Chirality and Helicity In Depth
(and How They Merge When v = c)

Robert D. Klauber www.quantumfieldtheory.info
December 7, 2016 revision (Original Nov 2007)

Refs:
Quantum Field Theory, F. Mandl and G. Shaw, (Wiley, 2nd ed 2010)
Student Friendly Quantum Field Theory, R.D. Klauber (2nd ed, 2015 version)

We will work in the standard representation (Dirac-Pauli rep) of the spinor solutions to the Dirac equation. We will also keep it simple by showing examples rather than a usually harder-to-see theoretical treatment, and by focusing on only one of the spinor solutions (since the others will follow in similar fashion.)

Solutions to the Dirac Equation

There are four independent solutions to the Dirac equation, which in the standard rep can be represented as

$$_S\psi = {_S\psi^{(1)}} + {_S\psi^{(2)}} + {_S\psi^{(3)}} + {_S\psi^{(4)}} \ , \tag{1}$$

where subscript S stands for "standard rep", the first two solutions destroy particles of different spin, and the last two create antiparticles of different spin. We will focus primarily on the first of these, which turns out to be of form

$$\begin{aligned} {_S\psi^{(1)}} &= \sum_{\mathbf{p}} \left(\frac{E_{\mathbf{p}}+m}{2VE_{\mathbf{p}}} \right)^{1/2} c_1(\mathbf{p})\, u_1(\mathbf{p})\, e^{-ipx} \\ &= \sum_{\mathbf{p}} \left(\frac{E_{\mathbf{p}}+m}{2VE_{\mathbf{p}}} \right)^{1/2} c_1(\mathbf{p}) \underbrace{\begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E_{\mathbf{p}}+m} \\ \dfrac{p_x+ip_y}{E_{\mathbf{p}}+m} \end{pmatrix}}_{\text{spinor } u_1(\mathbf{p})} e^{-ipx} \ , \end{aligned} \tag{2}$$

where $c_1(\mathbf{p})$ destroys a particle of spin $u_1(\mathbf{p})$.

<u>Aside</u>: The coefficients in front in (2) are chosen to yield a probability density for a single fermionic particle equal to $1/V$. See Klauber, Sect. 3.1.4, pg 44 and Sect. 4.1.7, pg 91. The spinor normalization and coefficient must be such that the probability density $\rho = j^0 = \psi^\dagger \psi$ equals N/V, where N is the number operator $c_1^\dagger(\mathbf{p})\, c_1(\mathbf{p})$, so a single particle has probability density $1/V$. If you have seen other coefficients used by other authors (including me in Klauber, (4-20), pg. 89), then the $u_1(\mathbf{p})$ is defined there with a different normalization than in (2) such that $\psi^\dagger \psi$ still equals N/V. We have chosen the particular spinor normalization in (2) because the resulting form of the spinor will be more convenient for our present purposes.

<u>Exercise</u>: Find $\psi^\dagger \psi$ for a given $\mathbf{p}$ term in (2), and note that $u_1^\dagger(\mathbf{p})\, u_1(\mathbf{p}) = 2E_{\mathbf{p}} /(E_{\mathbf{p}} + m)$.

In Klauber (see (4-25) we assumed, as Mandl & Shaw and others do, that the spinor was normalized so that $u_1^{\dagger}(\mathbf{p})\, u_1(\mathbf{p}) = E_{\mathbf{p}} /m$, and $u_1(\mathbf{p})$ included the coefficient $\left(\frac{E_{\mathbf{p}}+m}{2VE_{\mathbf{p}}}\right)^{1/2}$ inside it.

For $v = 0$

Note that for v=0, i.e. $p_x = p_y = p_z = 0$ and $E_{\mathbf{p}} = m$, the spinor has form

$$
{}_S\psi^{(1)} = \sum_{\mathbf{p}} \left(\frac{1}{V}\right)^{1/2} c_1(\mathbf{p}) \underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{u_1(\mathbf{p})} e^{-ipx} \qquad (\text{for } v=0), \tag{3}
$$

and would be called a spin up field. The other three solution spinors would each have unity in a different one of the other three component slots and zero elsewhere. However, we are most interested in the case where $v = c$.

