
Ans. We will make use of the 2\textsuperscript{nd} row of (17-118), i.e.,

$$M_{B2} = -ie^4 \mu'_{S}(p_1) \gamma_\nu u_{S_1}(p_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(p_2) \gamma_\nu v_{s'_2}(p'_2).$$  

(17-118)

With that, we have

$$\frac{1}{4} \sum_{\text{spins}} |M_{B2}|^2 = \frac{1}{4} \sum_{\text{spins}} M_{B2} M_{B2}^* = \frac{1}{4} \sum_{\text{spins}} \left(-ie^4 \mu'_{S}(p_1) \gamma_\nu u_{S_1}(p_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(p_2) \gamma_\nu v_{s'_2}(p'_2)\right) \times \left(-ie^4 \mu'_{S}(p'_1) \gamma_\nu u_{S_1}(p'_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(p_2) \gamma_\nu v_{s'_2}(p'_2)\right)^* .$$

Or, rearranged,

$$\frac{1}{4} \sum_{\text{spins}} |M_{B2}|^2 = \frac{1}{4} \sum_{\text{spins}} \frac{1}{(p_2 - p'_2)^4} \left(\bar{\mu}'_{S_1}(p_1) \gamma_\alpha u_{S_1}(p_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(p_2) \gamma_\alpha v_{s'_2}(p'_2)\right) \times \left(\bar{\mu}'_{S_1}(p'_1) \gamma_\beta u_{S_1}(p'_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(p_2) \gamma_\beta v_{s'_2}(p'_2)\right) .$$

$$\begin{align*}
A_{\alpha\beta} &= \sum_{S_1} \sum_{S_2} \left(\bar{\mu}'_{S_1}(p_1) \gamma_\alpha u_{S_1}(p_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(p_2) \gamma_\alpha v_{s'_2}(p'_2)\right) \frac{1}{m^2} + \frac{m}{2m} \gamma_\alpha \gamma_\beta \gamma_\beta \gamma_\beta . \\
B^{\alpha\beta} &= \sum_{S_1} \sum_{S_2} \left(\bar{\mu}'_{S_1}(p_1) \gamma_\alpha u_{S_1}(p_1) \frac{1}{(p_2 - p'_2)^2} \bar{v}_{s_2}(p_2) \gamma_\alpha v_{s'_2}(p'_2)\right) \frac{1}{m^2} + \frac{m}{2m} \gamma_\alpha \gamma_\beta \gamma_\beta \gamma_\beta .
\end{align*}$$

This is the form we will want to use here

As an aside, this agrees with Wholeness Chart 17-5, pg. 460 last column

All traces of odd numbered gamma matrices = 0.

We assume relativistic speeds where $E \approx |p| >> m$ for all particles. Then,

$$A_{\alpha\beta} = \frac{1}{4m^2} \Tr \left\{ \gamma_\gamma \gamma_\beta \gamma_\beta \gamma_\beta \gamma_\beta \right\} = \frac{1}{4m^2} \left(\gamma_\gamma \gamma_\beta \gamma_\beta \gamma_\beta \gamma_\beta \right) \frac{1}{m^2} + \frac{m}{2m} \gamma_\alpha \gamma_\beta \gamma_\beta .$$

$$= \frac{1}{m^2} \left(\gamma_\gamma \gamma_\beta \gamma_\beta \gamma_\beta \gamma_\beta \right) \frac{1}{m^2} + \frac{m}{2m} \gamma_\alpha \gamma_\beta .$$

17-1
Chapter 17 Problem Solutions

\[ B^{\alpha \beta} = \frac{1}{4m^2} \text{Tr} \left\{ p_2^2 \gamma^\alpha p_2 \gamma^\beta \right\} \]

is like \( A^{\alpha \beta} \) above except that \( 1 \rightarrow 2, \; \alpha \leftrightarrow \beta \), and \( \alpha, \beta \) are raised. So, we can extrapolate our final result for \( A^{\alpha \beta} \) above directly to get the final result for \( B^{\alpha \beta} \).

