

Chapter 18 *print vers 3/23/14 Copyright of Robert D. Klauber*

***Path Integrals in Quantum Theories:
A Pedagogic First Step***

*The universe in each dimension
is vast beyond all comprehension.*

*A myriad of mysteries,
a multitude of histories ...*

From *Divine Intentions*
by R. Klauber

18.0 Preliminaries

As I mentioned on the first page of the book, I strongly believe it far easier, and more meaningful, for students to learn quantum field theory (QFT) first by the canonical quantization method, and once that has been digested, move on to the path integral (functional integral, many paths, or sum over histories) approach (functional quantization). The rest of the book is devoted to the first of these; the present chapter, to a brief introduction to the second.

*Two approaches
to (ways to
quantize) QFT:
1) canonical
2) path integral*

18.0.1 Chapter Overview

This chapter was composed so it can be read independently of (without reading) the rest of the book. So, some things may be defined/discussed again herein that are covered elsewhere in the text.

*#1 simpler, rest
of book;
#2 introduced in
this chapter*

In this chapter, we will define

- the functional and
 - the functional integral,
- then, with regard to non relativistic quantum mechanics (NRQM),
- transition amplitudes for position eigenstates,
 - the role of the Lagrangian and the wave function peak,
 - the central idea in Feynman's path integral approach,
 - expressing that idea mathematically, including Feynman's three postulates,
 - comparing the path integral approach in NRQM to Schrödinger and Heisenberg's,
 - determining the transition amplitude from the functional integral, and
 - applying the theory to an example.

Then, with regard to QFT, we will investigate

- comparing particle theory (NRQM) to field theory (QFT)
- "derivation" of the many paths approach to QFT, and
- deducing the form of the transition amplitude for QFT

*We'll examine
path integrals:
- math behind
- NRQM
- QFT*

18.1 Background Math

18.1.1 Integrating Functions of a Function

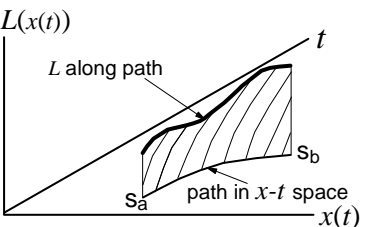
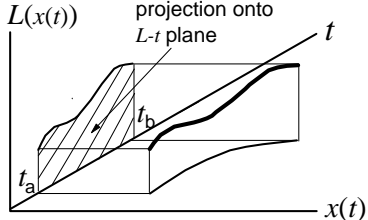
Functionals form the mathematical roots of Feynman's many paths approach to quantum theories. To help in understanding the concept, consider first a function of another function, such as

the Lagrangian of a particle, which is typically a function of particle position x and its time derivative \dot{x} . x , in turn, is a function of time t , i.e., $x(t)$, and finding that functional dependence on time comprises typical problems to be solved.

There are several ways we can integrate such a function of another function, two being shown in Wholeness Chart 18-1 (Part A) below. The figures and comments in that chart should be self explanatory. Mathematically, L can be any function of a function, but for our purposes, it will generally be the Lagrangian.

Integrating a function of a function

Wholeness Chart 18-1. From a Function of a Function to the Functional Integral– Part A

	<u>Process</u>	<u>Graphically</u>	<u>Math</u>	<u>Comment</u>
1.	Integration over the path in $x(t)$ vs t space = area shown		$\int_{S_a}^{S_b} L ds$ where s is spacetime distance along path	L is a function of the function x (and \dot{x}), and the functional dependence of x on t is typically the problem to be solved. Integration shown is not relevant for us.
2.	Integration over t = projection of the area in #1 onto the L - t plane		$F = \int_{t_a}^{t_b} L dt$ F is a <u>functional</u>	If L is the Lagrangian, then this integral $F = S$, the action. Classically, S = minimum (or stationary) for physical paths

18.1.2 Defining “Functional”

In the path integral approach to quantum physics, we use a narrower definition of a functional than the general mathematical definition¹. We define integration #2 above, the integral of the function (L) of a function ($x(t)$) with respect to the independent variable (t) between fixed limits t_a and t_b as a functional, and designate it as F . It is a number that depends on the form of the function $x(t)$, on t_a , and on t_b . It is different for different paths.

Our definition of a functional F tailored to quantum physics

$$F = \int_{t_a}^{t_b} L dt \quad (\text{for a particular path}) \quad (18-1)$$

That definition

Functionals are symbolized by enclosing their arguments in square brackets.

Symbolism: $F[x(t)]$ or $F[x]$, (18-2)

though, nevertheless, you may see functionals written with normal, rather than square, brackets.

If L is the Lagrangian, then the functional $F = S$, the action.

