16.4.6 QFT's Second Order Electron Magnetic Moment

QED's 2nd order determination of g

Note: A summary of this section can be found in the chapter summary on pg. 429.

As I'm sure you understand, we get the first refinement to the magnetic moment calculation by adding in the next higher order terms (associated with next higher order Feynman diagrams). When Julian Schwinger[1] first calculated this in 1948, and it matched experiment, the physics community was electrified. It was dramatic confirmation of QED.

Ways to Calculate the Second Order Correction to *g*

We will use a clever analysis that makes calculations much easier

There are two ways we can calculate the gyromagnetic ratio *g* to second order, the first of which is probably what you would first assume we would do.

1. Take all the amplitude terms from the various 2nd order Feynman diagrams (the last four diagrams in Fig. 16-4 below), find all the terms therein containing $\sigma^{\mu\nu}$ (using, in some cases, Gordon's identity to convert terms in γ^{μ}), add those terms, and use them in the way we did in the prior section to find *g*.
2. Use a quicker, simpler trick (to be described below.)

Method #1 above can be done, but it is extraordinarily long and messy. We will use method #2.

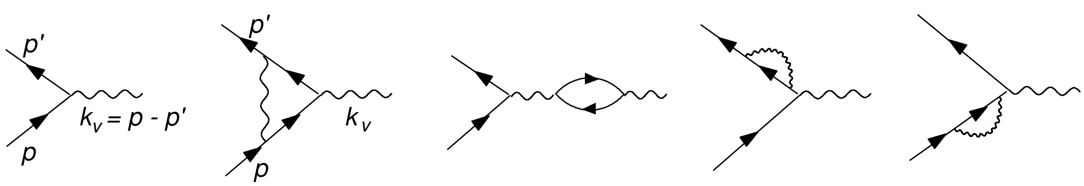

Figure 16-4. Contributions at Second Order to Single Vertex Interaction

The Method of Choice for Finding *g* to Second Order

Steps in our analysis

The steps using method #2 above that we will follow are:

Step #1: Express the 2nd order amplitude in its most general form in terms of initially unknown functions $F_1(k_\nu^2)$ and $F_2(k_\nu^2)$. (We use k_ν for the 4-momentum of the external photon. We used *k* in the first order case, but in higher order cases, we typically use *k* for internal photon corrections such as the vertical photon in the 2nd diagram, Fig. 16-4.)

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[1] Schwinger, J., Quantum Electrodynamics III. The Electromagnetic Properties of the Electron – Radiative Corrections to Scattering, *Phys. Rev.* **76**(6), 790-817 (Sept 15, 1948).

Step #2: Examine the non-relativistic limit case where $k_\nu \to 0$ corresponding to experimentally determined gyromagnetic ratio g, and show $F_1(0) = 1$,

Step #3: Show $F_2(0)$ relates to g, so all we have to do is find $F_2(0)$ and the form of relevant F_2 terms.

Step #4: Because the term with $F_2(0)$ has no γ^μ factor, recognize that we don't need to evaluate any 2nd order Feynman diagrams that give us a term with a γ^μ factor.

Step #5: In taking $\gamma^\mu \to \gamma^\mu_{\substack{mod \\ 2nd}}(p,p') = \gamma^\mu + e^2\Lambda_c^\mu$ for 2nd order amplitude, we only need to consider terms in Λ_c^μ that have no γ^μ factor. Find the sum of those terms.

Step #6: Use the sum of terms in #5 in #3 to find g to 2nd order.

