where $A^\mu = (\phi, \mathbf{A})$ with ϕ being the electric field scalar potential and $\mathbf{A}$ being the magnetic field vector potential. For RQM, $a_s(\mathbf{k})$, $a_s^\dagger(\mathbf{k})$ are constants (usually represented by $A_s(\mathbf{k})$, $A_s^\dagger(\mathbf{k})$ in that theory), For QFT, they are operators (represented by lower case) that destroy and create photons. The plane wave solution form (16-2) is well suited to many problems and experiments.

16.1.1 Classical/RQM Derivation of the Coulomb Potential

But spherical coords best for Coulomb potential

Coulomb's potential, however, is different in that it describes a field extending radially outward from a source (charge), so using a spherical coordinate system would be far simpler. In addition, Maxwell's equation inside the charged region (see Chap. 7, relation (7-18), pg. 186) becomes

$$\partial^\alpha \partial_\alpha A^\mu = s^\mu = -e\bar{\psi}\gamma^\mu\psi \ . \tag{16-3}$$

For N (negatively charged) fermions occupying the charged region (such as electrons in a metallic sphere), we can use a modified form of (16-3),

$$\partial^\alpha \partial_\alpha A^\mu = -N\, e\bar{\psi}\gamma^\mu\psi \ \left(= Ze\bar{\psi}\gamma^\mu\psi \text{ for } Z \text{ positively charged fermions}\right). \tag{16-4}$$

Sourceless Maxwell's eq in spherical coords

However, for the Coulomb potential this becomes simplified because that potential is measured in the region *outside* the charged region, where no charged fermion field ψ exists. That is, the fermion field carrying the charge extends throughout the source particle/object to its surface, but no further. Outside the surface $\psi = 0$, and that is our region of interest.

So (16-1) governs in that region, and we prefer a spherical, rather than Cartesian, 3D coordinate system. In such coordinates, (16-1) can be expressed, with $\mu = 0, 1, 2, 3$ representing t, r, θ, ϕ, as

$$\partial_\alpha \partial^\alpha A^\mu = \frac{\partial^2}{\partial t^2}A^\mu - \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(rA^\mu\right) - \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}A^\mu\right) - \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}A^\mu = 0\,. \tag{16-5}$$

Reduces to simple form for A^μ a static function of r

But since the field is symmetric spherically about the origin, where the charge is located, A^μ can only be a function of r and t. The Coulomb field is static (not a function of t), so (16-5) reduces to

$$\frac{\partial^2}{\partial r^2}\left(rA^\mu\right) = 0 \quad \text{for spherically symmetric, static source}\,. \tag{16-6}$$

Solution to that simple form

The general solution to (16-6), readily shown by substitution, is $A^\mu = \varepsilon_s{}^\mu\, C/r + \varepsilon_s{}^\mu\, D$, where C and D are constants. Physically, the field must vanish at infinity, so $D = 0$, and

$$A^\mu \propto \frac{1}{r}\varepsilon_s^\mu \quad \text{or as column matrix,} \quad A^\mu = \begin{bmatrix} A^t \\ A^r \\ A^\theta \\ A^\phi \end{bmatrix} = \frac{1}{r}\begin{bmatrix} A_0^t \\ A_0^r \\ A_0^\theta \\ A_0^\phi \end{bmatrix} = \frac{1}{r}\begin{bmatrix} \Phi_0 \\ A_0^1 \\ A_0^2 \\ A_0^3 \end{bmatrix} = \begin{bmatrix} \Phi(r) \\ \mathbf{A}(r) \end{bmatrix}. \tag{16-7}$$

Magnetic field (curl of A^μ) in spherical coordinates

From the physical symmetry, the 3D vector potential can only have a radial direction, so it cannot have any component in the angular directions θ or ϕ, i.e., $A_0{}^\theta = A_0{}^\phi = 0$. Thus,

$$\mathbf{B} = \nabla\times\mathbf{A} = \varepsilon_s^r\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(A^\phi\sin\theta\right) - \frac{\partial A^\theta}{\partial\phi}\right) - \varepsilon_s^\theta\left(\frac{1}{r\sin\theta}\frac{\partial A^r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rA^\phi\right)\right) + \varepsilon_s^\phi\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA^\theta\right) - \frac{\partial A^r}{\partial\theta}\right) = \mathbf{0}. \tag{16-8}$$

$\mathbf{B} = 0$ for spherically symmetric case

So without loss of generality can assume $\mathbf{A} = 0$

and no magnetic field is produced. To keep things simple, we can therefore just take $\mathbf{A} = 0$, (i.e., $A_0{}^r = 0$ also) without loss of generality (in this spherically symmetric case.)

