For this frame, we have, where we note that because photons are massless (and $k^\mu k_\mu = 0$), $|\mathbf{p}_1| = |\mathbf{k}| = \omega$, and $|\mathbf{p}'_1| = |\mathbf{k}'| = \omega'$,

$$p_1 = k = \begin{pmatrix}\omega \\ \mathbf{k}\end{pmatrix} = \begin{pmatrix}\omega \\ 0 \\ 0 \\ |\mathbf{k}|\end{pmatrix} = \begin{pmatrix}\omega \\ 0 \\ 0 \\ \omega\end{pmatrix} \qquad p_2 = p = \begin{pmatrix}m \\ 0\end{pmatrix} = \begin{pmatrix}m \\ 0 \\ 0 \\ 0\end{pmatrix}$$

$$p'_1 = \begin{pmatrix}E'_1 \\ \mathbf{p}'_1\end{pmatrix} = k' = \begin{pmatrix}\omega' \\ \mathbf{k}'\end{pmatrix} = \begin{pmatrix}\omega' \\ |\mathbf{k}'| \sin\theta \\ 0 \\ |\mathbf{k}'|\cos\theta\end{pmatrix} \qquad p'_2 = \begin{pmatrix}E'_2 \\ \mathbf{p}'_2\end{pmatrix} = p' = \begin{pmatrix}E' \\ \mathbf{p}'\end{pmatrix} = \begin{pmatrix}\omega + m - \omega' \\ -|\mathbf{k}'| \sin\theta \\ 0 \\ |\mathbf{k}| - |\mathbf{k}'|\cos\theta\end{pmatrix} \quad \left\{ p' \text{ from conserv laws.} \right. \tag{17-134}$$

4-momenta in terms of initial ω, **k**, *m, and final* ω', $\mathbf{k}'$, θ

From (17-134), and/or Fig. 17-21, we see

$$\mathbf{k} \bullet \mathbf{k}' = |\mathbf{k}||\mathbf{k}'| \cos\theta \quad (= \omega\omega' \cos\theta). \tag{17-135}$$

We will look at obtaining the Compton cross section for three cases in the common test frame case where the initial electron is stationary in the lab.

3 cases: depending on spin and polarization measurements

1. Spins and polarizations measured
2. Spins and polarizations both unmeasured
3. Polarizations measured, but spins unmeasured

The first of these is difficult to do experimentally, but its analysis provides a good starting point. The second and third of these are common in experiments. In the third case, photon polarizations are measured, which is considerably easier to do than measuring fermion spins.

1. Spins and Polarizations Measured

Case 1: all spins and polarizations measured

(17-69), when one of the final particles is a photon, repeated below as the first line of (17-136) is

Start with basic differential cross section expression with #2 particle stationary

$$\left(\frac{d\sigma}{d\Omega'_1}\right)_{\substack{\#2 \\ stat}} = \frac{1}{64\pi^2 m_2 E'_1 E'_2}\left(\prod_l^{\substack{\text{extern} \\ \text{ferms}}} 2m_l\right)|\mathcal{M}|^2 \frac{|\mathbf{p}'_1|^2}{|\mathbf{p}_1|}\left(\frac{\partial E'_1}{\partial|\mathbf{p}'_1|} + \frac{\partial E'_2}{\partial|\mathbf{p}'_1|}\right)^{-1} \quad \left\{ \begin{array}{l}\text{\#2 station,} \\ \text{2 initial, 2} \\ \text{final parts,} \\ \text{elast/inelas}\end{array}\right.$$

$$\rightarrow \left(\frac{d\sigma}{d\Omega'_1}\right)_{\substack{e^-\gamma \to e^-\gamma \\ 1st\ e^-\ stat}} = \frac{1}{64\pi^2 m\,\omega' E'} 4m^2 |\mathcal{M}|^2 \frac{\omega'^2}{\omega}\left(\frac{\partial E'_1}{\partial|\mathbf{k}'|} + \frac{\partial E'_2}{\partial|\mathbf{k}'|}\right)^{-1} \tag{17-136}$$

$$= \frac{m}{16\pi^2 E'}\frac{\omega'}{\omega}\left(\frac{\partial\omega'}{\partial\omega'} + \frac{\partial E'}{\partial|\mathbf{k}'|}\right)^{-1}|\mathcal{M}|^2 . = \frac{m}{16\pi^2 E'}\frac{\omega'}{\omega}\left(1 + \frac{\partial E'}{\partial|\mathbf{k}'|}\right)^{-1}|\mathcal{M}|^2 .$$

