Definition: The transition amplitude is that complex number, the square of the absolute magnitude of which is the probability of measuring a transition from a given initial state to a specific final state. (As discussed in Chaps. 1, 7, 8, etc.)

Square of absolute value of transition amplitude = probability of transition

Symbolism: The transition amplitude for a time of interaction approaching infinity, as in the canonical quantization approach, is typically written as S_{fi} (see chapters cited above). However, in the path integral approach, where elapsed time T between measurements of the initial state ψ_i and final state ψ_f is commonly finite, it is more typical to write

$$U\left(\psi_i,\psi_f;T\right) \qquad \left(\text{for } T\to\infty, U=S_{fi} \text{ of canonical quantization}\right). \tag{18-7}$$

Symbol U for path integrals with T finite; for $T\to\infty$, $U = S_{fi}$ of canonical case

This terminology carries over to inelastic cases (where particles change types). (Most of QFT, as seen in the rest of this book, is devoted to determining the transition amplitudes for the different possible interactions between particles.)

Schrödinger Approach – Transition Amplitudes

The Schrödinger approach to QM leads to an expression of the transition amplitude of form (note the parallel with (7-62), pg. 198)

$$U\left(\psi_i,\psi_f;T\right)=\underbrace{\left\langle\psi_f\right|}_{\substack{\text{final state}\\ \text{measured}\\ \text{at } T+t_a}}\underbrace{e^{-iHT/\hbar}\underbrace{\left|\psi_i\right\rangle}_{\text{inital state at } t_a}}_{\text{evolved state at } T+t_a}\quad, \tag{18-8}$$

U for NRQM Schrödinger wave mechanics approach

where H is the Hamiltonian operator, and we retain the symbol $\hbar$ even though $\hbar$ = 1 in natural units.

Alternative nomenclature: The transition amplitude U is sometimes called the propagator (though *not* the QFT Feynman propagator). It projects the wave function at $T + t_a$ that evolved from the initial state $|\psi_i\rangle$ at t_a onto the final state $|\psi_f\rangle$ at time $T + t_a$. It "propagates" the particle from i to f.

18.3.2 Position Eigenstates

When the particle has a definite position, e.g., x_i, the ket is an eigenstate of position, written $|x_i\rangle$. The transition amplitude for measuring a particle initially at x_i, and finally at x_f, would take the form

$$U\left(x_i,x_f;T\right)=\left\langle x_f\right|\underbrace{e^{-iHT/\hbar}\left|x_i\right\rangle}_{\substack{\text{evolved state,}\\ \text{in } x \text{ space } = \psi}}\quad. \tag{18-9}$$

$|x_i\rangle$, eigenstate of position, in x space rep, is a delta function; which can be effectively represented by a steep, narrow wave packet

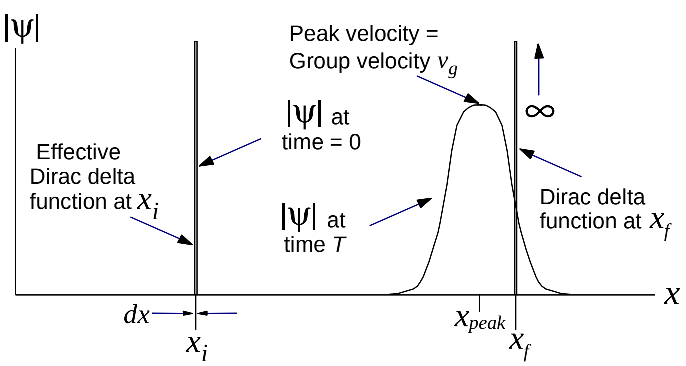

Figure 18-1. Propagation of an Effectively Initial Position Eigenstate Quantum Wave

A position eigenstate such as $|x_f\rangle$ is, in x space, a delta function of form[1] $\delta(x - x_f)$, schematically represented on the right in Fig. 18-1. As the initial state at x_i evolves into ψ, however, it, like wave packets generally do, spreads, and its peak diminishes (wave function envelope at right in Fig. 18-1.) The amplitude for measuring the particle at time T at x_f, i.e., for measuring $|x_f\rangle$ that collapsed from ψ, is (18-9).

It spreads as it evolves

When measured at x_f, wave packet collapses to $|x_f\rangle$, eigenstate of position, i.e., a delta function

We can re-write (18-9), in wave mechanics notation, as

$$U(x_i,x_f;T)=\int_{-\infty}^{+\infty}\delta\left(x-x_f\right)\psi\left(x,T\right)dx=\psi(x_f,T) \tag{18-10}$$

So U for position eigenstate at x_f, $\to |U|^2$= probability density at x_f

[1] There are different ways to normalize position eigenstates. Here we use what is easiest to understand for our purposes. Also, in practice, a position measurement is always over finite Δx, not dx, so our initial delta function actually corresponds to a very narrow, very high real world wave packet (with the standard normalization, such that the square of its absolute value is probability density).

Thus, $$\left|U\left(x_i, x_f; T\right)\right|^2 = \left|\psi\left(x_f, T\right)\right|^2 = \psi^*(x_f, T)\,\psi(x_f, T) = \begin{cases}\text{probability density of measuring} \\ \text{particle at } x_f \text{ at time } T.\end{cases} \tag{18-11}$$

Modification to definition: Hence, from (18-10), the square of the absolute value of the transition amplitude for eigenstates of position (with the chosen normalization and considering the initial state a very high, very narrow wave packet), is *probability density, not probability*, as was the case for energy eigenstate wave functions of form (18-5).[1]

As we will see, the value found using the RHS of (18-9), i.e., that of the Schrödinger approach, is the same as the value found using Feynman's many paths approach.

