15.4.2 Relations for Arbitrary Dimension Spacetime

We will take $D \neq 4$ relations mathematicians give us without deriving them

The mathematics behind *D* dimension spaces is extensive and delving into it to any depth would consume considerable time and effort. Instead, we will simply cite certain results the mathematicians have provided to us physicists for use in renormalization. We start with integer values for *D*, and then will extrapolate, rather uncritically, to non-integer values.

First for integer D

Metrics in *D* Integer Dimensions

Metric contraction for general D spacetime

For *D* any integer, $g_{\mu\nu}$ is a *D*x*D* matrix. Parallel to $g_{\mu\nu}g^{\mu\nu} = 4$ for *D* = 4 spacetime, we have

$$g_{\mu\nu}g^{\mu\nu} = D . \tag{15-61}$$

Gamma Matrices in *D* Integer Dimensions

Gamma matrices relation for general D spacetime

In *D* dimensions, where *D* is an integer, there are *D* gamma matrices labeled $\gamma^0, \gamma^1, ...\gamma^{D-1}$. These are $f(D) \times f(D)$ matrices, where $f(D)$ is an integer that depends on *D*. For *D* = 4, $f(D) = 4$.

γ matrices for integer *D* satisfy anti-commutation relations similar those seen before,

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} . \tag{15-62}$$

From these, one can derive contraction and trace relations parallel to (15-20) to (15-23). That is

$$\gamma_{\lambda}\gamma^{\lambda} = D, \qquad \gamma_{\lambda}\gamma^{\alpha}\gamma^{\lambda} = -(D-2)\gamma^{\alpha}$$
$$\gamma_{\lambda}\gamma^{\alpha}\gamma^{\beta}\gamma^{\lambda} = -(D-4)\gamma^{\alpha}\gamma^{\beta} + 4g^{\alpha\beta} \qquad \text{etc.} \tag{15-63}$$

$$\text{Tr}\left(\gamma^{\alpha}\gamma^{\beta}\right) = f(D)g^{\alpha\beta} , \tag{15-64}$$

$$\text{Tr}\left(\gamma^{\sigma}\gamma^{\delta}\gamma^{\mu}...\right) = 0 \quad \text{for any odd number of gamma matrices} , \tag{15-65}$$

$$\text{Tr}\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\rho}\right) = f(D)\left(g^{\alpha\beta}g^{\gamma\rho} - g^{\alpha\gamma}g^{\beta\rho} + g^{\alpha\rho}g^{\beta\gamma}\right)$$

or as we will see the indices later $\text{Tr}\left(\gamma^{\mu}\gamma^{\delta}\gamma^{\nu}\gamma^{\sigma}\right) = f(D)\left(g^{\mu\delta}g^{\nu\sigma} - g^{\mu\nu}g^{\delta\sigma} + g^{\mu\sigma}g^{\delta\nu}\right)$. (15-66)

Key Integrals in *D* Integer Dimensions

Demo: how we use Wick rotation to turn Euclidean D space integral into D spacetime integral

For a Euclidean space of arbitrary integer dimension *D*, the mathematicians have provided us with the integral

$$\int \frac{1}{(2\pi)^D} \frac{d^D p_E}{\left(p_E^2 - s\right)^2} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)} \frac{1}{s^{2-\frac{D}{2}}} . \tag{15-67}$$

We can use this to find its equivalent in *D* dimensional spacetime. First perform an inverse Wick rotation transformation (3rd step below) on the LHS of (15-67) to get

$$\int \frac{1}{(2\pi)^D} \frac{1}{\left(p_E^2 - s\right)^2} d^D p_E = \frac{1}{(2\pi)^D} \int \frac{1}{\left(-p_E^2 + s\right)^2} d^D p_E = \frac{-i}{(2\pi)^D} \int \frac{1}{\left(p^2 + s\right)^2} d^D p . \tag{15-68}$$

From (15-67),

$$\frac{1}{(2\pi)^D} \int \frac{1}{\left(p^2 + s\right)^2} d^D p = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)} \frac{1}{s^{2-\frac{D}{2}}} \tag{15-69}$$

In similar fashion, one can deduce other relations for *D* dimensional spacetime parallel to (15-3) to (15-12). We list the most relevant of these (with the (2π) factors arranged differently from (15-69)) below where we use *q* instead of *p* to represent the general case. Note that for *n* = 2, (15-70) is (15-69).

