So finally, where terms noted in under brackets refer to (7-111), and we use (7-112)

$$\downarrow\ A_1B_2C_3 \text{ re-arranged using full commutation relations } \downarrow$$

$$T\{A_1B_2C_3\} = \underbrace{N\{A_1B_2C_3\}}_{\text{1st 8 terms}} + \underbrace{N\{\underbracket{A_1B_2}C_3\}}_{\text{1st term in last row}} + \underbrace{N\{A_1\underbracket{B_2C_3}\}}_{\text{2nd term in last row}} + \underbrace{N\{\underbracket{A_1B_2C_3}\}}_{\text{last 2 terms}} \quad \text{for } t_3 < t_2 < t_1. \tag{7-113}$$

Result: Wick's theorem for three fields, this time order case

Other cases other than $t_3 < t_2 < t_1$

For cases where $t_3 < t_2 < t_1$ is not true, we would have the same result for the RHS of (7-105), i.e., it would equal the RHS of (7-113). But the time ordered side would be ordered using the T_c operator (using the commutator/anti-commutator relations when switching field positions.) As with the two field case, we would get commutator/anti-commutator relations on the time ordered side that would cancel with identical relations on the N_c ordered side. You can do Prob. 15 to prove the case for $t_2 < t_3 < t_1$, and then, if you wish, play around with other time sequences to prove it in general. The final result is that (7-113) holds for any time sequence for any three fields.

Same result for all time orders = Wick's theorem for three fields

$$T\{A_1B_2C_3\} = N\{A_1B_2C_3\} + N\{\underbracket{A_1B_2}C_3\} + N\{A_1\underbracket{B_2C_3}\} + N\{\underbracket{A_1B_2C_3}\} \quad \text{for any time order}. \tag{7-114}$$

7.11.2 Wick's Theorem via Induction

Comparing (7-114) for three fields to (7-97) for two fields, we can see a pattern emerging. If we were to carry out one more example with four fields, we would see additional types of terms entailing two contractions. This pattern would be fully reflected by (7-82), and thus by Wick's theorem in full, (7-78).

A pattern emerges for more fields. That pattern is Wick's theorem

7.12 Appendix B: Operators in Exponentials and Time Ordering

7.12.1 Math Reference

For what follows we recall (hopefully, you have seen this relation before) the Baker-Campbell-Hausdorff formula, where, for A and B as operators,

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]-\frac{1}{24}[B,[A,[A,B]]]+\ldots}. \tag{7-115}$$

If A and B commute, or are c numbers, we get the familiar simple addition of exponents result.

7.12.2 Solution of Differential Equation with Operator

Consider (7-59), which we repeat below for convenience,

$$i\frac{dS_{oper}}{dt} = H_I^I S_{oper}, \tag{7-59}$$

and consider what we might naïvely expect to be the solution, (7-60),

$$S_{oper} = e^{-i\int_{t_i}^{t} H_I^I dt'}. \tag{7-60}$$

Then, the LHS of (7-59) can be found via (with (7-65) in the last line below)

$$\begin{aligned} i\frac{dS_{oper}(t,t_i)}{dt} &= i\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(S_{oper}(t+\Delta t,t_i) - S_{oper}(t,t_i)\right) = i\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{-i\int_{t_i}^{t+\Delta t}H_I^I(t')dt'} - e^{-i\int_{t_i}^{t}H_I^I(t')dt'}\right) \\ &= i\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{-i\int_{t}^{t+\Delta t}H_I^I(t')dt' - i\int_{t_i}^{t}H_I^I(t')dt'} - e^{-i\int_{t_i}^{t}H_I^I(t')dt'}\right) \end{aligned} \tag{7-116}$$

where in QED, $H_I^I(t') = \int \mathcal{H}_I^{1/2,1}(t')d^3x$ and $\mathcal{H}_I^{1/2,1}(t') = e\bar{\psi}(x')\gamma^\mu\psi(x')A_\mu(x')$.