From boundary conditions on the surface of the charged spherical source ($A^t = \Phi$ just on either side of the surface must be equal, though we won't go through the formal mathematics of it all), we obtain the constant $A_0{}^t$. We then end up with (16-7) having the well known Coulomb potential (in Heaviside-Lorentz units) as the timelike component of A^μ,

$$A^{\mu}=\begin{bmatrix}\Phi\\0\\0\\0\end{bmatrix}=\begin{bmatrix}-eN/(4\pi r)\\0\\0\\0\end{bmatrix}\quad\left(=\begin{bmatrix}Ze/(4\pi r)\\0\\0\\0\end{bmatrix}\begin{array}{l}\text{for a nucleus of } Z \text{ positively}\\ \text{charged protons approximated}\\ \text{by Coulomb potential}\end{array}\right). \tag{16-9}$$

16.1.2 Using the Coulomb Potential in Relativistic Hydrogen Atom

Recall from Chap. 7 (Sects. 7.1.4, and 7.2.1 pgs. 184-186) that our governing interaction equations for coupled photon-fermion fields are (16-3) and the full (interacting) Dirac equation

RQM solution to H atom assumes Coulomb potential, though this is only an approximation

$$\left(i\gamma^{\mu}\partial_{\mu}-m\right)\psi=-e\gamma^{\mu}\psi A_{\mu}\,. \tag{16-10}$$

To solve the H atom case exactly, we would need to solve coupled equations like (16-3)/(16-4) and (16-10), because the fermion field of the nucleus, the fermion field of the orbital electron, and the electromagnetic (photon) field all interact with one another throughout the entire region of the atom. Finding a closed form solution for this is essentially impossible.

To get a good approximation for the relativistic atom, however, we can assume the A^{μ} field, which really results from both the nucleus and the orbital electron (and for which the nuclear fermion field extends outside a clear spherical boundary of the nucleus), is just due to the nucleus and has Coulomb potential form as in (16-9). With that approximation substituted into (16-10), one then goes about solving the resulting equation. This is just the procedure we outlined in the first of the above referenced sections.

Doing that provides a more accurate solution (the relativistic solution) to the hydrogen atom, the orbital energy levels (eigenvalues), and thus the spectral line distribution seen in measurements. However, one might expect the resulting solution, due to the approximation (16-9) does not precisely match experiment. One would be right. One such discrepancy, a subtle but distinct one, is known as the Lamb shift, a slight shifting of the spectral lines in their actual measurement from that predicted by the above analysis approach. We discuss the Lamb shift and its successful prediction via QFT later in this chapter.

16.2 Coulomb Potential in QFT

Coulomb potential in QFT

One could simply assume in QFT that the form of Coulomb potential is same as that in RQM, since we found throughout our development of both theories that they paralleled one another in terms of the governing equations and solution forms, and differed only in the interpretation of the solution coefficients as constants or operators.

Doing so in the above described hydrogen atom analysis, for example, would have provided field eigen solutions of particular form, the same form as the state eigen solutions of RQM. The operators of those fields would create and destroy states mirroring those solutions, i.e., with the same eigen energies, spins, etc., and thus the same spectral line predictions.

However, for the sake of completeness, and to justify the parallel solutions argument for QFT, we present a derivation of the Coulomb potential from the perspective of QFT.

16.2.1 Repulsive Coulomb Scattering Equivalence to Møller Scattering

QFT repulsive Coulomb potential

Repulsive Coulomb scattering can be represented by Møller scattering as shown in Fig. 16-1, where the source charge particle is spherical (has a radial distribution of its radiation.)

Assume distinguishable particle to keep things simple

If the incoming particles in Fig. 16-1 are indistinguishable, such as two electrons, we need to include both diagrams to determine the amplitude. But, if they are distinguishable, such as an electron and a muon, then we only need to consider the LH diagram. (Because there is no indeterminancy in which original particle mutated into which final particle.) Further, the classical Coulomb potential is always between macro (distinguishable) objects. So, to make things simpler, we will assume the particles are distinguishable and examine the transition amplitude for only the LH diagram in Fig. 16-1.

Assume non-relativistic speeds for electrons

We will also assume non-relativistic speeds of our incoming and outgoing particles, as that is typically the case for Coulomb scattering. (And it makes our calculations simpler.) One can think of the particle labeled #1 as the source, whose radiated virtual particle affects the particle labeled #2.