To evaluate $\partial E'/\partial|\mathbf{k}'|$, we need E' as a function of $|\mathbf{k}'|$. With $\mathbf{p}_2 = 0$ in $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2$,

Evaluate energy expression in this frame

$$(E'_2)^2 = \left((m_2)^2 + |\mathbf{p}'_2|^2\right) = \left((m_2)^2 + |\mathbf{p}_1 - \mathbf{p}'_1|^2\right) \;\rightarrow\; (E')^2 = \left(m^2 + |\mathbf{p}'|^2\right) = \left(m^2 + |\mathbf{k} - \mathbf{k}'|^2\right)$$

$$\rightarrow\; E' = \left(m^2 + |\mathbf{k}|^2 + |\mathbf{k}'|^2 - 2\mathbf{k}\bullet\mathbf{k}'\right)^{1/2} = \left(m^2 + |\mathbf{k}|^2 + |\mathbf{k}'|^2 - 2|\mathbf{k}||\mathbf{k}'|\cos\theta\right)^{1/2}. \tag{17-137}$$

From (17-137), we find (using the last part of (17-134) in the second line below)

$$\frac{\partial E'}{\partial|\mathbf{k}'|} = \frac{1}{2}\underbrace{\left(m^2 + |\mathbf{k}|^2 + |\mathbf{k}'|^2 - 2|\mathbf{k}||\mathbf{k}'|\cos\theta\right)^{-1/2}}_{E'^{-1}}\left(2|\mathbf{k}'| - 2|\mathbf{k}|\cos\theta\right) = \frac{\left(|\mathbf{k}'| - |\mathbf{k}|\cos\theta\right)}{E'}$$

$$\rightarrow\; 1 + \frac{\partial E'}{\partial|\mathbf{k}'|} = \frac{E' + \left(|\mathbf{k}'| - |\mathbf{k}|\cos\theta\right)}{E'} = \frac{\omega + m - \omega' + \omega' - \omega\cos\theta}{E'} = \frac{m + \omega(1 - \cos\theta)}{E'}. \tag{17-138}$$

Then, we want to find a simpler form for the numerator of the last part of (17-138). To do this, we use our conservation relation

$$p + k = p' + k', \tag{17-139}$$

along with (which you can prove by doing Prob. 8)

$$pk = p'k' \qquad \text{i.e.,} \qquad p^{\mu}k_{\mu} = p'^{\mu}k'_{\mu}\,, \tag{17-140}$$

A useful relation

to show $$pk = p'k' = p'^{\mu}k'_{\mu} = \left(p^{\mu} + k^{\mu} - k'^{\mu}\right)k'_{\mu} = p^{\mu}k'_{\mu} + k^{\mu}k'_{\mu} - \underbrace{k'^{\mu}k'_{\mu}}_{0} = pk' + kk'\,. \tag{17-141}$$

(17-141) is true in any frame, but in our frame where p and k are as found in (17-134), we have

With the useful relation, we can simply things

$$pk = m\omega - \underbrace{|\mathbf{p}|}_{0}\bullet|\mathbf{k}| = pk' + kk' = p^{\mu}k'_{\mu} + k^{\mu}k'_{\mu}$$
$$\rightarrow \; m\omega = m\omega' - 0 + \omega\omega' - \mathbf{k}\bullet\mathbf{k}' = m\omega' + \omega\omega' - \omega\omega'\cos\theta \tag{17-142}$$
$$\rightarrow \; \frac{m\omega}{\omega'} = m + \omega(1 - \cos\theta).$$

We now use the last row of (17-142) in the numerator of the last part of (17-138) to get

$$\left(1 + \frac{\partial E'}{\partial|\mathbf{k}'|}\right)_{\theta\phi} = \frac{m\omega}{E'\omega'}\,. \tag{17-143}$$

With (17-143), (17-136) becomes

Result for Case 1

$$\left(\frac{d\sigma}{d\Omega'_1}\right)_{\substack{e^-\gamma\to e^-\gamma \\ 1st\ e^-\ stat}} = \frac{m}{16\pi^2 E'}\frac{\omega'}{\omega}\frac{E'\omega'}{m\omega}|\mathcal{M}|^2 = \frac{1}{(4\pi)^2}\left(\frac{\omega'}{\omega}\right)^2|\mathcal{M}|^2 \quad \begin{cases}\text{fully polarized, i.e.,}\\ \text{measured spins}\\ \text{and polarizations}\end{cases}. \tag{17-144}$$

All we need to do further is evaluate $|\mathcal{M}|^2$ assuming spins and polarizations are known. We will not do that here, but proceed to the more typical experimental condition where neither spins nor polarizations are measured.