18.4 Expressing the Wave Function Peak in Terms of the Lagrangian

18.4.1 Background

Path integral approach expresses wave function peak in terms of Lagrangian

One of Feynman's assumptions for his path integral approach to NRQM, RQM (relativistic quantum mechanics), and QFT was to express the wave function value at the peak of a wave packet (see Fig. 18-1) in terms of the Lagrangian (exact relation shown at the end of this section 18.4). I have never seen much justification for this in the literature, other than it is simply an assumption that works (so learn to live with it and move on!)

In the present section I take a different tack, by providing rationale for why we could expect Feynman's expression for the value of the wave function peak to work. The logic herein may well parallel what went on in Feynman's mind as he was developing his path integral approach.

18.4.2 Deducing Feynman's Phase Peak Relationship

The Simplified, Heuristic Argument

Heuristic way to deduce $\psi_{peak} = Ae^{i\int \frac{L}{\hbar}dt}$

In NRQM, the plane wave function solution to the Schrödinger equation,

$$\psi = Ae^{-i(Et - \mathbf{p}\cdot\mathbf{x})/\hbar} , \tag{18-12}$$

means the phase angle, at any given $\mathbf{x}$ and t, is

$$\phi = -\left(Et - \mathbf{p}\cdot\mathbf{x}\right)/\hbar \quad . \tag{18-13}$$

If we have a particle wave packet, it is an aggregate of many such waves, so it is not in an energy or momentum eigenstate. However, it does have energy and momentum expectation values that correspond to the classical values for the particle. The wave packet peak travels at the wave packet group velocity, which corresponds to the classical particle velocity.

Now, imagine that we approximate the wave packet with a (spatially short) wave function such as ψ, where E and $\mathbf{p}$ take on the values of the wave packet expectation values for energy and momentum, respectively. If $\mathbf{x}$ represents the position of the wave packet "peak" (the middle of our approximated wave function ψ), the time rate of change of phase at $\mathbf{x}$ is then

$$\frac{d\phi}{dt} = \frac{-\left(E - \mathbf{p}\cdot\mathbf{v}\right)}{\hbar} = \frac{-T - V + \mathbf{p}\cdot\mathbf{v}}{\hbar} , \tag{18-14}$$

where $\mathbf{v}$ is the velocity of the wave peak, T is kinetic energy, and V is potential energy. Non-relativistically,

$$T = \tfrac{1}{2}mv^2 \qquad \mathbf{p} = m\mathbf{v} \quad \rightarrow \quad \mathbf{p}\ \mathbf{v} = 2T , \tag{18-15}$$

so, in terms of the classical Lagrangian L, (18-14) becomes

$$\frac{d\phi}{dt} = \frac{T - V}{\hbar} = \frac{L}{\hbar} \quad . \tag{18-16}$$

More formally, using the Legendre transformation

$$H = p_i\dot{q}_i - L \quad \left(E = \mathbf{p}\cdot\mathbf{v} - L \quad \text{here}\right), \tag{18-17}$$

directly in (18-14), after the first equality, we get (18-16).

Thus, from (18-16), the phase difference between two events the particle traverses is

[1] This definition of U differs from that of other authors. We address this in Sect. 18.9.1 and the appendix.

We would then repeat that procedure for every other point on the screen. For a fixed source at (x_i, y_i) , and a fixed x_f for the screen, the amplitude would be spatially only a function of y_f, and we could express it simply as $U(y_f)$.

18.6.5 Finding the Proportionality Constant: By Example

Feynman result of summation is only proportional to $|U|^2$. Need to find proportionality constant another way.

The square of the absolute value of the amplitude U is the probability density. So we can normalize U over the length of the screen, i.e.,

$$\int_{y_f=-\infty}^{y_f=+\infty}\left|C\lim_{N\to\infty}\sum_{j=1}^{N}e^{iS_j/\hbar}\right|^2 dy_f=\int_{y_f=-\infty}^{y_f=+\infty}\left|U\left(y_f\right)\right|^2 dy_f=1\,, \qquad (18\text{-}27)$$

and thus, once the value of the limit is determined, readily find the proportionality constant C.

18.7 Summary of Approaches

18.7.1 Feynman's Postulates

Path integral four starting postulates

Richard Feynman was probably well aware of much of the foregoing when he speculated on the viability of the following four postulates for his many paths approach. Subsequent extensive analysis by Feynman and many others has validated his initial speculation.

The postulates of the many paths approach to quantum theories are:

1. A particle is assumed classical in the sense that it can be considered a point-like object, with both its position and its 3-momentum well defined along each individual path, so those values determine the Lagrangian at any point and time along any given path. However, the particle is assumed quantum mechanical in that, like a wave function, it has a phase (at the point).
2. The phasor value at any final event is equal to $e^{iS/\hbar}$ where the action S is calculated along a particular path beginning with a particular initial event.
3. The probability density for the final event is given by the square of the magnitude of a typically complex amplitude.
4. That amplitude is found by adding together the phasor values at that final event from all paths between the initial and final events, including classically impossible paths. The amplitude of the resultant summation must then be normalized relative to all other possible final events, and it is this normalized form of the amplitude referred to in 3.