For $v = c$

For simplicity, we take our coordinate system such that the z direction is aligned with the velocity and thus only p_z is non-zero. We need m=0 in (2) in order to have v=c. Thus, $p_z = E_{\mathbf{p}}$, and

$$
{}_S\psi^{(1)} = \sum_{\mathbf{p}} \left(\frac{1}{2V}\right)^{1/2} c_1(\mathbf{p}) \underbrace{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}_{u_1(\mathbf{p})} e^{-ipx} \qquad (\text{for } v=c). \tag{4}
$$

The second solution form

For future reference we note that the second term in (1) is

$$
\begin{aligned}
{}_S\psi^{(2)} &= \sum_{\mathbf{p}} \left(\frac{E_{\mathbf{p}}+m}{2VE_{\mathbf{p}}}\right)^{1/2} c_2(\mathbf{p})\, u_2(\mathbf{p})\, e^{-ipx} \\
&= \sum_{\mathbf{p}} \left(\frac{E_{\mathbf{p}}+m}{2VE_{\mathbf{p}}}\right)^{1/2} c_2(\mathbf{p}) \underbrace{\begin{pmatrix}0\\1\\ \dfrac{p_x - ip_y}{E_{\mathbf{p}}+m}\\ \dfrac{-p_z}{E_{\mathbf{p}}+m}\end{pmatrix}}_{\text{spinor } u_2(\mathbf{p})} e^{-ipx}\quad .
\end{aligned} \tag{5}
$$

Relativistic Spin Operator

We will start with the standard mathematical definition for the relativistic spin operator, then give it some justification by applying it to (3) and (4), and seeing what we get. The relativistic spin operator uses the 4X4 spinor space version of the old 2X2 non-relativistic QM Pauli spin matrices σ_i, which we used with two component (spin up and spin down) particle wave functions in non-relativistic quantum mechanics (NRQM). This relativistic spin operator is designated $\Sigma_i/2$, and is defined as (see Klauber, Sect. 4.1.10, pg 93)

$$\tfrac{1}{2}\Sigma_i = \tfrac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix} \qquad \text{or} \qquad \tfrac{1}{2}\vec{\Sigma} = \tfrac{1}{2}\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}, \tag{6}$$

so

$$\tfrac{1}{2}\Sigma_1 = \tfrac{1}{2}\begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix} \quad \tfrac{1}{2}\Sigma_2 = \tfrac{1}{2}\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix} \quad \tfrac{1}{2}\Sigma_3 = \tfrac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}. \tag{7}$$

Note the effect of operating the z direction spin operator on the field (3) (or (4)),

$$\tfrac{1}{2}\Sigma_3\, {}_S\psi^{(1)} = \tfrac{1}{2}\, {}_S\psi^{(1)}\ , \tag{8}$$

and thus (3) (and (4)) represents a spin up field (eigenvalue of +½ for positive direction spin ½.)

Exercise: Repeat (8) for (5) and draw conclusions (i.e., show that (5) for the same special cases is a spin down field.)

Helicity

Helicity Operator

Helicity is positive, and called right-handed (RH), if the spin (using RH rule) points in the direction of **p**. It is negative, or LH, if it points in the – **p** direction. Mathematically, helicity represents the component of the spin in the direction of momentum, so the helicity operator is the inner product of spin and the unit vector in the **p** direction (see Klauber, Sect. 4.1.11, pg 100), i.e.,

$$\tfrac{1}{2}\mathbf{\Sigma}\cdot\hat{\mathbf{p}} = \tfrac{1}{2}\mathbf{\Sigma}\cdot\frac{\mathbf{p}}{|\mathbf{p}|} = \frac{1}{2|\mathbf{p}|}\begin{pmatrix}p_3 & p_1-ip_2 & 0 & 0\\ p_1+ip_2 & -p_3 & 0 & 0\\ 0 & 0 & p_3 & p_1-ip_2\\ 0 & 0 & p_1+ip_2 & -p_3\end{pmatrix} = \text{spin component in } \mathbf{p} \text{ direction}\ . \tag{9}$$

