

Problem 7. Derive (17-122).

Ans. We will make use of the 2nd row of (17-118), i.e.,

$$\mathcal{M}_{B2} = -ie^2 \bar{u}_{s'_1}(\mathbf{p}'_1) \gamma_\nu u_{s_1}(\mathbf{p}_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(\mathbf{p}_2) \gamma^\nu v_{s'_2}(\mathbf{p}'_2). \quad (17-118)$$

With that, we have

$$\frac{1}{4} \sum_{spins} |\mathcal{M}_{B2}|^2 = \frac{1}{4} \sum_{spins} \mathcal{M}_{B2} \mathcal{M}_{B2}^* = \frac{1}{4} \sum_{spins} \left(-ie^2 \bar{u}_{s'_1}(\mathbf{p}'_1) \gamma_\alpha u_{s_1}(\mathbf{p}_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(\mathbf{p}_2) \gamma^\alpha v_{s'_2}(\mathbf{p}'_2) \right) \times \left(-ie^2 \bar{u}_{s'_1}(\mathbf{p}'_1) \gamma_\beta u_{s_1}(\mathbf{p}_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(\mathbf{p}_2) \gamma^\beta v_{s'_2}(\mathbf{p}'_2) \right)^*.$$

Or, rearranged,

$$\begin{aligned} \frac{1}{4} \sum_{spins} |\mathcal{M}_{B2}|^2 &= \frac{1}{4} \sum_{spins} \frac{\underbrace{-i^2 e^4}_{|\Gamma|^2}}{(p_2 - p'_2)^4} \underbrace{(\bar{u}_{s'_1}(\mathbf{p}'_1) \gamma_\alpha u_{s_1}(\mathbf{p}_1))}_{\text{part of } A_{\alpha\beta}} \underbrace{(\bar{v}_{s_2}(\mathbf{p}_2) \gamma^\alpha v_{s'_2}(\mathbf{p}'_2))}_{\text{part of } B^{\alpha\beta}} \underbrace{(\bar{u}_{s_1}(\mathbf{p}_1) \gamma_\beta u_{s'_1}(\mathbf{p}'_1))}_{\text{part of } A_{\alpha\beta}} \underbrace{(\bar{v}_{s'_2}(\mathbf{p}'_2) \gamma^\beta v_{s_2}(\mathbf{p}_2))}_{\text{part of } B^{\alpha\beta}}. \quad (A) \\ &= \frac{1}{4} |\Gamma|^2 A_{\alpha\beta} B^{\alpha\beta}, \text{ where the sum of spins is taken inside } A_{\alpha\beta} \text{ and } B^{\alpha\beta}. \end{aligned}$$

$$\begin{aligned} A_{\alpha\beta} &= \sum_{s'_1} \sum_{s_1} (\bar{u}_{s'_1 \delta}(\mathbf{p}'_1) (\gamma_\alpha)_{\delta\eta} u_{s_1 \eta}(\mathbf{p}_1)) (\bar{u}_{s_1 \rho}(\mathbf{p}_1) (\gamma_\beta)_{\rho\sigma} u_{s'_1 \sigma}(\mathbf{p}'_1)) \\ &= \left(\sum_{s'_1} u_{s'_1 \sigma}(\mathbf{p}'_1) \bar{u}_{s'_1 \delta}(\mathbf{p}'_1) \right) (\gamma_\alpha)_{\delta\eta} \left(\sum_{s_1} u_{s_1 \eta}(\mathbf{p}_1) \bar{u}_{s_1 \rho}(\mathbf{p}_1) \right) (\gamma_\beta)_{\rho\sigma} = \text{Tr} \left\{ \frac{\not{p}'_1 + m}{2m} \gamma_\alpha \frac{\not{p}_1 + m}{2m} \gamma_\beta \right\}. \\ &\quad \left(\frac{\not{p}'_1 + m}{2m} \right)_{\sigma\delta} \quad \left(\frac{\not{p}_1 + m}{2m} \right)_{\eta\rho} \end{aligned}$$

