$$T\left\{A_1B_2\right\} = N\left\{A_1B_2\right\} + \underbrace{A_1B_2}_{} \quad \text{for any time order} \,. \tag{7-97}$$

Result: Wick's theorem proven for two fields

7.8.3 More Than Two Fields

Three fields treated, and extended by induction to more fields, in Appendix A

Hopefully, we have gained some comfort with Wick's theorem by working through the examples with only two fields. And perhaps, we can simply accept that the mathematicians have proven Wick's theorem formally, for us. Much like integral tables, which we employ regularly without proving each relation we use, we can simply accept Wick's theorem, apply it to our work at hand, and move on.

Those wishing to dig deeper and understand a bit better can read Appendix A where we extend the above type of analysis to three fields. In that appendix, we then use induction to justify Wick's theorem for any number of fields.

For those who feel the need for a formal proof, see the original article "The Evaluation of the Collision Matrix" by G. C. Wicks (Phys. Rev. **80**, 268, 1950), or any of "Notes on Wick's Theorem in Many-Body Theory" by Luca Guido Molinari (wwwteor.mi.infn.it/~molinari/NOTES/Wick.pdf), *Quantum Field Theory for Mathematicians* by R.Ticciati (Cambridge University Press 1999, pg. 85-87), and *Field Quantization* by W. Greiner and J. Reinhardt (Springer, 1966, pg. 231-233).

7.8.4 The Issue of Equal Time Operators

Turning Our Different Times Relation into One with Some Equal Times

Operators at same time seemingly not treated in above

Readers may have noticed that (7-82) seemed to be stated for fields $A_1B_2C_3D_4\ldots$ where times t_1 of field A_1, t_2 of field B_2, t_3 of C_3, etc. are all different (none are the same time.) In contrast, our statement of Wick's theorem (7-78) was more general in the sense that it has several fields at the same time, such as $A_1B_1C_1D_1\ldots = (ABCD\ldots)_{x1}$ all at the same time, and $A_2B_2C_2D_2\ldots = (ABCD\ldots)_{x2}$ all at the same, but different from t_1, time.

But can generalize by taking $t_2 = t_1$, for example

We can generalize (7-82) by taking, for example, $t_1 = t_2$, so that $A_1B_2 \rightarrow A_1B_1$. That is, wherever we have different fields in (7-82), we can just assume some have equal times. We should thus be able to derive Wick's theorem (7-78) entailing more than one field at the same time from our relation (7-82). We would find (7-82) then looks like

$$T\left\{A_1B_1C_1..F_2G_2..\right\} = N\left\{A_1B_1C_1..F_2G_2..\right\} + N\left\{\underbrace{A_1B_1}_{}C_1..F_2G_2..\right\} + N\left\{\underbrace{A_1B_1C_1}_{}....F_2G_2..\right\} + ..$$

$$+ N\left\{\underbrace{A_1B_1}_{}C_1..\underbrace{F_2G_2}_{}..\right\} + N\left\{A_1B_1C_1..F_2G_2..\right\} + ... \tag{7-98}$$

+ (all normal ordered terms with three contractions) + etc.,

which with slightly different notation looks a lot like Wicks' theorem (7-78).

But then we get equal times contractions, which are not in Wick's theorem

The Fly in the Ointment

The one difference between (7-98) and Wicks' theorem (7-78) is that the latter (7-78) has no equal times contractions, whereas the former (7-98) does. How do we resolve this?

Resolving the Fly in the Ointment

i) The Traditional Resolution: Normal Ordering in Interaction Hamiltonian

Many treatments resolve this by assuming $\mathcal{H}_I^{\,I}$ is already normal ordered

In traditional QFT, we apply Wick's theorem using $\mathcal{H}_I^{\,I}$, which for QED takes the form (7-79), which we repeat below.

$$(AB...)_{x_1} = \mathcal{H}_I^{\,I}\left(x_1\right) = -e\left(\bar{\psi}\gamma^{\mu}\psi A_{\mu}\right)_{x_1} \qquad (AB...)_{x_2} = \mathcal{H}_I^{\,I}\left(x_2\right) = -e\left(\bar{\psi}\gamma^{\mu}\psi A_{\mu}\right)_{x_2} \quad \text{etc.} \tag{7-99}$$

In that approach it is common to assume the fields in each of $\mathcal{H}_I^{\,I}(x_1)$, $\mathcal{H}_I^{\,I}(x_2)$, etc are already normal ordered. That is,

$$\mathcal{H}_I^{\,I}(x_1) = -eN\left\{\bar{\psi}\gamma^{\mu}\psi A_{\mu}\right\}_{x_1} \qquad \mathcal{H}_I^{\,I}(x_2) = -eN\left\{\bar{\psi}\gamma^{\mu}\psi A_{\mu}\right\}_{x_2} \quad \text{etc.} \tag{7-100}$$

If that is so, then all equal time contractions on the RHS of (7-98) are zero, since each is arrived at by re-ordering the fields $\psi, \bar{\psi}, A_{\mu}$ for each $\mathcal{H}_I^{\,I}(x_i)$ so they are normal ordered. But if they are already normal ordered, no such re-ordering is required, and we have no equal times contractions.