Note the effect of operating the helicity operator on the field (4), where in that case $p_1 = p_2 = 0$,

$$\tfrac{1}{2}\mathbf{\Sigma}\cdot\hat{\mathbf{p}}\ {}_S\psi^{(1)} = \tfrac{1}{2}\ {}_S\psi^{(1)}\ . \tag{10}$$

So the field (4) has positive, or right hand, helicity, since the helicity eigenvalue is positive. For momentum in directions not aligned with spin, we would not have a helicity eigenvector field. (Take $p_1 \neq 0$, $p_2 \neq 0$ and use (9) to operate on (2).)

Helicity Projection Operator

If we have a general field of form (1), that is, a superposition of spin up and spin down fields with no definite helicity, then we may want to "pull" or "filter" or "project" out only the LH (or RH) helicity state. The operator that does this is

$$\Pi^{R}_{L}(\mathbf{p}) = \tfrac{1}{2}\left(1 \pm \frac{\Sigma\cdot\mathbf{p}}{|\mathbf{p}|}\right) = \tfrac{1}{2}\left(1 \pm \Sigma\cdot\hat{\mathbf{p}}\right). \tag{11}$$

Understanding this completely takes a fair amount of time playing around with (11) acting on all of the various forms of (2) and the other three solutions. However, to gain some insight, consider only the first two terms of (1) for the special case where $v=c$ ($m=0$) and operate on them with (11). To simplify, we align the z axis of our coordinate system with $\mathbf{p}$, so $p_1 = p_2 = 0$.

$$\begin{aligned}
\Pi^{R}(\mathbf{p})_S\psi &= \tfrac{1}{2}\left(1+\Sigma\cdot\hat{\mathbf{p}}\right)\left({}_S\psi^{(1)} + {}_S\psi^{(2)}\right) \qquad\qquad (\text{for } v=c \text{ below})\\
&= \tfrac{1}{2}\left({}_S\psi^{(1)} + {}_S\psi^{(2)}\right) + \\
&\qquad \sum_{\mathbf{p}}\left(\frac{1}{2V}\right)^{1/2}\left(c_1(\mathbf{p})\frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + c_2(\mathbf{p})\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right)e^{-ipx}\\
&= \tfrac{1}{2}\left({}_S\psi^{(1)} + {}_S\psi^{(2)}\right) + \sum_{\mathbf{p}}\left(\frac{1}{2V}\right)^{1/2}\left(c_1(\mathbf{p})\frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - c_2(\mathbf{p})\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right)e^{-ipx}\\
&= \tfrac{1}{2}\left({}_S\psi^{(1)} + {}_S\psi^{(2)}\right) + \tfrac{1}{2}\left({}_S\psi^{(1)} - {}_S\psi^{(2)}\right) = {}_S\psi^{(1)}\ ,
\end{aligned} \tag{12}$$

which we saw above in (10) is the RH helicity field. Thus Π^R does indeed project out the RH helicity component for this case.

Exercise: Repeat the above procedure for Π^L. (If you want to do it for all four terms in (1), you'll need to find the other two solution forms first. See Mandl and Shaw, 1st ed, Appendix A, eqs (A.75).)

Note: In classical relativity, one can show that for a spinning object, as $v\rightarrow c$, the rotation spin axis approaches alignment with the momentum vector. This can be visualized as due to the Lorentz-Fitzgerald shortening of the direction parallel to the momentum vector direction, as $v\rightarrow c$. This is illustrated in Klauber, Box 4-2, pg 95. That box shows a rotating wheel with axis not aligned at low speed to the velocity vector with the dimension in the velocity direction shrinking to zero as speed increases. The plane of the wheel effectively rotates into the plane perpendicular to velocity. So, any particle traveling with speed c would be in a pure helicity state. That is what we have shown quantum mechanically in (12).