For Commuting Variables in the Exponent

If the first term in the 2nd line of (7-116) contained only c numbers or commuting operators in H_I^I, then we would have, via (7-115),

$$
\begin{aligned}
i\frac{dS_{oper}\left(t,t_i\right)}{dt} &= i\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{-i\int_t^{t+\Delta t}H_I^I(t')dt'}e^{-i\int_{t_i}^{t}H_I^I(t')dt'}-e^{-i\int_{t_i}^{t}H_I^I(t')dt'}\right) \quad \left(\begin{array}{c}\leftarrow \text{only if } H_I^I \text{ at different}\\ \text{times commute}\end{array}\right)\\
&= i\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{-i\int_t^{t+\Delta t}H_I^I(t')dt'}-1\right)e^{-i\int_{t_i}^{t}H_I^I(t')dt'} = i\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(e^{-i\int_t^{t+\Delta t}H_I^I(t')dt'}-1\right)S_{oper}\left(t,t_i\right) \qquad \text{(7-117)}\\
&= i\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(1-iH_I^I(t)\Delta t-\tfrac{1}{2}\left(H_I^I(t)\Delta t\right)^2+\dots-1\right)S_{oper}\left(t,t_i\right)\\
&= i\left(-iH_I^I(t)\right)S_{oper}\left(t,t_i\right)=H_I^I(t)S_{oper}\left(t,t_i\right),
\end{aligned}
$$

which is the same as (7-59).

But, the first term in the 2nd line of (7-116) contains operators in H_I^I, that do not commute at different times (during the integration over time process), and so, it seems, due to (7-115), we cannot conclude that (7-60) is a solution of (7-59).

However, we can still make (7-60) meaningful by attaching a particular interpretation to the symbolism, as we show below.

<u>For Commuting or Non-commuting Variables in the Exponent</u>

To find a viable solution to (7-59) that is good regardless of the implications of (7-115), we start by noting that (7-59) can be solved with

$$S_{oper}\left(t,t_i\right)=1-i\int_{t_i}^{t}H_I^I\left(t_1\right)S_{oper}\left(t_1,t_i\right)dt_1, \qquad \text{(7-118)}$$

though this assumes we know the form of $S_{oper}(t_1,\, t_i)$. But we can use (7-118) repeatedly to solve (7-59) via iteration. That is, plug

$$S_{oper}\left(t_1,t_i\right)=1-i\int_{t_i}^{t_1}H_I^I\left(t_2\right)S_{oper}\left(t_2,t_i\right)dt_2 \qquad \text{(7-119)}$$

into (7-118), then express $S_{oper}(t_2,\, t_i)$ in terms of integration over t_3 using (7-118), etc. The resulting infinite series looks like

$$
\begin{aligned}
S_{oper}\left(t,t_0\right)=1-i\int_{t_0}^{t}H_I^I\left(t_1\right)dt_1+(-i)^2\overbrace{\int_{t_0}^{t}H_I^I\left(t_1\right)\left(\int_{t_0}^{t_1}H_I^I\left(t_2\right)dt_2\right)dt_1}^{\text{See LHS of Fig. 7-4}} \qquad \text{(7-120)}\\
+(-i)^3\int_{t_0}^{t}H_I^I\left(t_1\right)\left(\int_{t_0}^{t_1}H_I^I\left(t_2\right)\left(\int_{t_0}^{t_2}H_I^I\left(t_3\right)dt_3\right)dt_2\right)dt_1+\dots
\end{aligned}
$$

where no assumption need be made about the commutation properties of factors in the integrands. In this case, for the bottom line term above, we must integrate over t_3 first, then t_2, then t_1. In the term with an integration over t_n, the order is t_n, t_{n-1}, …. t_2, t_1. The LHS of Fig. 7-4 is a graphic representation of the integration regions involved in the double integral term in (7-120).