Step #1. Most General Form of the Amplitude

Step #1

Consider Fig. 16-4 for time vertically upward and the electron emitting a photon. We will take it from the mathematicians that the most general expression for the total amplitude for electron scattering from an external field to any order (though we will focus on 2nd order as in Fig. 16-4) is

$$\mathcal{M}_{mm}^{(nth)} = ie\bar{u}_{r'}(\mathbf{p}')\left(\gamma^\mu F_1\left(k_\nu^2\right) + i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_2\left(k_\nu^2\right)\right)u_r(\mathbf{p})A_\mu^e(\mathbf{k}_\nu)\,. \tag{16-76}$$

Most general form of amplitude

with $k_\nu = p - p'$, and functions F_1 and F_2, about which we know little right now, are known as form factors. (If we treated photon absorption instead, then $k_\nu = p' - p$, and we would get the same result.)

Note that by using Gordon's identity

$$\bar{u}_{r'}(\mathbf{p}')\gamma^\mu u_r(\mathbf{p}) = \bar{u}_{r'}(\mathbf{p}')\left(\frac{p^\mu + p'^\mu}{2m} - \frac{i\sigma^{\mu\nu}k_\nu}{2m}\right)u_r(\mathbf{p}) \tag{16-77}$$

to substitute for the γ^μ term in (16-76), we can express the amplitude of (16-76) solely as a function of a term that is a scalar in spinor space (having no γ^μ or $\sigma^{\mu\nu}$ factor) and a term having a $\sigma^{\mu\nu}$ factor. Alternatively, we could solve Gordon's identity for the $\sigma^{\mu\nu}$ factor term and substitute that in (16-76) to give us the amplitude solely as a function of a spinor space scalar term and a term in γ^μ.

With Gordon's identity, we can express the amplitude via any 2 of 3 possible types of terms

In other words, there are three possible types of terms we can have in the amplitude (16-76), but because of Gordon's identity, we can, as we find convenient, express (16-76) solely as a function of any two of them we like. We will use this to our advantage in what is coming.

Note also that we refer above to each part of (16-76) as a single term, but in actuality F_1 and F_2 are, in general, comprised of more than one term, so there are really more than two terms in (16-76) for all but the first order case.

We will from henceforth focus on the 2nd order case of (16-76). As an aside, we can see how the second order terms come in, if we temporarily use slightly different form factors F_A and F_B defined via

Temporarily re-express with slightly different form factors

$$F_1\left(k_\nu^2\right) = 1 + F_A\left(k_\nu^2\right) \qquad\qquad F_2\left(k_\nu^2\right) = F_B\left(k_\nu^2\right). \tag{16-78}$$

So, (16-76) becomes

$$\mathcal{M}_{mm}^{(2+4)} = \mathcal{M}_{mm}^{(2)} + \mathcal{M}_{mm}^{(4)} = ie\bar{u}_{r'}(\mathbf{p}')(\gamma^\mu + \underbrace{\gamma^\mu F_A\left(k_\nu^2\right) + i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_B\left(k_\nu^2\right)}_{\text{2nd order contibution}})u_r(\mathbf{p})A_\mu^e(\mathbf{k}_\nu)\,, \tag{16-79}$$

With that re-expression, 1st and 2nd order parts appear separately

where the superscripts, as used in this book, refer to the powers of e involved. When we use the term "second order", we mean second order in α, i.e., 4th order in e. (16-79) expresses the amplitude to 2nd order in terms of the first order and second order contributions.

We now return to using the form factors F_1 and F_2, instead of F_A and F_B.

Return to form factors F_1 and F_2

NOTE: The following Step #2 and the equations therein have been changed since the May 2014 version of this book. The revised Step #2 below should be easier to understand than the prior one.

