Problem 7. Derive (17-122).

Ans. We will make use of the 2nd row of (17-118), i.e.,

$$\mathcal{M}_{B2} = -ie^2 \bar{u}_{s_1'}(\mathbf{p}_1')\gamma_\nu u_{s_1}(\mathbf{p}_1)\frac{1}{(p_2 - p_2')^2}\bar{v}_{s_2}(\mathbf{p}_2)\gamma^\nu v_{s_2'}(\mathbf{p}_2'). \tag{17-118}$$

With that, we have

$$\frac{1}{4}\sum_{spins}\left|\mathcal{M}_{B2}\right|^2 = \frac{1}{4}\sum_{spins}\mathcal{M}_{B2}\mathcal{M}_{B2}^* = \frac{1}{4}\sum_{spins}\left(-ie^2\bar{u}_{s_1'}(\mathbf{p}_1')\gamma_\alpha u_{s_1}(\mathbf{p}_1)\frac{1}{(p_2 - p_2')^2}\bar{v}_{s_2}(\mathbf{p}_2)\gamma^\alpha v_{s_2'}(\mathbf{p}_2')\right)\times$$
$$\left(-ie^2\bar{u}_{s_1'}(\mathbf{p}_1')\gamma_\beta u_{s_1}(\mathbf{p}_1)\frac{1}{(p_2 - p_2')^2}\bar{v}_{s_2}(\mathbf{p}_2)\gamma^\beta v_{s_2'}(\mathbf{p}_2')\right)^*.$$

Or, rearranged,

$$\frac{1}{4}\sum_{spins}\left|\mathcal{M}_{B2}\right|^2 = \frac{1}{4}\sum_{spins}\underbrace{\frac{\overbrace{-i^2}^{1} e^4}{(p_2 - p_2')^4}}_{|\Gamma|^2}\underbrace{\left(\bar{u}_{s_1'}(\mathbf{p}_1')\gamma_\alpha u_{s_1}(\mathbf{p}_1)\right)}_{\text{part of } A_{\alpha\beta}}\underbrace{\left(\bar{v}_{s_2}(\mathbf{p}_2)\gamma^\alpha v_{s_2'}(\mathbf{p}_2')\right)}_{\text{part of } B^{\alpha\beta}}\underbrace{\left(\bar{u}_{s_1}(\mathbf{p}_1)\gamma_\beta u_{s_1'}(\mathbf{p}_1')\right)}_{\text{part of } A_{\alpha\beta}}\underbrace{\left(\bar{v}_{s_2'}(\mathbf{p}_2')\gamma^\beta v_{s_2}(\mathbf{p}_2)\right)}_{\text{part of } B^{\alpha\beta}}. \tag{A}$$

$= \frac{1}{4}|\Gamma|^2 A_{\alpha\beta}B^{\alpha\beta}$, where the sum of spins is taken inside $A_{\alpha\beta}$ and $B^{\alpha\beta}$.

$$A_{\alpha\beta} = \sum_{S_1'}\sum_{S_1}\left(\bar{u}_{s_1'\delta}(\mathbf{p}_1')(\gamma_\alpha)_{\delta\eta}u_{s_1\eta}(\mathbf{p}_1)\right)\left(\bar{u}_{s_1\rho}(\mathbf{p}_1)(\gamma_\beta)_{\rho\sigma}u_{s_1'\sigma}(\mathbf{p}_1')\right)$$
$$= \underbrace{\left(\sum_{S_1'}u_{s_1'\sigma}(\mathbf{p}_1')\bar{u}_{s_1'\delta}(\mathbf{p}_1')\right)}_{\left(\frac{\not{p}_1'+m}{2m}\right)_{\sigma\delta}}(\gamma_\alpha)_{\delta\eta}\underbrace{\left(\sum_{S_1}u_{s_1\eta}(\mathbf{p}_1)\bar{u}_{s_1\rho}(\mathbf{p}_1)\right)}_{\left(\frac{\not{p}_1'+m}{2m}\right)_{\eta\rho}}(\gamma_\beta)_{\rho\sigma} = \text{Tr}\left\{\frac{\not{p}_1'+m}{2m}\gamma_\alpha\frac{\not{p}_1+m}{2m}\gamma_\beta\right\}.$$