In similar fashion, other D spacetime integrals derived from Euclidean ones

$$\int \frac{1}{\left(q^2 + s\right)^n} d^D q = i\pi^{D/2} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)} \frac{1}{s^{n-\frac{D}{2}}} \tag{15-70}$$

$$\int \frac{q^{\mu}}{\left(q^2 + s\right)^n} d^D q = 0 \tag{15-71}$$

$$\int \frac{q^\mu q^\nu}{\left(q^2+s\right)^n}\, d^D q = i\pi^{D/2}\,\frac{\Gamma\left(n-1-\frac{D}{2}\right)}{2\Gamma(n)}\,\frac{g^{\mu\nu}}{s^{n-1-D/2}} \tag{15-72}$$

$$\int \frac{q^2}{\left(q^2+s+i\varepsilon\right)^n}\, d^D q = i\pi^{D/2}\,\frac{\Gamma\left(n-1-\frac{D}{2}\right)}{2\Gamma(n)}\,\frac{D}{s^{n-1-D/2}}\,. \tag{15-73}$$

Extrapolating to Non-Integer D Dimensions

Now assume above relations work in non-integer D spaces

Note that the gamma function Γ is also defined for non-integer D, so the RHS of integrals (15-69), (15-70), (15-71), and (15-73) remain valid in that case as well. So, we simply assume all of the relations (15-61) to (15-73) hold for both integer and non-integer D.

One may feel some unease with a metric $g_{\mu\nu}$ and γ matrices for dimension spaces where D is not an integer. However, when we use these entities in such cases, in our final result we always take $D \to 4$, so we can carry the symbols representing them along as we go, knowing that all will be OK in the end. Though this implies the seemingly weird process of integration over fractional dimension spaces, it does turn out to work, as we are about to see.

With this most general interpretation of the above relations, we can re-visit the simple example we regularized via the Pauli-Villars method, but this time, using dimensional regularization.

15.4.3 The Same Simple (Unphysical) Example Again

Dimensional regularization applied to Pauli-Villars example

We can use (15-69) to deduce the integral (15-56) in D dimensional spacetime and then take $D \to 4$. Ignore the ε for convenience and use $s = -m^2$. Note that the LHS of (15-69) with $D = 4$ is the same integral (15-56) we evaluated using Pauli-Villars regularization.

Using standard integral for D spacetime

$$\frac{1}{(2\pi)^4}\int \frac{1}{\left(p^2-m^2\right)^2}\, d^4 p \longrightarrow \frac{1}{(2\pi)^D}\int \frac{1}{\left(p^2-m^2\right)^2}\, d^D p = \frac{i}{(4\pi)^{D/2}}\,\frac{\Gamma\left(2-\frac{D}{2}\right)}{\Gamma(2)}\left(\frac{1}{m^2}\right)^{2-\frac{D}{2}} \tag{15-74}$$

Γ(z) has poles (goes to infinity) at 0, –1, –2, …., so (15-74) has poles at D = 4, 6, 8, … To examine the behavior around D = 4, define $\eta = 4 - D$ and use the approximation (which hopefully you can accept like an integral from a table)

$$\Gamma\left(2-\frac{D}{2}\right) = \Gamma\left(\frac{\eta}{2}\right) \underset{\substack{\eta\to 0\\ D\to 4}}{\to} \frac{2}{\eta} - \gamma + \mathcal{O}(\eta)\,, \tag{15-75}$$

Gamma function in limit

where γ here is the Euler-Mascheroni constant ≈ .5772, which will always cancel in observable quantities. We also use the standard relation

$$a^x = 1 + x\ln a + \frac{(x\ln a)^2}{2!} + \frac{(x\ln a)^3}{3!} + \ldots. \tag{15-76}$$

Expansion of a^x

with $a = 1/m^2$ and $x = \eta/2$ to obtain

$$\left(\frac{1}{m^2}\right)^{2-\frac{D}{2}} = \left(\frac{1}{m^2}\right)^{\frac{\eta}{2}} \underset{\substack{\eta\to 0\\ D\to 4}}{\to} = 1 + \frac{\eta}{2}\ln\frac{1}{m^2} + \mathcal{O}\left(\eta^2\right) = 1 - \frac{\eta}{2}\ln m^2 + \mathcal{O}\left(\eta^2\right) = 1 - \mathcal{O}(\eta)\,. \tag{15-77}$$

Using limiting values as D → 4

(15-74), with Γ(2) = 1, is then

$$\frac{1}{(2\pi)^D}\int \frac{1}{\left(p^2-m^2\right)^2}\, d^D p \underset{\substack{\eta\to 0\\ D\to 4}}{\to} \frac{i}{(4\pi)^2}\left(\frac{2}{\eta} - \gamma + \mathcal{O}(\eta)\right)\left(1-\mathcal{O}(\eta)\right). \tag{15-78}$$

In the limit $\eta \to 0$, this becomes (referring back to (15-68)

We get integral for D = 4

$$\frac{1}{(2\pi)^4}\int \frac{1}{\left(p^2-m^2\right)^2}\, d^4 p \underset{\substack{\eta\to 0\\ D\to 4}}{\to} \frac{i}{(4\pi)^2}\left(\frac{2}{\eta} - \gamma\right), \tag{15-79}$$

found via dimensional regularization.