Aside: Keep in mind that ${}_S\psi^{(1)}$ is a field, not a state, and because of the $c_1(\mathbf{p})$ operator, will destroy a particle (state) [in the case when $p_1=p_2=0$, it will be a spin up state.]

Chirality

While helicity is related to what the particle is like in physical space (what its spin and momentum are like), chirality is related to the γ^5 matrix in Dirac spinor space. It turns out (who knows why?) that nature has related the weak force to chirality, i.e., to the γ^5 matrix, so we need to investigate the effects of γ^5 on our Dirac equation solution terms (1).

In the standard representation,

$$\gamma^5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{13}$$

A field in a pure chirality state would be an eigenvector of (13). So let's look at the special massless case for our first solution (4) and see what we get.

$$\begin{aligned} \gamma^5{}_S\psi^{(1)} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} {}_S\psi^{(1)} = \sum_{\mathbf{p}} \left(\frac{1}{2V} \right)^{1/2} \left(c_1(\mathbf{p}) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right) e^{-ipx} \\ &= \sum_{\mathbf{p}} \left(\frac{1}{2V} \right)^{1/2} c_1(\mathbf{p}) \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-ipx} = {}_S\psi^{(1)} \quad (\text{for } v = c). \end{aligned} \tag{14}$$

Thus, ${}_S\psi^{(1)}$ in this special case is an eigenvector of chirality with a +1 eigenvalue for γ^5. The nomenclature that has evolved calls this a RH chirality field. Unfortunately, using the same nomenclature as for helicity causes considerable confusion, so we must use care when talking about RH or LH fields and states, and designate whether we mean helicity or chirality.

Note that, in the more general case,

$$\gamma^5{}_S\psi^{(1)} == \sum_{\mathbf{p}} \left(\frac{E_{\mathbf{p}}+m}{2VE_{\mathbf{p}}}\right)^{1/2} c_1(\mathbf{p}) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E_{\mathbf{p}}+m} \\ \frac{p_x+ip_y}{E_{\mathbf{p}}+m} \end{pmatrix} e^{-ipx}$$

$$= \sum_{\mathbf{p}} \left(\frac{E_{\mathbf{p}}+m}{2VE_{\mathbf{p}}}\right)^{1/2} c_1(\mathbf{p}) \begin{pmatrix} \frac{p_z}{E_{\mathbf{p}}+m} \\ \frac{p_x+ip_y}{E_{\mathbf{p}}+m} \\ 1 \\ 0 \end{pmatrix} e^{-ipx} \neq (\text{constant})\ {}_S\psi^{(1)} \quad (\text{for } v \neq c). \tag{15}$$

So generally, ${}_S\psi^{(1)}$ is not an eigenvector of γ^5, and this is true even when p_1=p_2=0. ${}_S\psi^{(1)}$ is a chiral eigenvector when v=c, as in (14).

Exercise: Show that when v=c and the z direction of our coordinate system is aligned with **v**, ${}_S\psi^{(2)}$ is an eigenvector of γ^5 with eigenvalue –1. We call this a LH chirality eigenfield. Show that in general ${}_S\psi^{(2)}$ is not a chiral eigenfield.