$$\begin{aligned} B^{\alpha\beta} &= \sum_{s'_2} \sum_{s_2} (\bar{v}_{s_2 \delta}(\mathbf{p}_2) (\gamma^\alpha)_{\delta\eta} v_{s'_2 \eta}(\mathbf{p}'_2)) (\bar{v}_{s'_2 \rho}(\mathbf{p}'_2) (\gamma^\beta)_{\rho\sigma} v_{s_2 \sigma}(\mathbf{p}_2)) \\ &= \left(\sum_{s'_2} v_{s'_2 \eta}(\mathbf{p}'_2) \bar{v}_{s'_2 \rho}(\mathbf{p}'_2) \right) (\gamma^\beta)_{\rho\sigma} \left(\sum_{s_2} v_{s_2 \sigma}(\mathbf{p}_2) \bar{v}_{s_2 \delta}(\mathbf{p}_2) \right) (\gamma^\alpha)_{\delta\eta} \\ &\quad \left(\frac{\not{p}'_2 - m}{2m} \right)_{\eta\rho} \quad \left(\frac{\not{p}_2 - m}{2m} \right)_{\sigma\delta} \\ &= \text{Tr} \left\{ \frac{\not{p}'_2 - m}{2m} \gamma^\beta \frac{\not{p}_2 - m}{2m} \gamma^\alpha \right\} = \text{Tr} \left\{ \frac{\not{p}_2 - m}{2m} \gamma^\alpha \frac{\not{p}'_2 - m}{2m} \gamma^\beta \right\}. \end{aligned}$$

This is the form we will want to use here

As an aside, this agrees with Wholeness Chart 17-5, pg. 460 last column

All traces of odd numbered gamma matrices = 0.

We assume relativistic speeds where $E \approx |\mathbf{p}| \gg m$ for all particles. Then,

$$\begin{aligned} A_{\alpha\beta} &\approx \frac{1}{4m^2} \text{Tr} \left\{ \not{p}'_1 \gamma_\alpha \not{p}_1 \gamma_\beta \right\} = \frac{1}{4m^2} (p'_1{}^\delta p_1{}^\eta) \underbrace{\text{Tr} \left\{ \gamma_\delta \gamma_\alpha \gamma_\eta \gamma_\beta \right\}}_{4 \left(\begin{array}{l} g_{\delta\alpha} g_{\eta\beta} - g_{\delta\eta} g_{\alpha\beta} \\ + g_{\delta\beta} g_{\alpha\eta} \end{array} \right)} \\ &= \frac{1}{m^2} (p'_1{}_\alpha p_{1\beta} - p'_1{}_\beta p_{1\alpha} + p'_1{}_\beta p_{1\alpha}) = \frac{1}{m^2} (p'_1{}_\alpha p_{1\beta} + p_{1\alpha} p'_1{}_\beta - p'_1{}_\beta p_{1\alpha}). \end{aligned}$$

Chapter 17 Problem Solutions

$B^{\alpha\beta} = \frac{1}{4m^2} \text{Tr} \{ \not{p}'_2 \gamma^\beta \not{p}'_2 \gamma^\alpha \}$ is like $A_{\alpha\beta}$ above except that $1 \rightarrow 2$, $\alpha \leftrightarrow \beta$, and α, β are raised. So, we

can extrapolate our final result for $A_{\alpha\beta}$ above directly to get the final result for $B_{\alpha\beta}$.

$$B^{\alpha\beta} = \frac{1}{m^2} (p_2'^\beta p_2^\alpha + p_2^\beta p_2'^\alpha - p_2' p_2 g^{\alpha\beta}).$$