Note two things.

Phasors are not weighted when summing them

First, there is no weighting of the various path phasors. The nearly classical paths are not weighted more heavily than the paths that are far from classical. That is, the different individual paths in the summation do not have different amplitudes (see (18-24) and Fig. 18-3). The correlation with the classical result comes from destructive interference among the paths far from classical, and constructive interference among the paths close to classical.

Second, time on all paths (all histories) must move forward. This is implicit in the exponent phase value of (18-19), where the integral of L is over time, with time moving forward. Our paths do not include particles zig-zagging backward and forward through time[1].

18.7.2 Comparison of Approaches to QM

Comparing 3 equivalent approaches to NRQM

Wholeness Chart 18-2 summarizes the major similarities and differences between alternative approaches to NRQM.

[1] Caveat: A famous quote by Freeman Dyson states that Feynman, while speculating on this approach, told him that one particle travels all paths, including those going backward in time. But the usual development of the theory (see Section 18.6) only includes paths forward in time. Perhaps all paths backward in time sum to zero and so are simply ignored. In such case, Dyson's quote would be accurate. But I have not personally investigated this and do not know for sure.

Wholeness Chart 18-2. Equivalent Approaches to Non-relativistic Quantum Mechanics

	<u>Schrödinger Wave Mechanics</u>	<u>Heisenberg Matrix Mechanics</u>	<u>Feynman Many Paths</u>
Probability Density of Position Eigenstates	$\lvert\text{amplitude}\rvert^2$		$\lvert\text{amplitude}\rvert^2$
Transition Amplitude	$U(x_i, x_f; T) = \langle x_f \lvert e^{-iHT/\hbar} \rvert x_i \rangle$	Same results as other two approaches.	$U(x_i, x_f; T) = C \lim_{N\to\infty} \sum_{j=1}^{N} e^{iS_j/\hbar}$ $= C\int_{x_i}^{x_f} e^{i\int_0^T \frac{L}{\hbar}dt} \mathcal{D}x(t)$
Comments	Above interpretation assumes $\lvert x_i\rangle$ is high narrow wave packet and $\lvert x_f\rangle$ is a pure delta function in position space		Need to determine C. Some others include C in definition of $\mathcal{D}x(t)$. We haven't done the integral part yet.

18.8 Finite Sums to Functional Integrals

18.8.1 Time Slicing: The Concept

Slicing time into "pieces" for discrete time analysis

After all of the foregoing groundwork, it is time to extend the phasor sum of a finite number of paths, such as we saw in Fig. 18-3 and (18-24), over into an infinite sum, or in other words, an integral. To do this, we first consider finite "slices" of time, for a finite number of paths in one spatial dimension, as shown in Fig. 18-5 where, for convenience, we plot time vertically and space horizontally. As opposed to our spatially 2D example in Fig. 18-3, different paths in Fig. 18-5 actually refer to the particle traveling along the x axis only between i and f, though at varying (both positive and negative) velocities. The paths between each slice are straight lines, but there is no loss in generality, as one can take the time between slices $\Delta t \rightarrow dt$, and thus, any possible shape path can be included.

A simple example

As noted earlier, for any <u>single path</u>, the

$$\text{phasor at } \mathbf{f} = \underbrace{e^{i\int_{t_i}^{t_f}\frac{L}{\hbar}dt} = e^{iS/\hbar}}_{\text{one path}} \quad , \tag{18-28}$$

The amplitude U for the transition from **i** to **f** is proportional to the sum of (18-28) for <u>all paths</u>,

$$\text{sum of } \infty \text{ phasors at } \mathbf{f} = \lim_{N\to\infty} \sum_{j=1}^{N} e^{iS_j/\hbar} \quad . \tag{18-29}$$

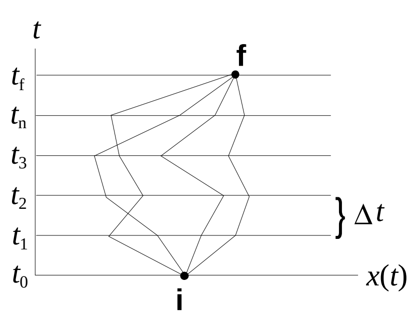

Figure 18-5. Time Slicing for Finite Number of Paths

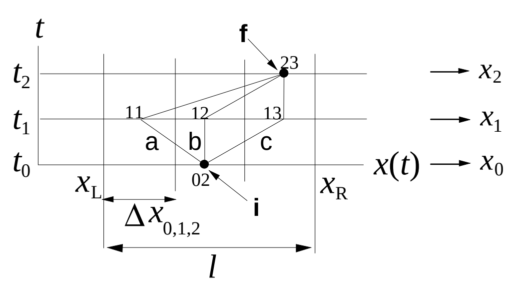

Figure 18-6. Space Slicing for Three Discrete Paths

18.8.5 Practicality and Calculations

Practically, for the first approximation addressed in Section 18.8.4, we really don't have to take l to infinity, as we know that paths outside of a reasonably large range from the initial and final spatial locations will sum to very close to zero. So we can live with significant, but not infinite, l.