Chirality Projection Operator

It turns out (again, who knows why?) that LH chirality fields are coupled to the weak force, but RH fields are not. (This is strictly true only above the electroweak symmetry breaking scale.) So we would like to be able to isolate, or project out, the pure LH (weak interacting) components of a general field containing both RH and LH components; and vice versa. In similar fashion to the helicity projection operator, we can do this by using the chirality projection operator

$$P^{\substack{R \\ L}} = \tfrac{1}{2}\left(1 \pm \gamma^5\right). \tag{16}$$

Like helicity, understanding completely how this works takes a fair amount of time playing around with the various solutions to the Dirac equation and the action of (16) on those solution fields. To get a feeling for how P^R works, we look at its action on the same special case fields as in (12).

$$
\begin{aligned}
P^R{}_S\psi &= \tfrac{1}{2}\left(1+\gamma^5\right)\left({}_S\psi^{(1)} + {}_S\psi^{(2)}\right) \qquad\qquad \text{(for } v=c \text{ below)} \\
&= \tfrac{1}{2}\left({}_S\psi^{(1)} + {}_S\psi^{(2)}\right) + \\
&\qquad \sum_{\mathbf{p}}\left(\frac{1}{2V}\right)^{1/2}\left(c_1(\mathbf{p})\frac{1}{2}\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + c_2(\mathbf{p})\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right)e^{-ipx} \\
&= \tfrac{1}{2}\left({}_S\psi^{(1)} + {}_S\psi^{(2)}\right) + \sum_{\mathbf{p}}\left(\frac{1}{2V}\right)^{1/2}\left(c_1(\mathbf{p})\frac{1}{2}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - c_2(\mathbf{p})\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right)e^{-ipx} \\
&= \tfrac{1}{2}\left({}_S\psi^{(1)} + {}_S\psi^{(2)}\right) + \tfrac{1}{2}\left({}_S\psi^{(1)} - {}_S\psi^{(2)}\right) = {}_S\psi^{(1)} \;,
\end{aligned}
\tag{17}
$$

where we see that the RH chirality part of the general field is projected out. In general, the same effect occurs for non special cases. P^R filters out the RH chirality component in any field; P^L, the left chirality component.

Nomenclature

We use the following symbols

$$
\begin{aligned}
&P^R\psi = \psi_R \qquad\qquad P^L\psi = \psi_L \\
&\Pi^R(\mathbf{p})\psi = \psi_{R\ helicity} \quad \Pi^L(\mathbf{p})\psi = \psi_{L\ helicity} \;.
\end{aligned}
\tag{18}
$$

Take care, however, that some authors may use the *R* and *L* subscripts alone for helicity. In general, however, those subscripts usually refer to chirality, as that is the more important of the two concepts in QFT.

ψ_L couples to the weak force, whereas ψ_R does not (again, strictly speaking, only above the electroweak symmetry breaking energy scale.)

Chirality and Helicity Merge at v = c

Note what happens to helicity and chirality relations as $v \to c$. We will use a frame where p_1=p_2=0 for convenience.

$$
\tfrac{1}{2}\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}} = \tfrac{1}{2}\left(\frac{p_1}{|\mathbf{p}|}\Sigma_1 + \frac{p_2}{|\mathbf{p}|}\Sigma_1 + \frac{p_3}{|\mathbf{p}|}\Sigma_1\right) = \frac{1}{2}\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix} \quad \begin{matrix}\text{at any speeed}\\ \text{in this frame}\end{matrix}
\tag{19}
$$

$$
\gamma^5 = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix} \quad \begin{matrix}\text{at any speeed}\\ \text{in this frame}\end{matrix}
\tag{20}
$$

So right off the bat from (19) and (20), we see

$$\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}} \text{ never } = \gamma^5 \qquad \text{even at } v = c\,. \tag{21}$$

Now look at the field.

$$u_1(\mathbf{p}) = \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E_{\mathbf{p}}+m} \\ \dfrac{p_x+ip_y}{E_{\mathbf{p}}+m} \end{pmatrix} \xrightarrow{\text{for } v=c} \qquad u_1(\mathbf{p}) = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{22}$$

