To add to appendix of Chapter 15

Going from (15-44) to (15-45)

To start, note the following.

$$\int_0^1 z(1-z)dz = \int_0^1 \left(z - z^2\right)dz = \left[\frac{z^2}{2} - \frac{z^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \tag{1}$$

$$\int_0^1 z\,ln\,z dz = \left[z^2\frac{ln\,z}{2} - \frac{z^2}{4}\right]_0^1 = \frac{ln 1}{2} - \frac{1}{4} = -\frac{1}{4} \qquad \int_0^1 z^2\,ln\,z dz = \left[z^3\frac{ln\,z}{3} - \frac{z^3}{9}\right]_0^1 = \frac{ln 1}{3} - \frac{1}{9} = -\frac{1}{9} \tag{2}$$

$$\underbrace{\int_0^1 z(1-z)ln(1-z)dz = -\int_1^0 z'(1-z')ln\,z'dz'}_{\text{substitute } z=1-z'} = \underbrace{\int_0^1 z'\,ln\,z'dz' - \int_0^1 z'^2\,ln\,z'dz'}_{\text{see above}} = -\frac{1}{4} + \frac{1}{9} = -\frac{5}{36} \tag{3}$$

Also note the following for logs of negative numbers, where $z > 0$,

$$ln(-1) = ln\left(e^{i\pi}\right) = i\pi \quad \rightarrow \quad ln(-z) = ln(-1)z = ln(-1) + ln\,z = i\pi + ln\,z\,. \tag{4}$$

In general, the log of a negative number is a complex number.

Now, look at the first two terms in the integrand of (15-44). First the first of those two terms, where after the first equal sign we use (4), and after the last equal sign we use (1) and (3),

$$\int_0^1 \left\{z(1-z)ln(-z)dz = \int_0^1 \left\{z(1-z)(i\pi + ln\,z)dz = i\pi\int_0^1 z(1-z)dz + \int_0^1 z(1-z)ln\,zdz = \frac{i\pi}{6} - \frac{5}{36}\,. \tag{5}$$

For the second of the two terms, with (3)

$$\int_0^1 \left\{z(1-z)ln(1-z)dz = -\frac{5}{36}\,. \tag{6}$$

Thus, adding (5) and (6), as we do in (15-44), we get

$$\int_0^1 \left\{z(1-z)\left(ln(-z) + ln(1-z)\right)dz = \frac{i\pi}{6} - \frac{5}{36} - \frac{5}{36} = \frac{i\pi}{6} - \frac{5}{18}\,. \tag{7}$$

The first two terms in the second line of (15-44) are identical to the first two terms in the first line of (15-44), so they all sum to $i\pi/3 - 5/9$, and since this is of order 1, we can ignore all of these terms.