Fig. 7-4 should be relatively self explanatory (a picture is worth a thousand words). From the last line in that figure, where we simply switch dummy variables in the last term of the first line below,

$$
\begin{aligned}
\int_{t_0}^{t}\int_{t_0}^{t}T\left\{H_I^I\left(t_1\right)H_I^I\left(t_2\right)\right\}dt_2dt_1 &= \underbrace{\int_{t_0}^{t}\left(\int_{t_0}^{t_1}H_I^I\left(t_1\right)H_I^I\left(t_2\right)dt_2\right)dt_1}_{t_2<t_1\text{ region}}+\underbrace{\int_{t_0}^{t}\left(\int_{t_0}^{t_2}H_I^I\left(t_2\right)H_I^I\left(t_1\right)dt_1\right)dt_2}_{t_1<t_2\text{ region}}\\
&= \underbrace{\int_{t_0}^{t}\left(\int_{t_0}^{t_1}H_I^I\left(t_1\right)H_I^I\left(t_2\right)dt_2\right)dt_1}_{t_2<t_1\text{ region}}+\underbrace{\int_{t_0}^{t}\left(\int_{t_0}^{t_1}H_I^I\left(t_1\right)H_I^I\left(t_2\right)dt_2\right)dt_1}_{t_2<t_1\text{ region}} \qquad \text{(7-121)}\\
&= 2\int_{t_0}^{t}\left(\int_{t_0}^{t_1}H_I^I\left(t_1\right)H_I^I\left(t_2\right)dt_2\right)dt_1.
\end{aligned}
$$

Thus, for the double integration term in (7-120), we can substitute

$$\int_{t_0}^{t}\left(\int_{t_0}^{t_1}H_I^I\left(t_1\right)H_I^I\left(t_2\right)dt_2\right)dt_1=\frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t}T\left\{H_I^I\left(t_1\right)H_I^I\left(t_2\right)\right\}dt_2dt_1 \qquad \text{(7-122)}$$

