^S e^{-iHt}$, i.e.,

$$\frac{\partial \mathcal{O}^H}{\partial t} = e^{iHt}\frac{\partial \mathcal{O}^S}{\partial t}e^{-iHt} \qquad \text{unorthodox definition defined in (2-37) and used elsewhere in text .} \tag{3}$$

The more orthodox, common mathematical use of this symbol is

$$\frac{\partial \mathcal{O}^H}{\partial t} = \frac{\partial e^{iHt}}{\partial t}\mathcal{O}^S e^{-iHt} + e^{iHt}\frac{\partial \mathcal{O}^S}{\partial t}e^{-iHt} + e^{iHt}\mathcal{O}^S\frac{\partial e^{-iHt}}{\partial t} \qquad \text{more orthodox definition .} \tag{4}$$

Although definition (3) is employed elsewhere in the literature[1], and streamlines notation in later relations, it, given (4), can be confusing. Thus, it should help to use the following alternative symbolism in (2-37) instead of that in the text. (Note the "hat" on the partial derivative symbol.)

$$\frac{\hat{\partial} \mathcal{O}^H}{\partial t} = e^{iHt}\frac{\partial \mathcal{O}^S}{\partial t}e^{-iHt} \qquad \text{new symbol for definition in (2-37) .} \tag{5}$$

Substituting (5) for (3) in the text means making the following changes. (Changes are shown in red. Sorry to have so many of them, but I think in the long run it will make things easier.)

[1] See, for example, Mandl, F. and Shaw, G., *Quantum Field Theory*, 2nd Ed.,(Wiley, 2010), pg. 21, eq (1.81)

1. Change (2-37) to

$$\frac{d}{dt}\left(U^{\dagger}\mathcal{O}^{S}U\right)=(iH)\underbrace{e^{iHt}\mathcal{O}^{S}e^{-iHt}}_{\mathcal{O}^{H}}+\underbrace{e^{iHt}\left(\frac{\partial\mathcal{O}^{S}}{\partial t}\right)e^{-iHt}}_{\text{defined as }\hat{\partial}\mathcal{O}^{H}/\partial t}+\underbrace{e^{iHt}\mathcal{O}^{S}e^{-iHt}}_{\mathcal{O}^{H}}(-iH)$$

$$=\frac{d\mathcal{O}^{H}}{dt}=-i\left[\mathcal{O}^{H},H\right]+\underbrace{\frac{\hat{\partial}\mathcal{O}^{H}}{\partial t}}_{=0\text{ in this book}}\quad . \qquad \text{new (2-37)}$$

2. Change (2-39) to

$$\frac{d\bar{\mathcal{O}}}{dt}={}_{S}\langle\psi|UU^{\dagger}\left(-i\left[\mathcal{O}^{S},H\right]\right)UU^{\dagger}|\psi\rangle_{S}+{}_{S}\langle\psi|UU^{\dagger}\frac{\partial\mathcal{O}^{S}}{\partial t}UU^{\dagger}|\psi\rangle_{S}$$

$$={}_{H}\langle\psi|\left(-i\left[\mathcal{O}^{H},H\right]\right)|\psi\rangle_{H}+{}_{H}\langle\psi|\frac{\hat{\partial}\mathcal{O}^{H}}{\partial t}|\psi\rangle_{H}\ . \qquad \text{new (2-39)}$$

3. Change first sentence after (2-39) to the following.

From which we see that the equation of motion for the expectation value of an operator has ~~exactly~~ the same form in both pictures.

4. Wholeness Chart 2-4, pg. 28, change the 3rd column, Heisenberg Picture row block to

$$\frac{d\mathcal{O}^{H}}{dt}=-i\left[\mathcal{O}^{H},H\right]+\underbrace{\frac{\hat{\partial}\mathcal{O}^{H}}{\partial t}}_{\substack{\text{usually}\\=0}}$$

5. Wholeness Chart 2-4, change the 4th column, Heisenberg Picture row block to

Same as Schrödinger picture above
with sub and superscript $S\rightarrow H$ and $\partial\mathcal{O}^{H}\rightarrow\hat{\partial}\mathcal{O}^{H}$
$|\psi\rangle_{H}$ const in time; $\mathcal{O}^{H}$ often changes in time

5. Change (2-41), pg. 29 to

$$\frac{du}{dt}=\frac{-i}{\hbar}[u,H]+\frac{\hat{\partial}u}{\partial t} \qquad \text{new (2-41)}$$

6. Line after (2-41), after "NRQM", add – in the Heisenberg picture –.

7. In Section 2.7.2, pg. 29, make the following changes.

3rd line after (2-41) change "only" to – fundamental –
Last paragraph, 1st line, after "one" insert – significant –
Last paragraph, 2nd line, after "remain" insert – effectively –

8. Wholeness Chart 2-5, pgs. 30-31, change the 3rd column, next to last row to

i) for $v = H$ $\quad\frac{du}{dt}=\frac{-i}{\hbar}[u,H]+\frac{\hat{\partial}u}{\partial t}$

9. Wholeness Chart 2-5, pgs. 30-31, change the last column, next to last row to

i) for $v = H$ $\quad\dot{u}=\frac{du}{dt}=\frac{-i}{\hbar}[u,H]+\frac{\hat{\partial}u}{\partial t}$