(IV) Most General Form of the Loop Integrals

The most general possible form for $\Pi^{\mu\nu}_{nth}(k)$ is

$$\Pi^{\mu\nu}_{nth}(k) = -g^{\mu\nu} A_{nth}\left(k^2\right) + \underbrace{k^\mu k^\nu B_{nth}\left(k^2\right)}_{\text{drops out}} . \qquad (14\text{-}56)$$

Column (IV) shows most general form of these

In the 2nd order development we simply stated that the B term (analogous to B_{nth} above) didn't show up when we did the formal integration using regularization. In general, for higher orders, this is also true for the B_{nth} term. In the Appendix to this chapter we show why, and how, this occurs. The bottom line, as shown there, is that it is the result of current conservation.

B_{nth} term drops out for any order n due to gauge invariance (symmetry)

Key Point

Note that the B_{nth} term dropping out enables the theory to be renormalizable. This drops out because of current conservation, and current conservation is a direct result of local gauge invariance (internal symmetry) of the Lagrangian. (See Chap. 11.) Renormalization depends on local symmetry. If we don't have a symmetric (gauge invariant) theory, we don't have a finite theory, i.e., we don't have a theory that works.

Gauge invariance necessary for a finite theory

For fermions, similar to the 2nd order case, we can express the most general form for $\Sigma_{nth}(p)$ in terms of $\left(\not p - m_0\right)$ or $\left(\not p - m\right)$, but using the form with m makes the development easier. Thus, we have, similar to the 2nd order case, where again the A_{nth} and B_{nth} of (14-57) are different from those in (14-56),

$$\Sigma_{nth}(p) = A_{nth} + \left(\not p - m\right) B_{nth} + \left(\not p - m\right)\Sigma_{c\,nth}\left(\not p - m\right). \qquad (14\text{-}57)$$

And again in parallel with the 2nd order case, the most general form for $\Lambda^{\mu}_{nth}(p,p')$ is

$$\Lambda^{\mu}_{nth}(p,p') = a_{nth}\gamma^\mu + b_{1\,nth}\, p^\mu + b_{2\,nth}\, p'^\mu . \qquad (14\text{-}58)$$

(V) Series Expansion of the Loop Integrals

The general forms of column (IV) are here expressed as Taylor series expansions. The photon relation is expanded in powers of k^2; the fermion one, in powers of $\left(\not p - m\right)$, just as we did in the 2nd order case. Like that case, these expansions contain divergent and convergent terms, the latter labeled with subscript "c".

Column (V) contains series expansion form

For the photon, (14-56) becomes

$$\Pi^{\mu\nu}_{nth}(k) = -g^{\mu\nu} A_{nth}\left(k^2\right) = -g^{\mu\nu}\left(\underbrace{A_{nth}(0)}_{=0} + k^2 A'_{nth}(0) + k^2 \Pi_{c\,nth}\left(k^2\right)\right). \qquad (14\text{-}59)$$

In the 2nd order case we simply stated that the term $A(0)$ turned out to be zero when the integration was carried out via regularization. We can show this in general, and thus make it applicable to any order case, i.e., $A_{nth}(0) = 0$. But we need to wait until part (VI) below to do this.

The vertex form in column (V) is not exactly a series expansion, but is the most general form in terms of L_{nth} and $\Lambda^{\mu}_{nth}(p,p')$, rather than the a_{nth}, $b_{1,\,nth}$, and $b_{2,\,nth}$ of the prior column. This form was found via the exact same steps used for the 2nd order vertex case in going from column (IV) to column (V) there, i.e., (14-58)becomes

$$\Lambda^{\mu}_{nth}(p,p') = L_{nth}\gamma^\mu + \Lambda^{\mu}_{c\,nth}(p,p'). \qquad (14\text{-}60)$$

(VI) Putting Expansions (V) into Propagator, Leg, Vertex Modifications (II)

Here we insert the expansions of the nth order integrals of column (V) into the 2nd order correction relations for the propagators and vertex of column (II).

Column (VI) puts expansions of (V) into math of (II)

nth Order Photon Propagator Expansion

For the photon, using (14-4), we take (where below our operators happen to be mere numbers)

$$\text{operator } \frac{1}{A} = \frac{-i}{k^2 + i\varepsilon} = \frac{1}{i\left(k^2 + i\varepsilon\right)}, \text{ and} \qquad (14\text{-}61)$$

$$\text{operator } B = -ie_0^2 A_{nth}\left(k^2\right) = -ie_0^2\left(A_{nth}(0) + k^2 A'_{nth}(0) + k^2\Pi_{c\,nth}\left(k^2\right)\right) \tag{14-62}$$

Using (14-56) in the second line below, and (14-61) and (14-62) in the third, (14-46) then becomes