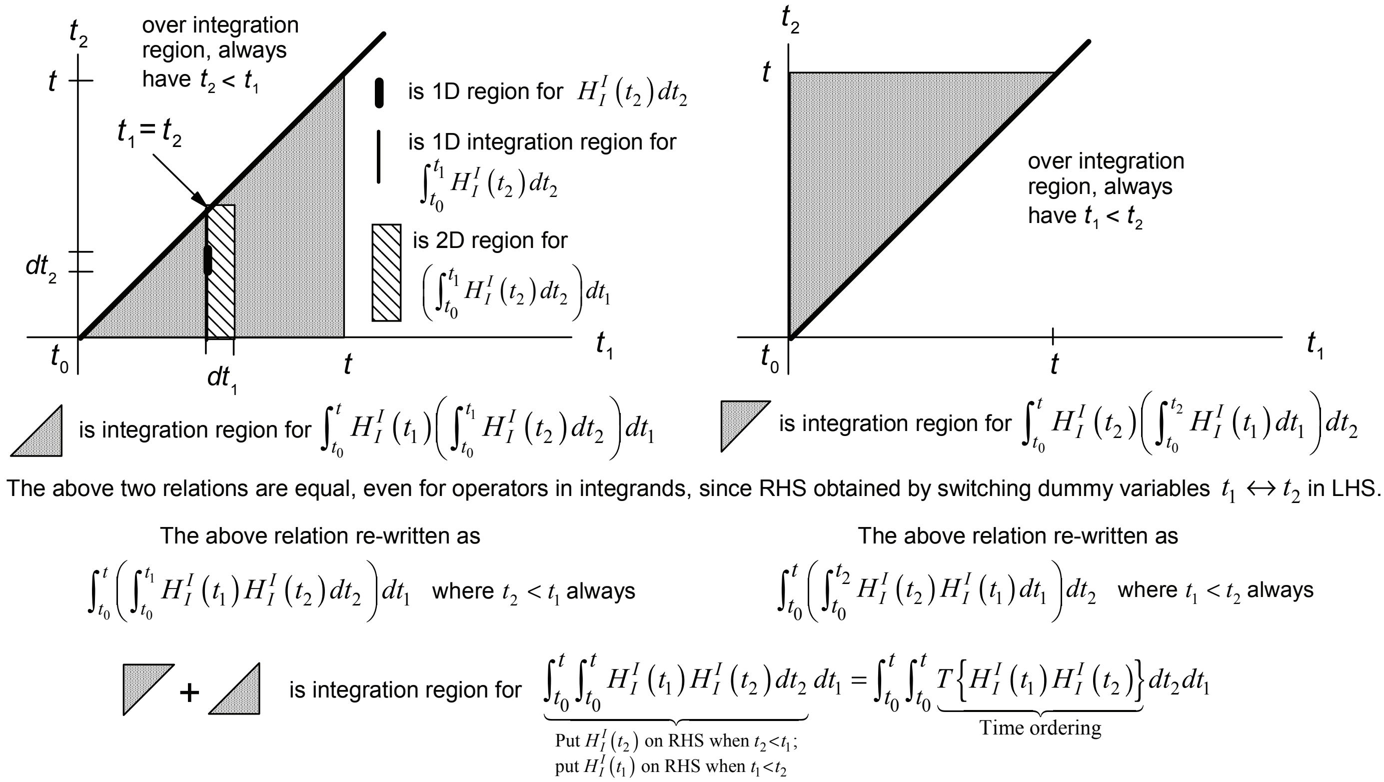

Figure 7-4. Regions of Integration Related to Double Integration Term in S_{oper} Expansion

For the triple integration term in the bottom row of (7-120), we get

$$
\begin{aligned}
&\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} T\left\{H_I^I(t_1)H_I^I(t_2)H_I^I(t_3)\right\}dt_3dt_2dt_1 \\
&=\int_{t_0}^{t} H_I^I(t_1)\left(\int_{t_0}^{t_1} H_I^I(t_2)\left(\int_{t_0}^{t_2} H_I^I(t_3)dt_3\right)dt_2\right)dt_1+\int_{t_0}^{t} H_I^I(t_1)\left(\int_{t_0}^{t_1} H_I^I(t_3)\left(\int_{t_0}^{t_3} H_I^I(t_2)dt_2\right)dt_3\right)dt_1 \\
&+\int_{t_0}^{t} H_I^I(t_2)\left(\int_{t_0}^{t_2} H_I^I(t_1)\left(\int_{t_0}^{t_1} H_I^I(t_3)dt_3\right)dt_1\right)dt_2+\int_{t_0}^{t} H_I^I(t_2)\left(\int_{t_0}^{t_2} H_I^I(t_3)\left(\int_{t_0}^{t_3} H_I^I(t_1)dt_1\right)dt_3\right)dt_2 \qquad (7\text{-}123) \\
&+\int_{t_0}^{t} H_I^I(t_3)\left(\int_{t_0}^{t_3} H_I^I(t_1)\left(\int_{t_0}^{t_1} H_I^I(t_2)dt_2\right)dt_1\right)dt_3+\int_{t_0}^{t} H_I^I(t_3)\left(\int_{t_0}^{t_3} H_I^I(t_2)\left(\int_{t_0}^{t_2} H_I^I(t_1)dt_1\right)dt_2\right)dt_3 \\
&=6\int_{t_0}^{t} H_I^I(t_1)\left(\int_{t_0}^{t_1} H_I^I(t_2)\left(\int_{t_0}^{t_2} H_I^I(t_3)dt_3\right)dt_2\right)dt_1,
\end{aligned}
$$

or

$$\int_{t_0}^{t} H_I^I(t_1)\left(\int_{t_0}^{t_1} H_I^I(t_2)\left(\int_{t_0}^{t_2} H_I^I(t_3)dt_3\right)dt_2\right)dt_1=\frac{1}{3!}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} T\left\{H_I^I(t_1)H_I^I(t_2)H_I^I(t_3)\right\}dt_3dt_2dt_1 \quad (7\text{-}124)$$

Repeating the procedure for higher integration number terms, we end up with (7-67) (with t_f there equal to t here).

Thus, we need to interpret the exponentials in (7-60), (7-61), (7-66), and (7-67) as being defined by (7-67), i.e., as implying time ordering in the expansion. Only by doing this can we avoid the issues that non-commutation would bring in via the Baker-Campbell-Hausdorff formula, (7-115).

Note that some authors use the time ordering symbol T to indicate this, as in

$$S_{oper}\left(t_f,t_i\right)=Te^{-i\int_{t_i}^{t_f} H_I^I(t)dt}. \qquad (7\text{-}125)$$

We don't do that in this book, but we need to keep in mind that by (7-61) we mean (7-67).