Practically can take l large enough to give accurate answer

For the second approximation, we only need small enough Δt such that taking a smaller value does not change our answer much.

If we use (18-38), with judicious choices for Δt and l, we can, in many cases, obtain valid closed form solutions for the amplitude. We can also obtain numerical solutions with a digital computer by using approximations for L between time slices, as we did previously. That is, we can approximate the RHS of (18-38) in the manner we did for the first line of (18-37), but extending the approximation of (18-37) from 3 to n time slices.

And can take Δt small as needed as well

XXXX 2 to 3 herel

18.9 An Example: Free Particle

We will first determine the amplitude (and thus detection probability density) of a free particle via the Schrödinger approach and then compare it to that for Feynman's many paths approach.

Free particle example in NRQM

18.9.1 Schrödinger Transition Amplitude

Recall, from Section 18.3.2, that, in the Schrödinger approach, a position eigenstate is effectively a delta function, and as it evolves, the wave function envelope spreads and the peak diminishes. $|U|^2$ for such functions is the probability density at the final point x_f, after time T. We should then expect $|U|^2$ to decrease as T increases, and to effectively equal infinity at x_{peak} when $T = 0$.

First, via Schrödinger wave mechanics approach

We start with the Schrödinger transition amplitude relation (18-9),

$$U\left(x_i, x_f; T\right) = \left\langle x_f \left| e^{-iHT/\hbar} \right| x_i \right\rangle \tag{18-41}$$

where we take the bra to be a pure delta function and the ket, a normalized wave packet approximation to a delta function. It is simpler mathematically to use a pure delta function to represent $|x_i\rangle$, but then we have to normalize it in a manner similar to a wave packet. There is a lot behind this that we summarize in the appendix (pg. 509), but here, we simply use A to represent the normalization factor in the ket of (18-41).

$$U\left(x_i, x_f; T\right) = \int_{-\infty}^{\infty} \left(\delta\left(x - x_f\right) e^{-iHT/\hbar} A\, \delta\left(x - x_i\right) \right) dx\,, \tag{18-42}$$

with the well-known relations

$$\delta\left(x - x_i\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ik\left(x - x_i\right)} dk = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\frac{p}{\hbar}\left(x - x_i\right)} dp\,. \tag{18-43}$$

(Box 2-3, pg. 27, explains the use of operators in exponents. In essence, one can express the exponential quantity as a Taylor series expanded about $T = 0$, i.e., $f(T) = e^{-iTH/\hbar} = 1 - iTH/\hbar - ½\, T^2H^2/\hbar^2 + \ldots$ Then, operate on the ket/state term by term [getting terms in $iET/\hbar$ to various powers], and finally re-express the resulting Taylor series as an exponential in $iET/\hbar$. We have taken the ket with time $t_i = 0$ to make things simpler, but even if you think of the Hamiltonian operator as a time derivative, when it acts on that ket, it functions as an energy operator and still yields the energy.)

XXXX + to – here before ½

For the exponential with the H operator acting on the initial state, and $E = p^2/2m$, (18-42) is

$$\begin{aligned} U\left(x_i, x_f; T\right) &= \int_{-\infty}^{\infty} \left(\left(\frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\frac{p'}{\hbar}\left(x - x_f\right)} dp' \right) \left(A \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{-\frac{i}{\hbar}TH} e^{i\frac{p}{\hbar}\left(x - x_i\right)} dp \right) \right) dx \\ &= \int_{-\infty}^{\infty} \left(\left(\frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\frac{p'}{\hbar}\left(x_f - x\right)} dp' \right) \left(A \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{-iTp^2/2m\hbar} e^{i\frac{p}{\hbar}\left(x - x_i\right)} dp \right) \right) dx\,, \end{aligned} \tag{18-44}$$

where we took $(x - x_f) \rightarrow (x_f - x)$ in the 2nd line on purpose. We then re-arrange (18-44) to get

$$U\left(x_i,x_f;T\right)=A\frac{1}{2\pi\hbar}\iint e^{-iTp^2/2m\hbar}\underbrace{\left(\frac{1}{2\pi\hbar}\int e^{i\frac{x}{\hbar}(p-p')}dx\right)}_{\delta(p-p')}e^{\frac{i}{\hbar}p'x_f}e^{-\frac{i}{\hbar}px_i}dp'dp \qquad (18\text{-}45)$$

$$=A\frac{1}{2\pi\hbar}\int e^{-iTp^2/2m\hbar}e^{\frac{i}{\hbar}p\left(x_f-x_i\right)}dp.$$

Using the integral formula

$$\int_{-\infty}^{+\infty}e^{-ax^2+bx}dx=\sqrt{\frac{\pi}{a}}e^{b^2/4a} \qquad Re(a)>0,\ , \qquad (18\text{-}46)$$

we find

$$U\left(x_i,x_f;T\right)=A\sqrt{\frac{m}{i2\pi\hbar T}}\,e^{\frac{i}{\hbar}\frac{m}{2T}\left(x_f-x_i\right)^2}\ . \qquad (18\text{-}47)$$

The astute reader may question whether (18-46), with complex a and b, converges. It does because the integrand oscillation rate increases with larger $|p|$ in such a way as to make successive cycles shorter. As $|p|$ gets very large, the cycles become so short that the contribution from each cycle (think area under a sine curve) tends to zero, and it does so in a manner that allows the integral to converge. Said another way, the smaller and smaller contributions as $|p|$ gets large alternate between positive and negative values (for both real and complex portions), and thus convergence is assured.