If L = Lagrangian, F = S, the action

18.2 Defining Functional Integral

A functional, as we have seen, is a definite integral, i.e., a number obtained by integrating between the end points of a certain path. Yet, because we get a different such number for each different path in x - t space, we can integrate those numbers over all possible paths. In other words, the functional, an integral, can itself be integrated. Such integrations are not simple, nor is their purpose at all obvious at this point. They are visualized in cases #4 and #7 below and are called functional integrals. We devote much of this chapter to explaining their origin, value, and means to evaluate. For now, just let it float easily through your head without straining to understand it.

The functional integral is an integral (over all paths) of the functional F (which is itself an integral)

¹ Mathematically, a functional is a function of a vector space to a scalar field, i.e., a functional maps a vector to a scalar. Spatial functions of time, i.e., paths, form a vector space by themselves, so our narrower definition is in line with the general definition. In our case, the mapping involves an integration.

Wholeness Chart 18-1 (continued). From a Function of a Function to the Functional Integral – Part B

Procedure	Graphically	Math	Comment
3. Sum F values as in #2 above for a number of discrete paths between a and b .		$\sum_{n=1}^4 F_n = \sum_{n=1}^4 \int_{t_a}^{t_b} L_n dt$	Not relevant for us.
4. Integrate F over all possible (continuous range of) paths between a and b .	Hard to show visually.	$\int_{x_a}^{x_b} F \mathcal{D}x(t)$	Not relevant for us. $\mathcal{D}x(t)$ implies all paths.
5. Another function of F (i.e., where F is the argument), e.g. exponentiation of F .	Not graphic. Raise e to i times value F for a given path.	$e^{iF[x(t)]} = e^{i \int_{t_a}^{t_b} L dt}$	Relevant for us.
6. Sum e^{iF} values for a number of discrete paths, like in #3 above.	Same paths as in #3.	$\sum_{n=1}^4 e^{iF_n} = \sum_{n=1}^4 e^{i \int_{t_a}^{t_b} L_n dt}$	Relevant for us.
7. Integration like #4 above over all possible paths in $x(t)$ vs t space.	Hard to show visually. Same paths as in #4.	$\int_{x_a}^{x_b} e^{iF} \mathcal{D}x(t)$	Feynman QM path integral approach. All paths, not just classical.

The chart above should be relatively self explanatory. In summary, we can add the values F_n for a discrete number of paths N ($= 4$ in #3). In the limit of adding all paths, we pass to an integral (don't worry how for now), where we use the symbol $\mathcal{D}x(t)$ to represent that functional integration.

$$\sum_{n=1}^N F_n = \sum_{n=1}^N \int_{t_a}^{t_b} L_n dt \xrightarrow[\text{total paths } N \rightarrow \infty]{\text{limit as}} \int_{x_a}^{x_b} F \mathcal{D}x(t) \quad (\text{not relevant for us}). \quad (18-3)$$

Adding F for all paths \rightarrow integration over all paths = functional integration

Alternatively, we can do the same thing for a function of F , such as e^{iF} (as in #6 and #7 above). Note that e^{iF} can itself be considered a functional, as it comprises a mapping from $x(t)$ to a (complex) scalar.

$$\sum_{n=1}^N e^{iF_n} = \sum_{n=1}^N e^{i \int_{t_a}^{t_b} L_n dt} \xrightarrow[\text{total paths } N \rightarrow \infty]{\text{limit as}} \int_{x_a}^{x_b} e^{iF} \mathcal{D}x(t) \quad (\text{will be relevant for us}). \quad (18-4)$$

Instead of F , do functional integration over e^{iF} . This is important in quantum physics

We will evaluate (18-4) for a quantum particle and see what that gives us later in this chapter.

Alternative nomenclature: Because functional integration involves integration over paths (in $x-t$ space), Feynman's approach is often also referred to as the path integral approach.

18.3 The Transition Amplitude

18.3.1 General Wave Functions (States)

Recall from QM wave mechanics, that for a general normalized wave function ψ equal to a superposition of energy eigenfunction waves (which are each also normalized),

$$\psi = A_1\psi_1 + A_2\psi_2 + A_3\psi_3, \quad (18-5)$$

A_1 is the amplitude of ψ_1 , so the probability of finding ψ_1 upon measuring is

$$A_1^* A_1 = |A_1|^2. \quad (18-6)$$

If we were to start with ψ initially, and measure ψ_1 later, the wave function would have collapsed, i.e., underwent a transition to a new state. (18-6) would be the transition probability.

Review of states, amplitudes, & probability

$\psi \rightarrow \psi_1$ transition amplitude = A_1

Probability of transition = $|A_1|^2$