$$B^{\alpha\beta} = \sum_{S_2'}\sum_{S_2}\left(\bar{v}_{s_2\delta}(\mathbf{p}_2)(\gamma^\alpha)_{\delta\eta}v_{s_2'\eta}(\mathbf{p}_2')\right)\left(\bar{v}_{s_2'\rho}(\mathbf{p}_2')(\gamma^\beta)_{\rho\sigma}v_{s_2\sigma}(\mathbf{p}_2)\right)$$
$$= \underbrace{\left(\sum_{S_2'}v_{s_2'\eta}(\mathbf{p}_2')\bar{v}_{s_2'\rho}(\mathbf{p}_2')\right)}_{\left(\frac{\not{p}_2'-m}{2m}\right)_{\eta\rho}}(\gamma^\beta)_{\rho\sigma}\underbrace{\left(\sum_{S_2}v_{s_2\sigma}(\mathbf{p}_2)\bar{v}_{s_2\delta}(\mathbf{p}_2)\right)}_{\left(\frac{\not{p}_2-m}{2m}\right)_{\sigma\delta}}(\gamma^\alpha)_{\delta\eta}$$
$$= \underbrace{\text{Tr}\left\{\frac{\not{p}_2'-m}{2m}\gamma^\beta\frac{\not{p}_2-m}{2m}\gamma^\alpha\right\}}_{\text{This is the form we will want to use here}} = \underbrace{\text{Tr}\left\{\frac{\not{p}_2-m}{2m}\gamma^\alpha\frac{\not{p}_2'-m}{2m}\gamma^\beta\right\}}_{\text{As an aside, this agrees with Wholeness Chart 17-5, pg. 460 last column}}.$$

All traces of odd numbered gamma matrices = 0.
We assume relativistic speeds where $E \approx |\mathbf{p}| >> m$ for all particles. Then,

$$A_{\alpha\beta} \approx \frac{1}{4m^2}\text{Tr}\left\{\not{p}_1'\gamma_\alpha\not{p}_1\gamma_\beta\right\} = \frac{1}{4m^2}\left(p_1'^{\delta}p_1^{\eta}\right)\underbrace{\text{Tr}\left\{\gamma_\delta\gamma_\alpha\gamma_\eta\gamma_\beta\right\}}_{4\left(\substack{g_{\delta\alpha}g_{\eta\beta}-g_{\delta\eta}g_{\alpha\beta}\\ +g_{\delta\beta}g_{\alpha\eta}}\right)}$$
$$= \frac{1}{m^2}\left(p_{1\alpha}'p_{1\beta} - p_1'p_1g_{\alpha\beta} + p_{1\beta}'p_{1\alpha}\right) = \frac{1}{m^2}\left(p_{1\alpha}'p_{1\beta} + p_{1\alpha}p_{1\beta}' - p_1'p_1g_{\alpha\beta}\right).$$

$B^{\alpha\beta} = \frac{1}{4m^2}\text{Tr}\left\{\not{p}_2'\,\gamma^\beta \not{p}_2 \gamma^\alpha\right\}$ is like $A_{\alpha\beta}$ above except that $1 \rightarrow 2$, $\alpha \leftrightarrow \beta$, and α,β are raised. So, we can extrapolate our final result for $A_{\alpha\beta}$ above directly to get the final result for $B_{\alpha\beta}$.

$$B^{\alpha\beta} = \frac{1}{m^2}\left(p_2'^{\beta} p_2{}^{\alpha} + p_2{}^{\beta} p_2'^{\alpha} - p_2' p_2 g^{\alpha\beta} \right).$$