Probability density at final event for free particle via Schrödinger approach

From (18-47), the probability density at event **f** is

$$\left|U\left(x_i,x_f;T\right)\right|^2=A^2\frac{m}{2\pi\hbar T}, \qquad (18\text{-}48)$$

which, as we said it must, decreases with increasing T, and equals infinity for $T = 0$[1].

18.9.2 Many Paths Transition Amplitude

Now, for free particle via path integral approach

We now seek to derive (18-47) using the many paths approach.

A free, non-relativistic particle has Lagrangian (all values are wave packet expectation values, e.g., $x_f = \bar{x} = x_{peak}$, $v = v_g$)

$$L=\tfrac{1}{2}mv^2\approx\tfrac{1}{2}m\left(\frac{x(t+\Delta t)-x(t)}{\Delta t}\right)^2, \qquad (18\text{-}49)$$

where the RHS is an approximation between adjacent time slices. Taking $t_i = 0$, and $l \to \infty$ (see Fig. 18-6, pg. 498), (18-38) becomes

$$U(i,f;T)\approx C\int_{x_n=-\infty}^{x_n=\infty}\ldots\int_{x_2=-\infty}^{x_2=\infty}\int_{x_1=-\infty}^{x_1=\infty}e^{i\int_{t_n}^{t_f=T}\frac{L}{\hbar}dt}e^{i\int_{t_{n-1}}^{t_n}\frac{L}{\hbar}dt}\ldots e^{i\int_{t_1}^{t_2}\frac{L}{\hbar}dt}e^{i\int_0^{t_1}\frac{L}{\hbar}dt}dx_1dx_2\ldots dx_n$$

$$\approx C\int_{x_n=-\infty}^{x_n=\infty}\ldots\int_{x_2=-\infty}^{x_2=\infty}\int_{x_1=-\infty}^{x_1=\infty}e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_f-x_n}{\Delta t}\right)^2\right\}\Delta t}\ldots e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_2-x_1}{\Delta t}\right)^2\right\}\Delta t}e^{\frac{i}{\hbar}\left\{\frac{1}{2}m\left(\frac{x_1-x_i}{\Delta t}\right)^2\right\}\Delta t}dx_1dx_2\ldots dx_n\,. \qquad (18\text{-}50)$$

$$=C\int_{x_n=-\infty}^{x_n=\infty}\underbrace{e^{\frac{im}{2\hbar(\Delta t)}\left(x_f-x_n\right)^2}}_{f_\zeta}\ldots\int_{x_2=-\infty}^{x_2=\infty}\underbrace{e^{\frac{im}{2\hbar(\Delta t)}\left(x_3-x_2\right)^2}}_{f_\gamma}\underbrace{\int_{x_1=-\infty}^{x_1=\infty}\underbrace{e^{\frac{im}{2\hbar(\Delta t)}\left(x_2-x_1\right)^2}}_{f_\beta}\underbrace{e^{\frac{im}{2\hbar(\Delta t)}\left(x_1-x_i\right)^2}}_{f_\alpha}dx_1}_{f(x_2)}dx_2\ldots dx_n\ ,$$

[1] Note that probability density for a wave function that is an exact delta function at time $T = 0$, is a straight line any time $T > 0$. This may seem confusing, but that is what (18-48) (with no x_f dependence) tells us. For a wave packet approximation to a delta function (instead of an exact delta function), for $T > 0$, we have the behavior as in Fig. 18-1, pg. 491.

$$U\left(x_i, x_f; T\right) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\tilde{U}\left(p\right)e^{\frac{i}{\hbar}p\left(x_f - x_i\right)}dp$$
$$\approx \frac{1}{2\pi\hbar}\left(\frac{i2\pi\hbar\left(\Delta t\right)}{m}\right)^{N/2} C\int_{-\infty}^{\infty} e^{-\frac{i}{\hbar}\frac{T}{2m}p^2} e^{\frac{i}{\hbar}p\left(x_f - x_i\right)}dp. \qquad (18\text{-}58)$$

In (18-58), we could have simply used x_f in the exponent, as we have been taking $x_i = 0$, and our result would have been in terms of x_f. In that case, x_f would have been the distance between x_i and x_f, i.e., $x_f - x_i$. In order to frame our final result in the most general terms, we re-introduced x_i as having any coordinate value in (18-58).

With (18-46) again, (18-58) becomes

$$U\left(x_i, x_f; T\right) \approx C\left(\frac{i2\pi\hbar\left(\Delta t\right)}{m}\right)^{N/2}\sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}\frac{m}{2T}\left(x_f - x_i\right)^2}. \qquad (18\text{-}59)$$

By comparison with (18-47), we see the phase and dependence on T is the same as in the wave mechanics approach. Using that comparison, we can see that the constant of proportionality is

$$C = A\left(\frac{m}{i2\pi\hbar\left(\Delta t\right)}\right)^{N/2}. \qquad (18\text{-}60)$$

And thus, the probability density at the final event **f** is the same as (18-48), i.e.,

$$\left|U\left(x_i, x_f; T\right)\right|^2 = A^2\frac{m}{2\pi\hbar T}, \qquad (18\text{-}61)$$

where the equal sign is appropriate for $N \to \infty$.