Step #2. Examine Non-relativistic Case Where $k_\nu \to 0$ and Show $F_1(0) = 1$

Step #2

Examine non-relativistic case where effectively $k_\nu \to 0$

The experimental value for g cited herein is determined for non-relativistic speeds of the incoming and outgoing electron. Thus, $p^\mu \approx (m,0,0,0)$, $p'^\mu \approx (m,0,0,0)$, and

$$\text{non-relativistic } k_\nu = p_\nu - p'_\nu \approx \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{16-80}$$

So we will consider $k_\nu \to 0$, and then of course, $k_\nu^2 \to 0$. We use the symbols $F_1(0)$ and $F_2(0)$ to represent our non-relativistic form factors. With this, (16-76) becomes, where we keep in mind that the 3-momentum label $\mathbf{k}_\nu$ is non-zero, but in relativistic terms, vanishingly small.

$$\mathcal{M}^{(2+4)}_{\substack{mm \\ k_\nu \to 0}} = ie\bar{u}_{r'}(\mathbf{p}')\left(\gamma^\mu F_1(0) + i\frac{\sigma^{\mu\nu}k_\nu}{2m}F_2(0)\right)u_r(\mathbf{p})A^e_\mu(\mathbf{k}_\nu). \tag{16-81}$$

Using Gordon's identity (16-77) for the γ^μ term,

Re-express again using Gordon's identity

$$\mathcal{M}^{(2+4)}_{\substack{mm \\ k_\nu \to 0}} = ie\bar{u}_{r'}(\mathbf{p}')\left(\frac{p^\mu + p'^\mu}{2m}F_1(0) + i\frac{\sigma^{\mu\nu}k_\nu}{2m}\left(-F_1(0) + F_2(0)\right)\right)u_r(\mathbf{p})A^e_\mu(\mathbf{k}_\nu). \tag{16-82}$$

We now employ our non-relativistic assumption and use $p^\mu = p'^\mu = (m,0,0,0)$ to get

Employ non-relativistic p^μ assumption

$$\mathcal{M}^{(2+4)}_{\substack{mm \\ k_\nu \to 0}} = ie\bar{u}_{r'}(\mathbf{p}')\left(A^e_0(\mathbf{k})F_1(0) + i\frac{\sigma^{\mu\nu}k_\nu}{2m}\left(-F_1(0) + F_2(0)\right)A^e_\mu(\mathbf{k}_\nu)\right)u_r(\mathbf{p}). \tag{16-83}$$

When we replace the zeroth component of the photon factor with Φ, representing the electric field potential, we have

With Φ for A_0^e, showing non spin and spin contributions

$$\mathcal{M}^{(2+4)}_{\substack{mm \\ k_\nu \to 0}} = i\bar{u}_{r'}(\mathbf{p}')\Bigg(\underbrace{e\Phi F_1(0)}_{\substack{\text{non spin} \\ \text{contribution}}} + \underbrace{ie\frac{\sigma^{\mu\nu}k_\nu}{2m}\left(-F_1(0) + F_2(0)\right)A^e_\mu(\mathbf{k}_\nu)}_{\text{spin contribution, as it has } \sigma^{\mu\nu} \text{ in it}}\Bigg)u_r(\mathbf{p}). \tag{16-84}$$

If we examine the non spin contribution, all second order effects must be included in the renormalized value for e. We have no vertex or propagator renormalizations in that term to worry about, and the external line factors $u_r(\mathbf{p})$ and $\bar{u}_{r'}(\mathbf{p}')$ remain unchanged under renormalization. Thus, if e is the electron charge value measured in experiment (as it is), i.e., the renormalized value, then second order effects in the non spin contribution of (16-84) are already included in the renormalization of e. Given that, the only relevant factors contributing to that part of the amplitude are e and Φ. Hence,

Conclude $F_1(0) = 1$

$$F_1(0) \text{ must } = 1. \tag{16-85}$$

Step #3. Form of Relevant Spin Terms

Step #3

With $F_1(0) = 1$ and our results from the previous Sect. 16.4.5 (where we showed that terms having factors in $\sigma^{0\nu}$ drop out and those with factors of σ^{ij} can be re-expressed in terms of the spin operator $\mathbf{\Sigma}$), relation (16-84), where the symbolism

$$\nabla \times \mathbf{A}^\varepsilon(\mathbf{k}_\nu) \text{ means } \nabla \times \mathbf{A}^\varepsilon(\mathbf{x}) \text{ Fourier transformed to momentum space,} \tag{16-86}$$