Operator relation yields $iD^{nth}_{F\alpha\beta}(k)$ in terms of integral expansion quantities

$$iD^{nth}_{F\alpha\beta}(k) = iD_{F\,\alpha\beta}(k) + iD_{F\,\alpha\mu}(k)\, ie_0^2 \Pi^{\mu\nu}_{nth}(k)\, iD_{F\,\nu\beta}(k) + \ldots$$

$$= \frac{g_{\alpha\beta}}{i\left(k^2+i\varepsilon\right)} + \frac{g_{\alpha\mu}}{i\left(k^2+i\varepsilon\right)} ie_0^2\left(-g^{\mu\nu}\right)A_{nth}\left(k^2\right)\frac{g_{\nu\beta}}{i\left(k^2+i\varepsilon\right)} + \ldots \tag{14-63}$$

$$= \frac{g_{\alpha\beta}}{i\left(k^2+i\varepsilon\right)} - \frac{g_{\alpha\beta}}{i\left(k^2+i\varepsilon\right)} ie_0^2 A_{nth}\left(k^2\right)\frac{1}{i\left(k^2+i\varepsilon\right)} + \ldots = g_{\alpha\beta}\left(\frac{1}{A} + \frac{1}{A}B\frac{1}{A} + \frac{1}{A}B\frac{1}{A}B\frac{1}{A} + \ldots\right).$$

With (14-4) and A_{nth} defined as in (14-59) (or (14-62)), we have

$$iD^{nth}_{F\alpha\beta}(k) = \frac{1}{A-B}g_{\alpha\beta} = \frac{g_{\alpha\beta}}{i\left(k^2+i\varepsilon\right) + i\left(e_0^2 A\left(k^2=0\right) + e_0^2 k^2 A'\left(k^2=0\right) + e_0^2 k^2 \Pi_c(k^2)\right)}. \tag{14-64}$$

Now consider the case where (14-64) approaches a representation of a real photon. That is, the virtual photon (propagator) it represents approaches being on shell, i.e., $k^2 \to 0$. The real photon propagator has a pole at $k^2 = 0$. That is, it goes to infinity in this limit. (This is verified by experiment, i.e., by the predictions made for virtual photons approaching such limits.)

Taking $iD^{nth}_{F\alpha\beta}(k)$ in limit $k^2 \to 0$ proves quantity $A(0) = 0$.

All of the terms in the denominator of (14-64) go to zero as $k^2 \to 0$ except the term with $A(0)$ in it. If that term were non-zero, it would keep (14-64) finite in the real photon propagator limit, which does not happen. Therefore, $A(0)$ must equal zero, as we note in (14-59). Thus, (14-64) becomes

$$iD^{nth}_{F\alpha\beta}(k) = \frac{g_{\alpha\beta}}{i\left(k^2+i\varepsilon\right) + i\left(e_0^2 k^2 A' + e_0^2 k^2 \Pi_c(k^2)\right)} = \frac{-ig_{\alpha\beta}}{\left(k^2+i\varepsilon\right)\left(1 + e_0^2 A' + e_0^2 \Pi_c(k^2)\right)}$$

$$\approx \frac{1}{1+e_0^2 A'} \underbrace{\frac{-ig_{\alpha\beta}}{\left(k^2+i\varepsilon\right)}}_{iD_{F\alpha\beta}(k)} \frac{1}{1+e_0^2\Pi_c(k^2)}. \tag{14-65}$$

Most useful form of $iD^{nth}_{F\alpha\beta}(k)$

<u>*n*th Order Fermion Propagator Expansion</u>

Operator relation $\to iS^{nth}_F(p)$ in terms of integral expansion quantities

In (14-51) we take the relation (14-4) operators to be $1/A = iS_F(p) = i/\left(\not{p} - m_0 + i\varepsilon\right)$ and $B = ie_0^2\Sigma_{nth}(p)$. (See column (V)). This yields

Renormalize mass in the process (see (VIII)

$$iS^{nth}_F(p) \approx \frac{1}{A-B} = \frac{i}{\not{p}\underbrace{-m_0 + e_0^2 A_{nth}}_{-m} + e_0^2\left(\not{p}-m\right)B_{nth} + e_0^2\left(\not{p}-m\right)\Sigma_{c\,nth} + i\varepsilon}$$

$$= \frac{i}{\not{p} - m + e_0^2\left(\not{p}-m\right)B_{nth} + e_0^2\left(\not{p}-m\right)\Sigma_{c\,nth} + i\varepsilon} \tag{14-66}$$

$$= \frac{i}{\left(\not{p}-m+i\varepsilon\right)\left(1+e_0^2 B_{nth} + e_0^2\Sigma_{c\,nth}\right)} \approx \frac{1}{1+e_0^2 B_{nth}} \underbrace{\frac{i}{\not{p}-m+i\varepsilon}}_{iS_F(p)} \frac{1}{1+e_0^2\Sigma_{c\,nth}}.$$

Most useful form of $iS^{nth}_F(p)$

In the first line above, we renormalized mass in the same way we did it for the 2nd order case. More on this in (VIII).

<u>*n*th Order External Particles in Terms of Expansion Quantities</u>

Following logic exactly parallel to what we did for the 2nd order case, we find

$$u^{nth}_r(\mathbf{p}) = \frac{1}{\left(1+e_0^2 B_{nth}\right)^{1/2}} u_r(\mathbf{p}) \quad \text{same for } \bar{u}^{nth}_r(\mathbf{p}),\ v^{nth}_r(\mathbf{p}),\ \bar{v}^{nth}_r(\mathbf{p}), \text{ and} \tag{14-67}$$

External line corrections to nth order