We can find the normalization factor A by integrating $|U|^2$ over all space and setting the result to one, as is usual in NRQM. (See the appendix, pg. 509, for more on this.)

Path integral approach = Schrödinger approach

Note that for $v = (x_f - x_i)/T$, the amplitude (18-59) can be expressed in terms of the classical action as

$$U\left(x_i, x_f; T\right) = A\sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}\frac{mv^2}{2}T} = A\sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}LT} = A\sqrt{\frac{m}{i2\pi\hbar T}}\, e^{\frac{i}{\hbar}S} \qquad x_{peak} = \bar{x} = x_f. \qquad (18\text{-}62)$$

18.9.3 The Message

Pluses and minuses of path integral method

It has probably not escaped the reader that the evaluation of a free particle using Feynman's many paths approach is considerably more complicated and lengthy than the Schrödinger approach. This is true for most, if not all, problems in NRQM and RQM.

The <u>disadvantages</u> of the many paths approach in NRQM and RQM are these.

1. It is generally more mathematically cumbersome and time consuming than the wave mechanics approach.

More limited and generally harder

2. The quantity calculated is only proportional to the amplitude, and further analysis is required to determine the precise amplitude.

3. The approach is suitable primarily for position eigenstates and is not readily amenable to more general states, so it is generally not as encompassing in nature.

The <u>advantages</u> of the many paths approach are these.

Has some advantages for QFT

1. The approach also applies to QFT. In a number of instances therein, development of the theory is more direct, and calculation of amplitudes is easier, than with the alternative approach (canonical quantization).

2. Philosophically, we see that there is more than one way to skin a cat. We learn anew that the physical world can be modeled in different, equivalent ways. We learn caution with regard to interpreting a given model as an actual picture of reality.

	Particle Theory	**Field Theory**
	Quantum Theories	
	NRQM and RQM via Wave Mechanics	QFT via Wave Mechanics = Canonical Quantization
Quantum character change	x and all dynamical variables $\rightarrow$ operators	ϕ and all dynamical variables $\rightarrow$ operators
New quantum entity	state $\lvert\psi\rangle$ = wave function ψ	state $\lvert\phi\rangle$ different from (operator) field ϕ
Note		Fields create & destroy states. States can be multi-particle ($\lvert\phi_1,\phi_2,\ldots\rangle$)
Operators	functions of $x,\dot{x},t$	functions of $\phi,\phi_{,\mu},t$
Expectation values of operators	$\bar{E}=\langle\psi\lvert H\rvert\psi\rangle$ etc. for other operators	$\bar{E}=\langle\phi\lvert H\rvert\phi\rangle$ or for multi-particle state $\bar{E}=\langle\phi_1,\phi_2\ldots\lvert H\rvert\phi_1,\phi_2\ldots\rangle$
Equations of motion	For wave function ψ QM: Schrödinger eq RQM: Klein-Gordon, Dirac, Maxwell, Proça eqs or equivalently, Euler-Lagrange formulations	For quantum field ϕ QFT: Klein-Gordon, Dirac, Maxwell, Proça eqs or equivalently, Euler-Lagrange formulations
Macro equations of motion	Deduced from above and expectation values of force, acceleration	Deduced from above and expectation values of relevant quantities
Transition amplitude U (finite T)	$U\left(x_i,x_f;T\right)=\langle x_f\lvert e^{-iHT}\rvert x_i\rangle$ i & f are eigenstates of position	$U\left(\phi_i,\phi_f;T\right)=\langle\phi_f\lvert e^{-iHT}\rvert\phi_i\rangle$ i & f states can be multi-particle
$\lvert U\rvert^2$ =	probability density (for normalizations chosen herein)	probability

XXX deleted comma before "eqs" in last column

18.10.2 "Derivation" of Many Paths Approach for QFT

Extend the same analogies to path integrals

From the next to last row of Wholeness Chart 18-4, we see that the transition amplitude for the QFT canonical approach, which is essentially a wave mechanics approach for relativistic fields, is similar in form to that of the NRQM/RQM wave mechanics approach, given that we note the correspondence $x \rightarrow \phi$ between NRQM/RQM and QFT. An additional fundamental difference between NRQM and QFT is the form of the Hamiltonian H. In NRQM, H is a non-relativistic function of x, dx/dt, and (rarely) t. In QFT, it is a relativistic function of ϕ, $d\phi/dt$, and (rarely) t.

Since the canonical (wave mechanics) QFT approach mirrors the wave mechanics NRQM/RQM approach, one could postulate (and Feynman probably did) that the many paths approach in QFT would mirror the many paths approach in NRQM/RQM. (See Wholeness Chart 18-2 in Section 18.7.2 for the corresponding NRQM transition amplitudes using each approach.) Simply using the same correspondences $x \rightarrow \phi$ and $H_{nonrel} \rightarrow H_{rel}$ (and thus, $L_{nonrel} \rightarrow L_{rel}$) for the many paths approach yields Wholeness Chart 18-5.

18.10.4 More Ahead in Path Integral QFT

Note that we have only scratched the surface of the many paths approach to QFT. There is a great deal more, including some fairly fundamental concepts. However, hopefully, all of the above will provide a solid foundation for that, by explaining more simply, more completely, and in smaller steps of development what traditional introductions to the subject often treat more concisely.