Then

$$\begin{aligned} A_{\alpha\beta} B^{\alpha\beta} &= \frac{1}{m^4} (p_2'^\beta p_2^\alpha + p_2^\beta p_2'^\alpha - p_2' p_2 g^{\alpha\beta}) (p_1'_\alpha p_{1\beta} + p_{1\alpha} p_1'_\beta - p_1' p_1 g_{\alpha\beta}) \\ &= \frac{1}{m^4} \left\{ \underbrace{(p_1' p_2)(p_1 p_2')}_{\tilde{X}} + \underbrace{(p_1' p_2')(p_1 p_2)}_{\tilde{Y}} - \underbrace{(p_1' p_1)(p_2' p_2)}_{\tilde{Z}} + \underbrace{(p_1 p_2)(p_1' p_2')}_{\tilde{Y}} + \underbrace{(p_1 p_2')(p_1' p_2)}_{\tilde{X}} \right. \\ &\quad \left. - \underbrace{(p_1' p_1)(p_2 p_2')}_{\tilde{Z}} - \underbrace{(p_1' p_1)(p_2' p_2)}_{\tilde{Z}} - \underbrace{(p_1' p_1)(p_2 p_2')}_{\tilde{Z}} + \underbrace{4(p_1' p_1)(p_2' p_2)}_{4\tilde{Z}} \right\}. \end{aligned}$$

In the above, the terms labeled Z all cancel, leaving us with

$$A_{\alpha\beta} B^{\alpha\beta} = \frac{1}{m^4} \left\{ 2(p_1' p_2)(p_1 p_2') + 2(p_1' p_2')(p_1 p_2) \right\}. \quad (\text{B})$$

From (17-110), in the COM frame where all four particles, initial and final, have the same energy (see Fig. 17-16, pg. 465).

$$p_1 p_1' = E^2 - |\mathbf{p}| |\mathbf{p}'| \cos \theta \quad p_1 p_2' = E^2 + |\mathbf{p}| |\mathbf{p}'| \cos \theta \quad p_1 p_2 = E^2 + |\mathbf{p}|^2 \quad p_1' p_2' = E^2 + |\mathbf{p}'|^2. \quad (17-110)$$

Because the interaction is elastic (same initial and final particles, so no mass converted to KE), $|\mathbf{p}'| = |\mathbf{p}|$. Also, at relativistic speeds, in natural units, $E \approx |\mathbf{p}|$. This makes (17-110), for our purposes,

$$p_1 p_1' = E^2 - E^2 \cos \theta \quad p_1 p_2' = E^2 + E^2 \cos \theta \quad p_1 p_2 = E^2 + E^2 \quad p_1' p_2' = E^2 + E^2.$$

Using the above in (B), we find

$$\begin{aligned} A_{\alpha\beta} B^{\alpha\beta} &\approx \frac{1}{m^4} \left\{ 2(E^2 + E^2 \cos \theta)^2 + 2(2E^2)(2E^2) \right\} = \frac{2E^4}{m^4} \left\{ (1 + \cos \theta)^2 + 4 \right\} \\ &= \frac{2E^4}{m^4} \left\{ \left(2 \cos^2 \frac{\theta}{2} \right)^2 + 4 \right\} = \frac{2E^4}{m^4} \left(4 \cos^4 \frac{\theta}{2} + 4 \right). \end{aligned}$$

Inserting the above into (A) gives us

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{B2}|^2 = \frac{1}{4} |\Gamma|^2 A_{\alpha\beta} B^{\alpha\beta} = \frac{1}{4} |\Gamma|^2 \frac{2E^4}{m^4} \left(4 \cos^4 \frac{\theta}{2} + 4 \right) = \frac{e^4}{(p_2 - p_2')^4} \frac{2E^4}{m^4} \left(\cos^4 \frac{\theta}{2} + 1 \right) \quad (\text{C})$$

Using (17-129),

$$(p_2 - p_2')^2 \approx -4E^2 \sin^2(\theta/2), \quad (17-129)$$

(C) becomes

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{B2}|^2 \approx \frac{e^4}{(-4E^2 \sin^2(\theta/2))^2} \frac{2E^4}{m^4} \left(1 + \cos^4 \frac{\theta}{2} \right).$$

Or finally, where the approximation sign is due to our relativistic assumption $E \gg m$,

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}_{B2}|^2 = \frac{1}{4} \sum_{s_1'=1}^2 \sum_{s_2'=1}^2 \sum_{s_1=1}^2 \sum_{s_2=1}^2 |\mathcal{M}_{B2}|^2 \approx \frac{e^4}{8m^4 \sin^4(\theta/2)} \left(1 + \cos^4 \frac{\theta}{2} \right). \quad (17-122)$$