At any v (in this frame where **v** is aligned with z axis), we have

$$\tfrac{1}{2}\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\; u_1(\mathbf{p}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E_{\mathbf{p}}+m} \\ \dfrac{p_x+ip_y}{E_{\mathbf{p}}+m} \end{pmatrix} == \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E_{\mathbf{p}}+m} \\ \dfrac{-p_x-ip_y}{E_{\mathbf{p}}+m} \end{pmatrix} \tag{23}$$

$$\gamma^5 u_1(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \dfrac{p_z}{E_{\mathbf{p}}+m} \\ \dfrac{p_x+ip_y}{E_{\mathbf{p}}+m} \end{pmatrix} = \begin{pmatrix} \dfrac{p_z}{E_{\mathbf{p}}+m} \\ \dfrac{p_x+ip_y}{E_{\mathbf{p}}+m} \\ 1 \\ 0 \end{pmatrix} \tag{24}$$

$$\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\, u_1(\mathbf{p}) \text{ in general not } = \gamma^5 u_1(\mathbf{p}) \tag{25}$$

But at $v = c$, because $u_1(\mathbf{p})$ takes the form of the RHS of (22), (23) and (24) become

$$\tfrac{1}{2}\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\; u_1(\mathbf{p}) = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{for } v = c \tag{26}$$

$$\gamma^5 u_1(\mathbf{p}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{for } v = c\,. \tag{27}$$

So in that case,

$$\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}\,u_1(\mathbf{p}) = \gamma^5 u_1(\mathbf{p}) \qquad \text{when } v = c\,. \tag{28}$$

But that is only true because of the form the field takes (not the forms the operators take) as $v \to c$.

In general, $\boldsymbol{\Sigma}\cdot\hat{\mathbf{p}}$ does not equal γ^5 and this is true even when v=c (see (26) and (27))[1]. In the general case, the effect they have when operating on the general field solution ${}_S\psi$ (see (1),(2), and (5)) is different, so they are hardly the same thing.

However when v=c, the field solution components ((2) and (5), for examples) take a special form in which the effects of the helicity operator and the chirality operator on those fields are the same. See (10) and (14); and also (26) and (27). A given field solution component $\psi^{(i)}$ is then an eigenfield of both $\frac{(\boldsymbol{\Sigma}\bullet\mathbf{P})}{|\mathbf{P}|}$ and γ^5 with the same eigenvalues for both. RH helicity and RH chirality fields are the same, and the Π^R and P^R operators project out the exact same components of a general field solution. See (12) and (17). Similar conclusions hold for LH fields.

So at v=c, pure RH chirality (LH chirality) fields are the same as pure RH helicity (LH helicity) fields. That is,

$$\text{as } v \to c, \qquad \psi_L \to \psi_{L\,helicity} \qquad \psi_R \to \psi_{R\,helicity}\,. \tag{29}$$

Summary

We have shown special case examples, using the standard representation but not the Weyl or Majorana reps, of fields, as well as the action of the helicity and chirality operators on those fields. From those examples, we gain some foundation for accepting that (18) holds in all cases, and that it holds in any representation (standard, Weyl, or Majorana).

We saw that $\frac{(\boldsymbol{\Sigma}\bullet\mathbf{P})}{|\mathbf{P}|}$ and γ^5 are not the same matrices, under any circumstances, but that the fields in (18) take on particular forms when v=c, such that those fields become eigenfields of both $\frac{(\boldsymbol{\Sigma}\bullet\mathbf{P})}{|\mathbf{P}|}$ and γ^5, and that a particular field has the same eigenvalues for both $\frac{(\boldsymbol{\Sigma}\bullet\mathbf{P})}{|\mathbf{P}|}$ and γ^5.

Thus, we conclude that helicity and chirality are different things, although when v=c, they become effectively the same thing (because a RH [LH] helicity eigenfield is then also a RH [LH] chirality eigenfield.)

To return to main page, click here Student Friendly Quantum Field Theory.

---

[1] In Mandl & Shaw, page 457, after relation A.42, the authors state that when m = 0 (so particle speed is c), these two operators are equal. I humbly submit this is not the case.