\[ B^{\alpha \beta} = \frac{1}{m^2} \left( p_2^2 p_2^\alpha + p_2^2 p_2^\alpha - p_2^2 p_2 g^{\alpha \beta} \right) . \]

Then

\[ A^{\alpha \beta} B^{\alpha \beta} = \left( p_2^\alpha p_2^\alpha + p_2^\alpha p_2^\alpha - p_2^2 p_2 g^{\alpha \beta} \right) \left( p_2^\alpha p_2^\alpha + p_2^\alpha p_2^\alpha - p_2^2 p_2 g^{\alpha \beta} \right) \]

\[ = \frac{1}{m^2} \left\{ \left( \frac{p_2^\alpha p_2^\alpha}{X} \right) + \left( \frac{p_2^\alpha p_2^\alpha}{Y} \right) - \left( \frac{p_2^\alpha p_2^\alpha}{Z} \right) \right\} . \]

In the above, the terms labeled \( Z \) all cancel, leaving us with

\[ A^{\alpha \beta} B^{\alpha \beta} = \frac{1}{m^2} \left\{ 2 \left( p_2^\alpha p_2^\alpha \right) \right\} . \] (B)

From (17-110), in the COM frame where all four particles, initial and final, have the same energy (see Fig. 17-16, pg. 465).

\[ p_1 p_2 = E^2 - |p| |p| \cos \theta \quad p_1 p_2 = E^2 + |p| |p| \cos \theta \quad p_1 p_2 = E^2 + |p| ^2 \quad p_1 p_2 = E^2 + |p| ^2 . \] (17-110)

Because the interaction is elastic (same initial and final particles, so no mass converted to \( KE \)), \( |p| ^2 = |p| . \) Also, at relativistic speeds, in natural units, \( E \approx |p| . \) This makes (17-110), for our purposes,

\[ p_1 p_2 = E^2 - E^2 \cos \theta \quad p_1 p_2 = E^2 + E^2 \cos \theta \quad p_1 p_2 = E^2 + E^2 \quad p_1 p_2 = E^2 + E^2 . \]

Using the above in (B), we find

\[ A^{\alpha \beta} B^{\alpha \beta} = \frac{1}{m^2} \left\{ 2 \left( E^2 + E^2 \cos \theta \right) ^2 + \left( E^2 \right) \left( E^2 \right) \right\} = \frac{2E^4}{m^4} \left\{ \left( 1 + \cos \theta \right) ^2 + 4 \right\} \]

\[ = \frac{2E^4}{m^4} \left\{ \left( 2 \cos ^2 \frac{\theta}{2} \right) ^2 + 4 \right\} = \frac{4 \cos ^4 \frac{\theta}{2}}{2} + 4 \right\} . \]

Inserting the above into (A) gives us

\[ \frac{1}{4} \sum_{\text{spins}} |M_{B2}|^2 = \frac{1}{4} \sum_{\text{spins}} A^{\alpha \beta} B^{\alpha \beta} = \frac{1}{4} \sum_{\text{spins}} \frac{2E^4}{m^4} \left( 4 \cos ^4 \frac{\theta}{2} + 4 \right) = \frac{e^4}{(2 - 2)^4} \frac{2E^4}{m^4} \left( \cos ^4 \frac{\theta}{2} + 1 \right) . \] (C)

Using (17-129),

\[ (p_2 - p_2')^2 = -4E^2 \sin ^2 \left( \theta / 2 \right) , \] (17-129)

(C) becomes

\[ \frac{1}{4} \sum_{\text{spins}} |M_{B2}|^2 = \frac{e^4}{(4E^2 \sin ^2 \left( \theta / 2 \right)) ^2} \frac{2E^4}{m^4} \left( \cos ^4 \frac{\theta}{2} + 1 \right) . \]

Or finally, where the approximation sign is due to our relativistic assumption \( E \gg m \),

\[ \frac{1}{4} \sum_{\text{spins}} |M_{B2}|^2 = \frac{e^4}{8m^4 \sin ^4 \left( \theta / 2 \right)} \frac{2E^4}{m^4} \left( 1 + \cos ^4 \frac{\theta}{2} \right) . \] (17-122)