Then

$$A_{\alpha\beta}B^{\alpha\beta} = \left(p_2'^{\beta} p_2{}^{\alpha} + p_2{}^{\beta} p_2'^{\alpha} - p_2' p_2 g^{\alpha\beta} \right)\left(p_{1\alpha}' p_{1\beta} + p_{1\alpha} p_{1\beta}' - p_1' p_1 g_{\alpha\beta} \right)$$

$$= \frac{1}{m^4}\Big\{ \underbrace{(p_1' p_2)(p_1 p_2')}_{X} + \underbrace{(p_1' p_2')(p_1 p_2)}_{Y} - \underbrace{(p_1' p_1)(p_2' p_2)}_{Z} + \underbrace{(p_1 p_2)(p_1' p_2')}_{Y} + \underbrace{(p_1 p_2')(p_1' p_2)}_{X}$$
$$- \underbrace{(p_1' p_1)(p_2 p_2')}_{Z} - \underbrace{(p_1' p_1)(p_2' p_2)}_{Z} - \underbrace{(p_1' p_1)(p_2 p_2')}_{Z} + \underbrace{4(p_1' p_1)(p_2' p_2)}_{4Z} \Big\}.$$

In the above, the terms labeled Z all cancel, leaving us with

$$A_{\alpha\beta}B^{\alpha\beta} = \frac{1}{m^4}\left\{ 2(p_1' p_2)(p_1 p_2') + 2(p_1' p_2')(p_1 p_2) \right\}. \tag{B}$$

From (17-110), in the COM frame where all four particles, initial and final, have the same energy (see Fig. 17-16, pg. 465).

$$p_1 p_1' = E^2 - |\mathbf{p}||\mathbf{p}'|\cos\theta \qquad p_1 p_2' = E^2 + |\mathbf{p}||\mathbf{p}'|\cos\theta \qquad p_1 p_2 = E^2 + |\mathbf{p}|^2 \qquad p_1' p_2' = E^2 + |\mathbf{p}'|^2 \;. \tag{17-110}$$

Because the interaction is elastic (same initial and final particles, so no mass converted to KE), $|\mathbf{p}'| = |\mathbf{p}|$. Also, at relativistic speeds, in natural units, $E \approx |\mathbf{p}|$. This makes (17-110), for our purposes,

$$p_1 p_1' = E^2 - E^2\cos\theta \qquad p_1 p_2' = E^2 + E^2\cos\theta \qquad p_1 p_2 = E^2 + E^2 \qquad p_1' p_2' = E^2 + E^2 \;.$$

Using the above in (B), we find

$$A_{\alpha\beta}B^{\alpha\beta} \approx \frac{1}{m^4}\left\{ 2\left(E^2 + E^2\cos\theta\right)^2 + 2\left(2E^2\right)\left(2E^2\right) \right\} = \frac{2E^4}{m^4}\left\{ (1+\cos\theta)^2 + 4 \right\}$$
$$= \frac{2E^4}{m^4}\left\{ \left(2\cos^2\frac{\theta}{2} \right)^2 + 4 \right\} = \frac{2E^4}{m^4}\left(4\cos^4\frac{\theta}{2} + 4 \right).$$

Inserting the above into (A) gives us

$$\frac{1}{4}\sum_{spins}\left|\mathcal{M}_{B2}\right|^2 = \frac{1}{4}|\Gamma|^2 A_{\alpha\beta}B^{\alpha\beta} = \frac{1}{4}|\Gamma|^2 \frac{2E^4}{m^4}\left(4\cos^4\frac{\theta}{2} + 4 \right) = \frac{e^4}{(p_2 - p_2')^4}\frac{2E^4}{m^4}\left(\cos^4\frac{\theta}{2} + 1 \right) \tag{C}$$

Using (17-129),

$$(p_2 - p_2')^2 \approx -4E^2\sin^2(\theta/2), \tag{17-129}$$

(C) becomes

$$\frac{1}{4}\sum_{spins}\left|\mathcal{M}_{B2}\right|^2 \approx \frac{e^4}{\left(-4E^2\sin^2(\theta/2)\right)^2}\frac{2E^4}{m^4}\left(1 + \cos^4\frac{\theta}{2} \right).$$

Or finally, where the approximation sign is due to our relativistic assumption $E >> m$,

$$\frac{1}{4}\sum_{spins}\left|\mathcal{M}_{B2}\right|^2 = \frac{1}{4}\sum_{s_1'=1}^{2}\sum_{s_2'=1}^{2}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}\left|\mathcal{M}_{B2}\right|^2 \approx \frac{e^4}{8m^4\sin^4(\theta/2)}\left(1 + \cos^4\frac{\theta}{2} \right). \tag{17-122}$$