2. Unpolarized (Spins and Polarizations Not Measured)

Case 2: no spins nor polarizations measured

We need to take both spin sums and polarization sums into account. Using (17-95), we find

$$\underbrace{\frac{1}{2}\sum_{s'=1}^{2}\sum_{s=1}^{2}}_{\text{spins}}\;\underbrace{\frac{1}{2}\sum_{r'=1}^{2}\sum_{r=1}^{2}}_{\text{polariz}}|\mathcal{M}|^2 = \frac{1}{4}\underbrace{\sum_{s'=1}^{2}\sum_{s=1}^{2}}_{\text{spins}}\mathcal{M}^{\mu\nu}\mathcal{M}^{*}_{\mu\nu}\,, \tag{17-145}$$

Need to do spin and polarization sums

where for Compton scattering, $\mathcal{M}^{\mu\nu}$ is defined in (17-79) and (17-80), as

Compton scattering amplitude

$$\mathcal{M}^{\mu\nu} = \underbrace{-e^2\bar{u}_{S'}(\mathbf{p}')\gamma^{\mu}iS_F(p+k)\gamma^{\nu}u_S(\mathbf{p})}_{\mathcal{M}_1^{\mu\nu}} \; \underbrace{-\; e^2\bar{u}_{S'}(\mathbf{p}')\gamma^{\nu}iS_F(p-k')\gamma^{\mu}u_S(\mathbf{p})}_{\mathcal{M}_2^{\nu\mu}}$$
$$= \bar{u}_{S'}(\mathbf{p}')\underbrace{\left(\underbrace{-e^2\gamma^{\mu}iS_F(p+k)\gamma^{\nu}}_{\Gamma_1=\Gamma_1^{\mu\nu}}\underbrace{-e^2\gamma^{\nu}iS_F(p-k')\gamma^{\mu}}_{\Gamma_2=\Gamma_2^{\nu\mu}}\right)}_{\Gamma=\Gamma^{\mu\nu}=\Gamma_1^{\mu\nu}+\Gamma_2^{\nu\mu}}u_S(\mathbf{p}). \tag{17-146}$$

From the 1st row of Wholeness Chart 17-5, pg. 460, with Γ there = $\Gamma^{\mu\nu}$ here, (17-145) becomes

Spin and polarization sums in terms of traces → 4 terms

$$\frac{1}{4}\sum_{s'=1}^{2}\sum_{s=1}^{2}\mathcal{M}^{\mu\nu}\mathcal{M}^{*}_{\mu\nu} = \frac{1}{4}\text{Tr}\left\{\left(\frac{\not{p}'+m}{2m}\right)\Gamma^{\mu\nu}\left(\frac{\not{p}+m}{2m}\right)\tilde{\Gamma}_{\mu\nu}\right\}$$
$$= \frac{1}{4}\text{Tr}\left\{\left(\frac{\not{p}'+m}{2m}\right)\left(\Gamma_1^{\mu\nu}+\Gamma_2^{\nu\mu}\right)\left(\frac{\not{p}+m}{2m}\right)\left(\tilde{\Gamma}_{1\mu\nu}+\tilde{\Gamma}_{2\nu\mu}\right)\right\}$$
$$= \underbrace{\frac{1}{4}\text{Tr}\left\{\left(\frac{\not{p}'+m}{2m}\right)\Gamma_1^{\mu\nu}\left(\frac{\not{p}+m}{2m}\right)\tilde{\Gamma}_{1\mu\nu}\right\}}_{X_{11}} + \underbrace{\frac{1}{4}\text{Tr}\left\{\left(\frac{\not{p}'+m}{2m}\right)\Gamma_2^{\nu\mu}\left(\frac{\not{p}+m}{2m}\right)\tilde{\Gamma}_{2\nu\mu}\right\}}_{X_{22}} \tag{17-147}$$
$$+ \underbrace{\frac{1}{4}\text{Tr}\left\{\left(\frac{\not{p}'+m}{2m}\right)\Gamma_1^{\mu\nu}\left(\frac{\not{p}+m}{2m}\right)\tilde{\Gamma}_{2\nu\mu}\right\}}_{X_{12}} + \underbrace{\frac{1}{4}\text{Tr}\left\{\left(\frac{\not{p}'+m}{2m}\right)\Gamma_2^{\nu\mu}\left(\frac{\not{p}+m}{2m}\right)\tilde{\Gamma}_{1\mu\nu}\right\}}_{X_{21}}.$$

Label the 4 traces