is

Amplitude with $F_1(0) = 1$ and prior results

$$\mathcal{M}^{(2+4)}_{\substack{mm \\ k_\nu \to 0}} = i\bar{u}_{r'}(\mathbf{p}')\Bigg(\underbrace{e\Phi}_{\substack{\text{non spin} \\ \text{contribution}}} \underbrace{-\frac{\overbrace{e}^{\mu_B}}{2m}\overbrace{i\sigma^{\mu\nu}k_\nu A^e_\mu(\mathbf{k}_\nu)}^{-\overset{g}{\overbrace{2}}\mathbf{\Sigma}\cdot\nabla\times\mathbf{A}^e(\mathbf{k}_\nu)}}_{\substack{g=2 \text{ part of spin} \\ \text{contribution}}} + \underbrace{F_2(0)\frac{\overbrace{e}^{\mu_B}}{2m}\overbrace{i\sigma^{\mu\nu}k_\nu A^e_\mu(\mathbf{k}_\nu)}^{-2\mathbf{\Sigma}\cdot\nabla\times\mathbf{A}^e(\mathbf{k}_\nu)}}_{\substack{\text{higher order spin part} \\ \text{contribution to } g}}\Bigg)u_r(\mathbf{p}), \tag{16-87}$$

or

$$\mathcal{M}^{(2+4)}_{\substack{mm \\ k_\nu \to 0}} = i\bar{u}_{r'}(\mathbf{p}')\big(e\Phi + (2 \underbrace{-2F_2(0)}_{\substack{\text{higher order} \\ \text{correction} \\ \text{to 1st order } g}})\mu_B \boldsymbol{\Sigma} \bullet \nabla \times \mathbf{A}^e(\mathbf{k}_\nu)\big)u_r(\mathbf{p}). \qquad (16\text{-}88)$$

Note that 2^{nd} order part of g in amplitude has factor of F_2 (0)

We want to find $F_2(0)$. To help us do this, we need to note one more thing.

We only need to find F_2 (0)

We employ Gordon's identity (16-77) again by solving it for the term with $\sigma^{\mu\nu}$ in it and substitute that into (16-81) to get

$$\mathcal{M}^{(2+4)}_{\substack{mm \\ k_\nu \to 0}} = ie\bar{u}_{r'}(\mathbf{p}')\left(\gamma^\mu - \gamma^\mu F_2(0) + \frac{p^\mu + p'^\mu}{2m}F_2(0)\right)u_r(\mathbf{p})A^e_\mu(\mathbf{k}_\nu). \qquad (16\text{-}89)$$

Use Gordon's identity to re-express amplitude

We are only interested in F_2 (0), as it is all that contributes to the higher order spin correction to g at low k_ν. So, we can ignore all terms proportional to γ^μ and focus entirely on terms having form like that of the last term in (16-89). That is, we use Gordon's identity to re-arrange our amplitude into terms with a single gamma matrix γ^μ and terms that are spinor scalars. The factor $F_2(0)$ we seek is then the sum of every spinor scalar term divided by $(p^\mu + p'^\mu)/2m$. When we find $F_2(0)$, we can simply substitute it into (16-88) and determine the higher order contribution to the gyromagnetic ratio g.

To find F_2 (0) we can ignore terms in γ^μ and only evaluate those in $(p^\mu + p'^\mu)/2m$.

NOTE: FOR REMAINDER OF CHAPTER, WHERE PRIOR VERSION HAD F_B, SUBSTITUTE F_2. THE TWO ARE ACTUALLY EQUAL ANYWAY, FROM (16-78), BUT IN THE ABOVE REVISION, WE SWITCHED TO USING F_B.

NOTE ALSO: PROBLEM 9 HAS BEEN DELETED AS IT RELATED TO A PART OF SECT 16.4.6 THAT WAS REMOVED.