But, as I've said before, normal ordering assumes all fields commute (or for fermions, anti-commute), and since QFT is grounded in, and only exists because of, non-commutation (non-anti-commutation) relations, there seems to be an inconsistency. So, I prefer the following resolution.

But this may contradict basic postulates of QFT

ii) Another Resolution without Invoking Normal Ordering in $\mathcal{H}_I^I$:

Consider three fields (like $\psi, \bar{\psi}, A_\mu$) operating at the same time that we will label A_1, B_1, C_1, each composed of a construction plus a destruction operator. Superscripts c,d imply construction and destruction, respectively. With our N_c and T_c reordering, one such component of $A_1 B_1 C_1$ yields

Other resolution for equal times contractions = 0

$$N_c\left\{A_1^c B_1^d C_1^d\right\} = A_1^c B_1^d C_1^d = N\left\{A_1^c B_1^d C_1^d\right\} \quad = \quad T_c\left\{A_1^c B_1^d C_1^d\right\} = A_1^c B_1^d C_1^d = T\left\{A_1^c B_1^d C_1^d\right\}$$
$$T\left\{A_1^c B_1^d C_1^d\right\} = N\left\{A_1^c B_1^d C_1^d\right\} \quad \rightarrow \text{no equal time commutator (contraction) in Wick theorem.} \tag{7-101}$$

Another component, where we note that if all operators operate at the same time, we can time re-order them any way we like (as long as we include the proper commutation relation), yields

$$\left.\begin{array}{l} N_c\left\{A_1^d B_1^c C_1^d\right\} = B_1^c A_1^d C_1^d + \left[A_1^d, B_1^c\right]C_1^d = N\left\{A_1^d B_1^c C_1^d\right\} + \left[A_1^d, B_1^c\right]C_1^d \\ T_c\left\{A_1^d B_1^c C_1^d\right\} = \underbrace{B_1^c A_1^d C_1^d}_{\substack{\text{can time order} \\ \text{any way we like}}} + \underbrace{\left[A_1^d, B_1^c\right]C_1^d}_{\substack{\text{as long as we} \\ \text{include commutator}}} = \underbrace{T\left\{B_1^c A_1^d C_1^d\right\}}_{=T\left\{A_1^d B_1^c C_1^d\right\}} + \left[A_1^d, B_1^c\right]C_1^d \end{array}\right\} \begin{cases}\text{first line} \\ \text{equal} \\ \text{to second}\end{cases} \tag{7-102}$$

$$T\left\{A_1^d B_1^c C_1^d\right\} = N\left\{A_1^d B_1^c C_1^d\right\} \rightarrow \text{no equal time commutator (contraction) in Wick theorem.}$$

For simplifying choice for order of equal time factors, equal times contractions cancel out, and we can leave them out of Wick's theorem

Repeating for each component of $A_1 B_1 C_1$, we can always choose the time order T we want, since all operators operate at the same time. With the right choice, we get a commutation relation on one side of the equation that cancels with one on the other. Parallel logic holds for fermions/anti-commutators.

This choice of time ordering results in the simplest form for our theory (always the preferred starting point in any theory development) and as we shall see, correctly predicts experiment.

Thus, $N\left\{A_1 B_1 C_1\right\} = T\left\{A_1 B_1 C_1\right\} \rightarrow$ no equal time commutator (contraction) in Wick theorem. (7-103)

To those who might contend that the above is simply sleight-of-hand use of normal ordering, I reply that equal-time commutators, if included anyway, lead to non-physical situations. For example, conservation of 4-momentum in certain associated interactions would only be possible for particles having zero energy, i.e., for particles that do not exist, and thus can be ignored. We will see this in the appendix of Chap. 8, which has been added to the revision of the second edition and posted on the corrections page at the book web site. (See URL on pg. xvi, opposite pg. 1.)

If we included them anyway, they would lead to non-physical situations

<u>The bottom line:</u> Equal-time contractions don't play a role in Wick's theorem (7-78) for QFT.

7.8.5 Summary of Wick's Theorem

To get Wick's theorem, we start with a series of operator fields, operating in arbitrary order and set it equal to itself, i..e, $A_1 B_2 C_3 D_4 \ldots = A_1 B_2 C_3 D_4 \ldots$

On the LHS, we then re-arrange operator fields using commutation/anti-commutation relations such that earlier times are to the right of later times. We herein use the symbol T_c to represent this re-ordering procedure. The final result of the LHS equals the original LHS expression, since at each step, we simply substituted equivalent relations for the original pair of adjacent operators.

Summary of how Wick theorem arises from commutator/anti-commutator relations

On the RHS, we re-arrange operator fields using commutation/anti-commutation relations such that destruction operators are all to the right of creation operators. We herein use the symbol N_c to represent this re-ordering procedure. The final result of the RHS equals the original RHS expression. Thus, the final RHS equals the final LHS.

The final result of these operations is the same as employing Wick's theorem (7-78).

In Wick's theorem, the time ordering operation T re-orders operators with earlier times to the right of later times, but assumes we can switch orders of adjacent operators as if they commuted (or for two fermions, anti-commuted). Similarly, the normal ordering N operator re-orders with destruction operators all on the right, but assumes we can switch orders of adjacent operators as if they commuted (or for two fermions, anti-commuted).

Using the T_c and N_c procedures, we find contractions arising in the final result. Using the T and N operations, we insert those same contractions, as designated in Wick's theorem.