There is more to learn about QFT path integrals. This was an intro.

18.11 Chapter Summary

It is time to try your hand at creating a wholeness chart summary by doing Prob. 1.

18.12 Appendix XXX WHOLE APPENDIX IS NEW. SUBSTITUTED FOR PRIOR XXX

There are issues with normalization of position eigenstates that complicate the interpretation of the resulting transition amplitude for an initial position eigenstate transitioning to a final position eigenstate. To investigate this, we first consider A to be a normalization constant to be determined in

We will compare normalization constants for different methods of normalization for position eigenstates

$$\left|x_j\right\rangle \xrightarrow[\text{space}]{\text{in position}} = A\delta\left(x-x_j\right), \tag{18-65}$$

and then determine A for different ways to normalize.

18.12.1 Standard NRQM/RQM Normalization

In standard NRQM, eigenstates are generally orthonormal. For (18-66), this means

$$\left\langle x_j \middle| x_k \right\rangle = \delta_{jk} \xrightarrow[\text{space for } j=k]{\text{in position}} \int \underbrace{A^*\delta\left(x-x_j\right)A\overbrace{\delta\left(x-x_j\right)}^{f(x)}}_{\text{probability density}}dx = 1, \tag{18-66}$$

Standard NRQM normalization

where the square of the absolute value of the wave function equals probability density and the total probability of measuring the position eigenstate anywhere in space is one, as it should be. If we consider one of the delta functions to be just like any function of x, $f(x)$, then (18-66) leads to

XXX change "is" to "in"

$$|A|^2 f\left(x_j\right) = |A|^2\delta\left(x_j-x_j\right) = |A|^2\delta(0) = 1 \quad \to \quad A = \frac{1}{\sqrt{\delta(0)}}. \tag{18-67}$$

Standard NRQM normalization constant

While at first blush it may seem strange to have a factor with the square root of infinity in the denominator, it is not much different from having a wave function like $Ae^{-i(\omega t-kx)}$ that extends from – ∞ to + ∞ along the x axis. In that case, $A = 1/\sqrt{\infty}$ as well. So, if we can live with this hypothetically pure position eigenstate, then for NRQM, as usually done, (18-65) becomes

$$\left|x_j\right\rangle \xrightarrow[\text{in position space}]{\text{NRQM normalization}} = \frac{1}{\sqrt{\delta(0)}}\delta\left(x-x_j\right), \tag{18-68}$$

Standard NRQM normalized position eigenstate

and probability density is

$$\rho_{NRQM}(x) = \frac{\left(\delta\left(x-x_j\right)\right)^2}{\delta(0)} = \delta\left(x-x_j\right). \tag{18-69}$$

Standard NRQM total probability is one

(18-69) is infinite at $x = x_j$ and zero elsewhere. Total probability, its integral over all space, is one.

For this normalization of both bra and ket, the transition amplitude U and $|U|^2$ are

$$U_{NRQM}\left(x_i, x_f; T\right) = \left\langle x_f \middle| e^{-iHT/\hbar} \middle| x_i \right\rangle \quad \to \quad \left|U_{NRQM}\right|^2 = \text{total probability of transition}, \tag{18-70}$$

Standard NRQM $|U|^2$ is total transition probability

which is what we have come to expect $|U|^2$ to represent.

18.12.2 Normalization Found in Other QFT Texts

Other QFT texts, when discussing the path integral approach, use a different normalization[1],

[1] See Peskin. M. &Schroeder, D., *An Introduction to Quantum Field Theory* (Perseus 1995), pg. 277, (9-3) LHS and the first sentence in the paragraph beginning after (9-7) on pg. 279. See Zee, A., *Quantum Field Theory in a Nutshell* (Princeton 2010), pg. 10, 3rd line down under heading "Dirac's formulation".

$$\langle x_j | x_k \rangle = \delta(x_j - x_k) \xrightarrow[\text{space for } j=k]{\text{in position}} \int A^* \delta(x - x_j) A \delta(x - x_j) dx = \delta(0) . \qquad (18\text{-}71)$$

Other texts normalization

Taking one of the delta functions on the RHS of (18-71) as $f(x)$ as we did above, we find

$$|A|^2 \delta(x_j - x_j) = |A|^2 \delta(0) = \delta(0) \quad \rightarrow \quad A = 1 , \qquad (18\text{-}72)$$

Other texts normalization constant

so (18-65) becomes

$$|x_j\rangle \xrightarrow[\text{in position space}]{\text{other texts normalization}} = \delta(x - x_j) . \qquad (18\text{-}73)$$

Other texts normalized position eigenstate

Note that what we generally consider probability density is

$$\rho_{\substack{other \\ texts}}(x) = \left(\delta(x - x_j)\right)^2 , \qquad (18\text{-}74)$$

Other texts total probability is infinity

and the integral of (18-74) over all space, what we usually interpret as total probability, is infinite (and thus cannot represent total probability).

For this normalization of both bra and ket, the transition amplitude U and $|U|^2$ are

$$U_{\substack{other \\ texts}}(x_i, x_f ; T) = \langle x_f | e^{-iHT/\hbar} | x_i \rangle \xrightarrow[\text{for } T=0 \text{, i.e., } i=f]{\text{in position space}} \left| U_{\substack{other \\ texts, i=f}} \right|^2 = \infty^2 \quad \begin{pmatrix} \text{no physical} \\ \text{interpretation} \end{pmatrix} . \qquad (18\text{-}75)$$

Other texts $|U|^2$ has no physical meaning

If there is no physical interpretation when $T = 0$, it follows that there is none when $T \neq 0$.

18.12.3 Hybrid Normalization Found in This Text

This text assumes ket is normalized as in standard NRQM. i.e.,as if it were a wave packet

In Sect. 18.3.2, pg. 491, we considered a surrogate for the initial position eigenstate ket to be a high, narrow wave packet approximating a delta function, but, importantly, normalized as is usual in NRQM (see (18-66)), i.e.,

$$\langle x_i | x_i \rangle = 1 \qquad |x_i\rangle \text{ a high, narrow wave packet approx to position eigenstate} . \qquad (18\text{-}76)$$

That, along with considering the bra to be a pure delta function, as is (18-73), let us interpret $|U|^2$ of (18-11) as probability density. Had we taken both the ket and the bra as pure delta functions, such as in (18-73), we would have no readily comprehensible physical meaning for $|U|^2$. (See (18-75).) Since the path integral approach yields a quantity that is proportional to the probability density, I, the author, felt it best to present the background NRQM material in a manner amenable to correlating it with that approach.

But this text assumes bra is normalized as in other texts, i.e., as if it were a pure delta function

However, in Sect. 18.9.1, pg. 502, the math is greatly simplified by using an actual delta function for the initial ket, rather than a limiting case wave packet. But, we then need to normalize our initial delta function ket to satisfy (18-76). That is, we need the initial ket of delta function form to have $A = 1/\sqrt{\delta(0)}$, i.e., to be of form (18-68).

So, the normalization constant A in Sects. 18.9.1 and 18.9.2 equals $1/\sqrt{\delta(0)}$, but I felt that introducing the square root of infinity at that point would be inordinately confusing and take us away from the main purpose of the section.

This gives us a $|U|^2$ that is probability density and can thus be readily related to path integral result

These three approaches to normalization of position eigenstates, and the ramifications of each, are summarized in Wholeness Chart 18-7 on the next page.

18.12.4 Bottom Line

All of this appendix is focused on NRQM, and is not so relevant to QFT, except as part of an introduction to the path integral methodology. It is just background for the most important concept in the chapter, stated below.

Bottom line: The path integral approach result is proportional to probability density. We only discuss position eigenstates as an aid to developing that concept and later, to extrapolating it to QFT.

The important thing: what we find in path integral approach is proportional to probability density

Wholeness Chart 18-7. Comparing Normalization Methods for Position Eigenstates

	Standard	**Some Other Texts**	**This Text**
Position Eigenstate	$\left\vert x_j \right\rangle$	as at left	as at left
In position space	$\frac{1}{\sqrt{\delta_L(0)}}\delta\left(x-x_j\right)$	$\delta\left(x-x_j\right)$	ket: $\psi_{x_k}(x)$ = high, narrow normalized wave packet bra: $\delta\left(x-x_j\right)$
Normalization	$\left\langle x_j \vert x_k \right\rangle = \delta_{jk}$	$\left\langle x_j \vert x_k \right\rangle = \delta\left(x_j - x_k\right)$	$\left\langle x_j \middle\vert \psi_{x_k}(x) \right\rangle = \psi_{x_k}\left(x_j\right)$ = wave function value at x_j
Total probab of measuring (take $j = k$)	$\left\langle x_k \vert x_k \right\rangle$ = unity	$\left\langle x_k \vert x_k \right\rangle$ = infinity	$\left\langle x_k \middle\vert \psi_{x_k}(x) \right\rangle$ not total probability, but wave function value at x_k
Transition Amplitude *U*	$U\left(x_i, x_f, T\right) = \left\langle x_f \middle\vert e^{-iHT} \middle\vert x_i \right\rangle$ in each case		
$\vert U\vert^2$ represents	Total probability of measuring x_f state at time T.	Not total probability, nor probability density of measuring x_f at time T.	Probability density of evolved state.
Pro	1) Usual NRQM analysis. 2) Easy to visualize.	What most authors use.	1) Easy to visualize. 2) Easier to accept probability density than ∞ as substitute for total probability. 3) Easier to relate to path integral result, which is proportional to probability density.
Con	1) Not easy to see how total probability related to probability density, which is what we want to get from path integral approach. 2) Trying to do this with ∞ in denominator can be confusing.	1) ∞ probability, contradicts NRQM. 2) Impossible to visualize meaning of $\vert U\vert^2$. Students have no idea what U means.	1) Probability density not total probability, contradicts NRQM. 2) Not what most (any other?) authors use.
Bottom line	1) Path integral approach result is proportional to probability density. Don't need to use any of above approaches if set integral of that result over all space equal to one and solve for proportionality constant C. (But then no real proof that path integral approach yields what is claimed.) 2) Can compare any of above three wave mechanics interpretations to path integral approach to determine proportionality constant C. Just need to interpret final result as equal to that of the particular wave mechanics approach used, i.e., the meaning in "$\vert U\vert^2$ represents" row above. (Provides a proof that wave mechanics and path integral approaches are consonant.)		

18.13 Problem

1. Create a wholeness chart summarizing this chapter.